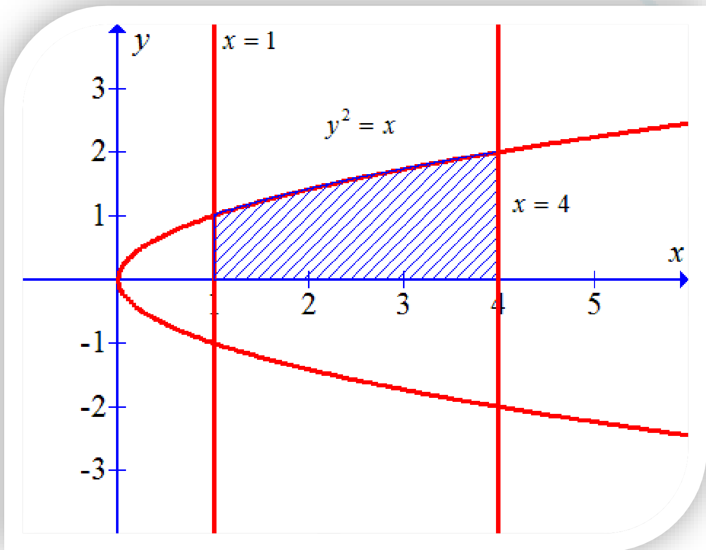
Chapter: Applications of integration

Exercise 8.1:

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the $x-$ axis in the first quadrant.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is the region bounded by the curve $y^2 = x$, the lines $x = 1, x = 4$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_1^4 \sqrt{x}\,dx$

$$A=\int_1^4 \sqrt{x}\,dx$$

$$=\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 \qquad \int x^n dx=\frac{x^{n+1}}{n+1};n\neq -1$$

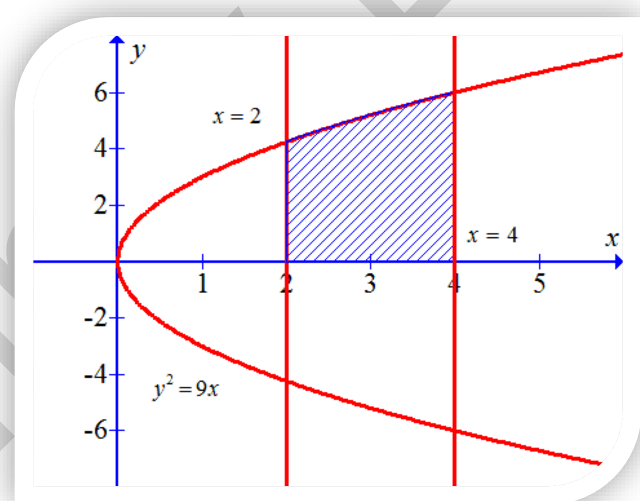
$$=\frac{2}{3}(8-1)$$

$$=\frac{14}{3}\text{square units}$$

Therefore, the area of the region bounded by the curve $y^2=x$ and the lines $x=1,x=4$ and the $x-$ axis in the first quadrant is $\frac{14}{3}$ square units

2. Find the area of the region bounded by $y^2=9x,x=2,x=4$ and the $x-$axis in the first quadrant.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a,x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The area of the shaded region is the region bounded by the curve $y^2=9x$, the lines $x=2,x=4$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A=\int_2^4 3\sqrt{x}dx$

$$A=3\int_2^4 \sqrt{x}dx$$

$$=3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

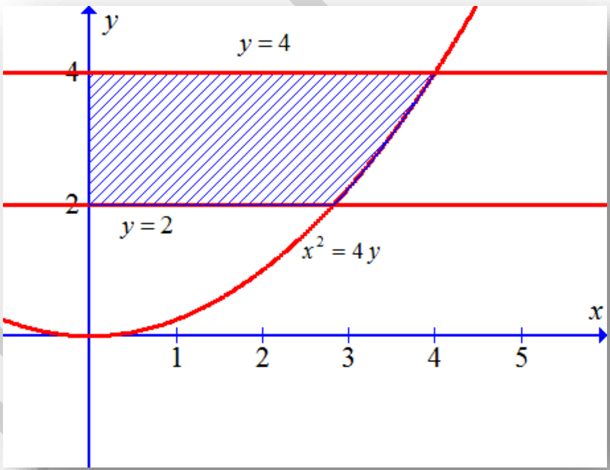
$$=3\left(\frac{2}{3}\right)\left(8-2^{\frac{3}{2}}\right)$$

$$=\left(16-4\sqrt{2}\right)\text{square units}$$

Therefore, the area of the region bounded by the curve $y^2=9x$ and the lines $x=1, x=4$ and the $x-$ axis in the first quadrant is $\left(16-4\sqrt{2}\right)$square units

3. Find the area of the region bounded by $x^2=4y, y=2, y=4$ and the $y-$axis in the first quadrant.

Solution: The required area of the region is the shaded region in the following figure



The area of the shaded region is the region bounded by the curve $x^2=4y$, the lines $y=2, y=4$ and the $y-$ axis in the first quadrant.

Hence, the required area is $A=\int_2^4 2\sqrt{y}dy$

$$A=2\int_2^4 \sqrt{y}\,dy$$

$$=2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

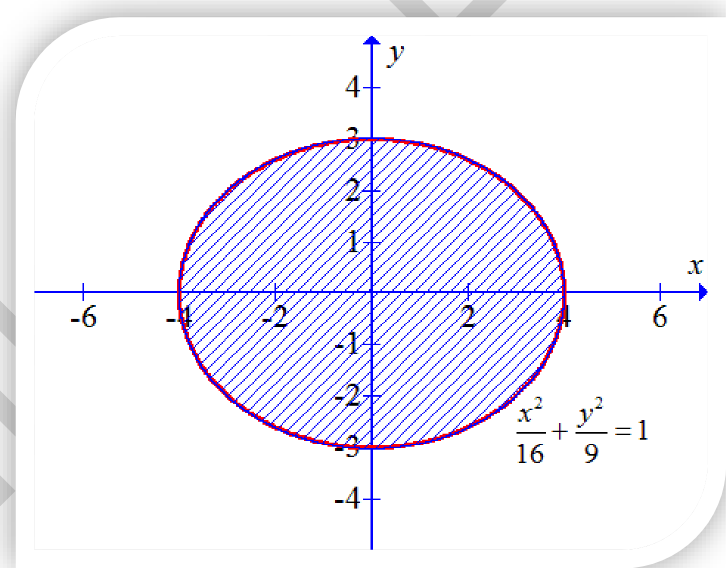
$$=2\left(\frac{2}{3}\right)\left(8-2^{\frac{3}{2}}\right)$$

$$=\frac{32-8\sqrt{2}}{3}\text{square units}$$

Therefore, the area of the region bounded by the curve $x^2=4y$, the lines $y=2, y=4$ and the $y-$ axis in the first quadrant is $\frac{32-8\sqrt{2}}{3}$ square units

4. Find the area of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The given equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ can be rewrite as below

$$\frac{y^2}{9}=1-\frac{x^2}{16}$$

$$y^2=9\left(1-\frac{x^2}{16}\right)$$

$$y=\frac{3}{4}\sqrt{16-x^2}$$

The area of the shaded region is 4 times the region bounded by the curve $y=\frac{3}{4}\sqrt{\left(16-x^2\right)}$, the lines $x=0, x=4$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A=4\left(\frac{3}{4}\int_0^4\sqrt{16-x^2}\,dx\right)$

$$A=3\int_0^4\sqrt{16-x^2}\,dx$$

$$=3\left\{\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right\}_0^4$$

$$=3\left(8\cdot\frac{\pi}{2}\right)$$

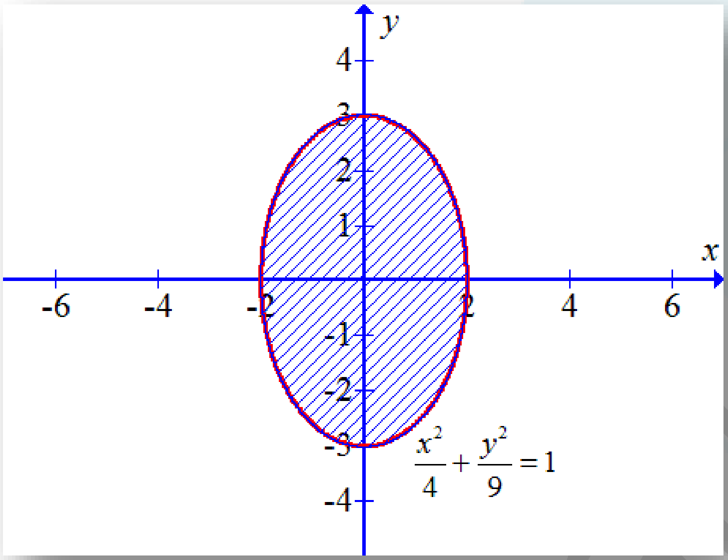
$$=12\pi$$

Therefore, the area of the region bounded by the curve $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

12π square units

5. Find the area of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The given equation $\frac{x^2}{4}+\frac{y^2}{9}=1$ can be rewrite as below

$$\frac{y^2}{9}=1-\frac{x^2}{4}$$

$$y^2=9\left(1-\frac{x^2}{4}\right)$$

$$y=\frac{3}{2}\sqrt{4-x^2}$$

The area of the shaded region is 4 times the region bounded by the curve $y=\frac{3}{2}\sqrt{\left(4-x^2\right)}$, the lines $x=0, x=4$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A=4\left(\frac{3}{2}\int_0^4\sqrt{4-x^2}dx\right)$

$$A=6\int_0^4\sqrt{4-x^2}\,dx$$

$$=6\left\{\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right\}_0^2$$

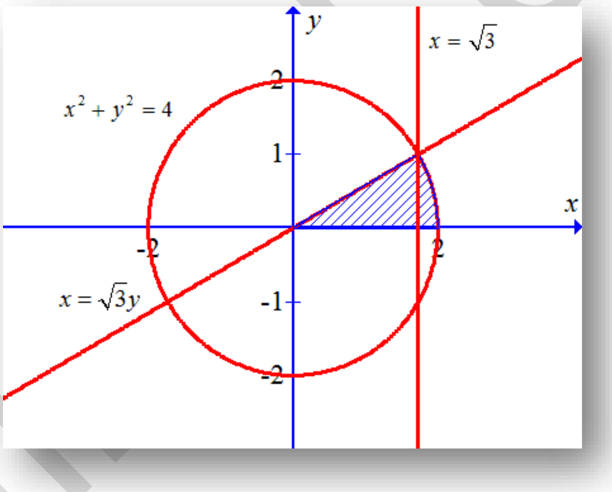
$$=6\left(2\cdot\frac{\pi}{2}\right)$$

$$=6\pi$$

Therefore, the area of the region bounded by the curve $\frac{x^2}{4}+\frac{y^2}{9}=1$ is 6π square units

6. Find the area of the region in the first quadrant enclosed by $x-$ axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is sum of the following regions

(i) The area of the shaded region bounded by the line $x=\sqrt{3}y$, lines $x=0, x=\sqrt{3}$ and the $x-$ axis in the first quadrant, and it is equal to $\frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} x\,dx$

(ii) The area of the shaded region bounded by the curve $y=\sqrt{4-x^2}$, lines $x=\sqrt{3}, x=2$ and the $x-$ axis in the first quadrant, and it is equal to $\int_{\sqrt{3}}^{2}\sqrt{4-x^2}dx$

Hence, the required area is

$$A=\frac{1}{\sqrt{3}}\int_{0}^{\sqrt{3}}xdx+\int_{\sqrt{3}}^{2}\sqrt{4-x^2}dx$$

$$=\frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}}+\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)_{\sqrt{3}}^{2}$$

$$=\frac{3}{2\sqrt{3}}+2\left(\frac{\pi}{2}\right)-\frac{\sqrt{3}}{2}-2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$=\frac{\sqrt{3}}{2}+\pi-\frac{\sqrt{3}}{2}-2\left(\frac{\pi}{3}\right)$$

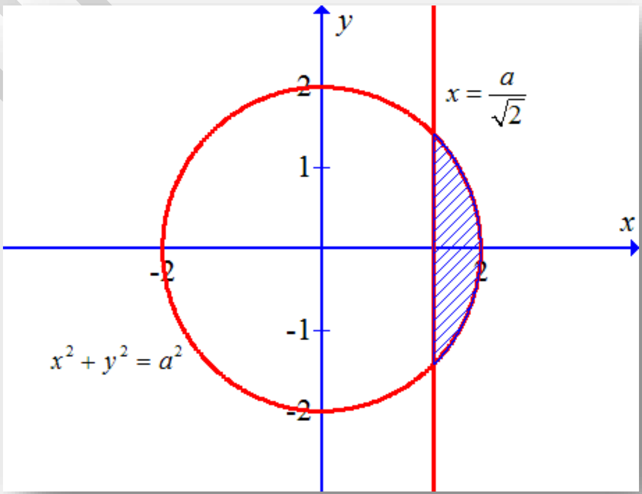
$$=\frac{\pi}{3}$$

Therefore, the area of the region in the first quadrant enclosed by $x-$ axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4$ is $\frac{\pi}{3}$ square units

7. Find the area of the smaller part of the circle $x^2+y^2=a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The area of the shaded region is two times the area of the region bounded by the curve $y=\sqrt{a^2-x^2}$, the lines $x=\frac{a}{\sqrt{2}}, x=a$ and $x-$ axis in the first quadrant.

Hence, the required area is

$$A=2\int_{\frac{a}{\sqrt{2}}}^{a}\sqrt{a^2-x^2}dx$$

$$=2\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)_{\frac{a}{\sqrt{2}}}^{a}$$

$$=2\left(\frac{a^2}{2}\left(\frac{\pi}{2}\right)-\frac{a^2}{2}-\frac{a^2}{2}\left(\frac{\pi}{4}\right)\right)$$

$$=2\left(\left(\frac{\pi a^2}{8}\right)-\frac{a^2}{2}\right)$$

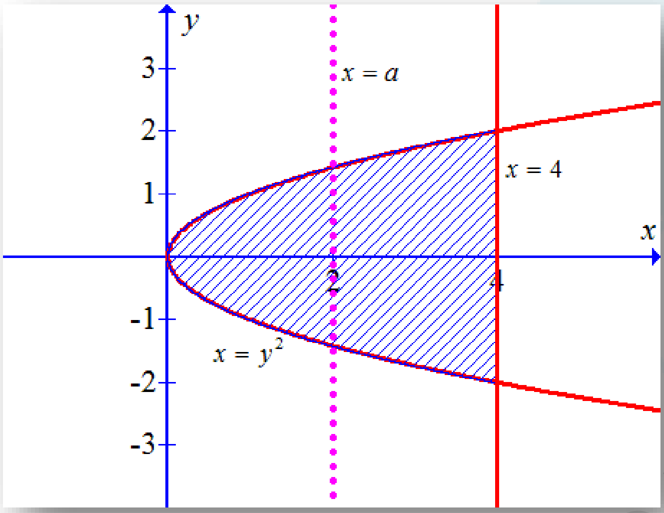
$$=\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$$

Therefore, the area of the smaller part of the circle $x^2+y^2=a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$ is $\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$square units

8. The area between $x=y^2$ and $x=4$ is divided into two equal parts by the line $x=a$ then find the value of a

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f\left(x\right)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f\left(x\right)dx\right|$.

Suppose that the line $x = a$ divide the area between $x = y^2$ and $x = 4$

Hence,

$$\int_0^a \sqrt{x}dx = \frac{1}{2}\int_0^4 \sqrt{x}dx$$

$$\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a = \frac{1}{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$\frac{2}{3}a^{\frac{3}{2}} = \frac{1}{2}\left(\frac{2}{3}\right)(8)$$

$$a^{\frac{3}{2}} = 4$$

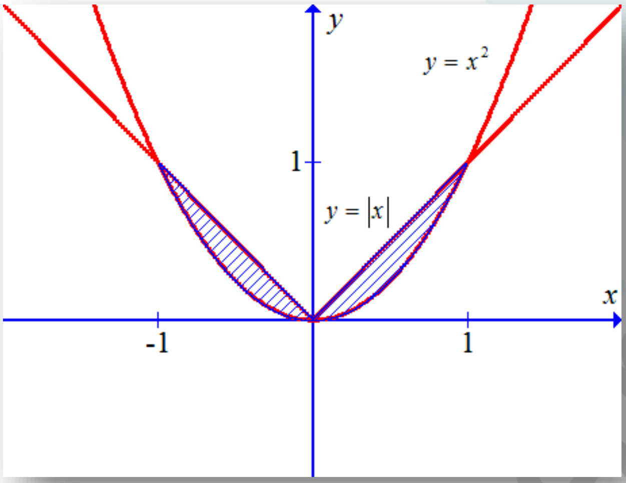
$$a = 4^{\frac{2}{3}}$$

Therefore, the value of a is $4^{\frac{2}{3}}$, such that the line $x = 4^{\frac{2}{3}}$ divide the region into two equal parts of the region bounded by $x = y^2$, the line $x = 4$

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is two times the area of the region bounded by the curve $y = x^2$, the line $y = x$, the lines $x = 0, x = 1$ and $x-$ axis in the first quadrant.

Hence, the required area is $A = 2\int_0^1 \left(x - x^2\right)dx$

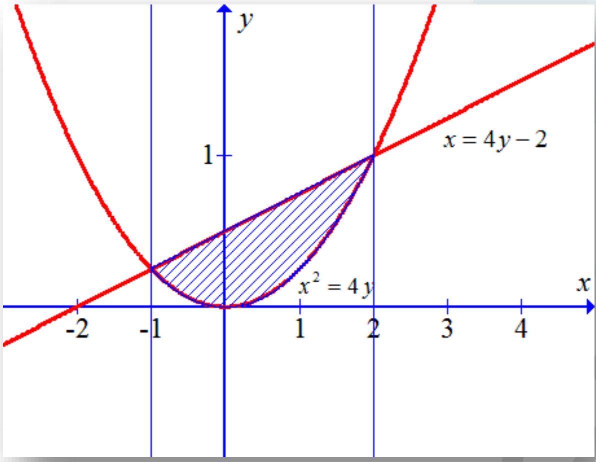
$$A = 2\int \left(x - x^2\right)dx$$
$$= 2\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right) + C$$
$$= \left(x^2 - \frac{2x^3}{3}\right)_0^1$$
$$= 1 - \frac{2}{3}$$
$$= \frac{1}{3}$$

Therefore, the area of the region bounded by the curve $y = x^2$ and the lines $y = |x|$ is $\frac{1}{3}$ square units

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

Points of intersection of the curve $x^2 = 4y$ and the line $x = 4y - 2$

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$x^2 - 2x + x - 2 = 0$$
$$x(x-2) + 1(x-2) = 0$$
$$(x+1)(x-2) = 0$$

Hence, the points of intersection of both curve and the line are $x = -1, x = 2$

The area of the shaded region is the area of the region bounded by the curve $x^2 = 4y$, the line $x = 4y - 2$, the lines $x = -1, x = 2$ and $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$

$$A=\int_{-1}^{2}\left(\frac{x+2}{4}-\frac{x^2}{4}\right)dx$$

$$=\frac{1}{4}\int_{-1}^{2}\left(x+2-x^2\right)dx$$

$$=\frac{1}{4}\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)_{-1}^{2}$$

$$=\frac{1}{4}\left(2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}\right)$$

$$=\frac{1}{4}\left(\frac{9}{2}\right)$$

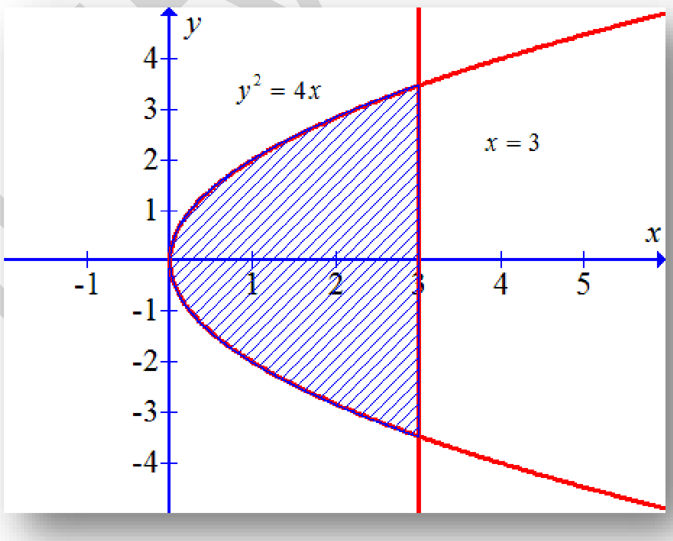
$$=\frac{9}{8}\text{square units}$$

Therefore, the area of the region bounded by the curve $x^2=4y$ and the line $x=4y-2$ is $\frac{9}{8}$ square units

11. Find the area of the region bounded by the curve $y^2=4x$ and the line $x=3$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $x=f(y)$, the lines $y=a, y=b$ and $y-$ axis is $\left|\int_a^b f(y)dy\right|$.

Points of intersection of the curve $y^2=4x$ and the line $x=3$

$$y^2 = 12 \Rightarrow y = \pm 2\sqrt{3}$$

The area of the shaded region is the area of the region bounded by the curve $x = \frac{y^2}{4}$, the line $y = 2\sqrt{3}, y = -2\sqrt{3}$ and $y-$ axis.

Hence, the required area is $A = \int_{-2\sqrt{3}}^{2\sqrt{3}} \left(3 - \frac{y^2}{4}\right) dx$

$$A = \int_{-2\sqrt{3}}^{2\sqrt{3}} \left(3 - \frac{y^2}{4}\right) dx$$

$$= \left(3y - \frac{y^3}{12}\right)_{-2\sqrt{3}}^{2\sqrt{3}}$$

$$= 6\sqrt{3} - \frac{24\sqrt{3}}{12} + 6\sqrt{3} - \frac{24\sqrt{3}}{12}$$

$$= 12\sqrt{3} - 4\sqrt{3}$$

$$= 8\sqrt{3}$$

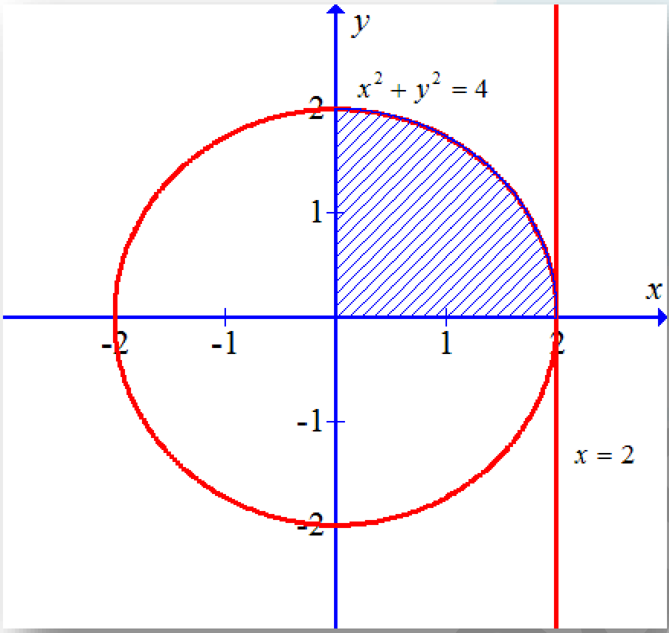
Therefore, the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$ is $8\sqrt{3}$ square units

12. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is in square units

1) π 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{4}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is the area of the region bounded by the curve $y = \sqrt{4 - x^2}$, the line $x = 0, x = 2$ and $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_0^2 \sqrt{4 - x^2}\,dx$

$$\begin{aligned} A &= \int_0^2 \sqrt{4 - x^2}\,dx \\ &= \left(\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right)_0^2 \\ &= \left(2\sin^{-1}(1)\right) \\ &= 2\left(\frac{\pi}{2}\right) \\ &= \pi \end{aligned}$$

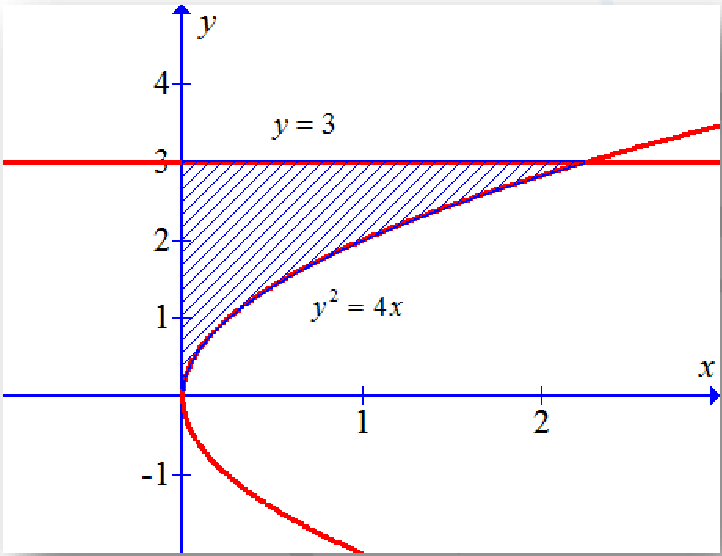
Therefore, the option 1 is correct

13. Area of the region bounded by the curve $y^2 = 4x$, $y-$ axis and the line $y=3$ in square units.

1) 2   2) $\frac{9}{4}$ 3) $\frac{9}{3}$ 4) $\frac{9}{2}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $x = f(y)$, the lines $y=a, y=b$ and $y-$ axis is $\left|\int_a^b f(y)\,dy\right|$.

The area of the shaded region is the area of the region bounded by the curve $x = \frac{y^2}{4}$, the line $y=0, y=3$ and $y-$ axis.

Hence, the required area is $A = \int_0^3 \left(\frac{y^2}{4}\right) dx$

$$A=\int_0^3\left(\frac{y^2}{4}\right)dx$$

$$=\left(\frac{y^3}{12}\right)_0^3$$

$$=\frac{27}{12}$$

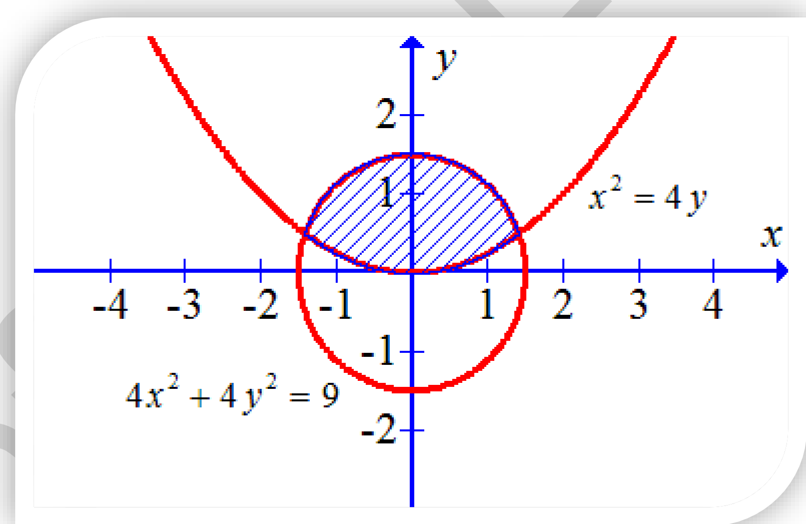
$$=\frac{9}{4}\text{square units}$$

Therefore, the option 2 is correct.

Exercise 8.2

1. Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$

The point of intersection of the circle $4x^2+4y^2=9$ and the curve $x^2=4y$

$$4x^2+\left(\frac{x^2}{2}\right)^2=9$$

$$16x^2+x^4=36$$

$$x^4+16x^2-36=0$$

$$x^4+18x^2-2x^2-36=0$$

$$x^2\left(x^2+18\right)-2\left(x^2+18\right)=0$$

$$\left(x^2-2\right)\left(x^2+18\right)=0$$

Hence, $x=-\sqrt{2},\sqrt{2}$

The area of the shaded region is the region bounded by the curve $y=\frac{1}{2}\sqrt{9-4x^2},y=\frac{x^2}{4}$, the lines $x=-\sqrt{2},x=\sqrt{2}$ and the $x-$ axis

Hence, the required area is $A=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\frac{1}{2}\sqrt{9-4x^2}-\frac{x^2}{4}\right)dx$

Hence,

$$A=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\frac{1}{2}\sqrt{9-4x^2}-\frac{x^2}{4}\right)dx$$

$$=2\left(\frac{1}{4}\left(\frac{2x}{2}\sqrt{9-4x^2}+\frac{9}{2}\sin^{-1}\frac{2x}{3}\right)-\frac{x^3}{12}\right)_0^{\sqrt{2}}$$

$$=2\left(\frac{1}{4}\left(\frac{2\sqrt{2}}{2}+\frac{9}{2}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right)-\frac{2\sqrt{2}}{12}\right)$$

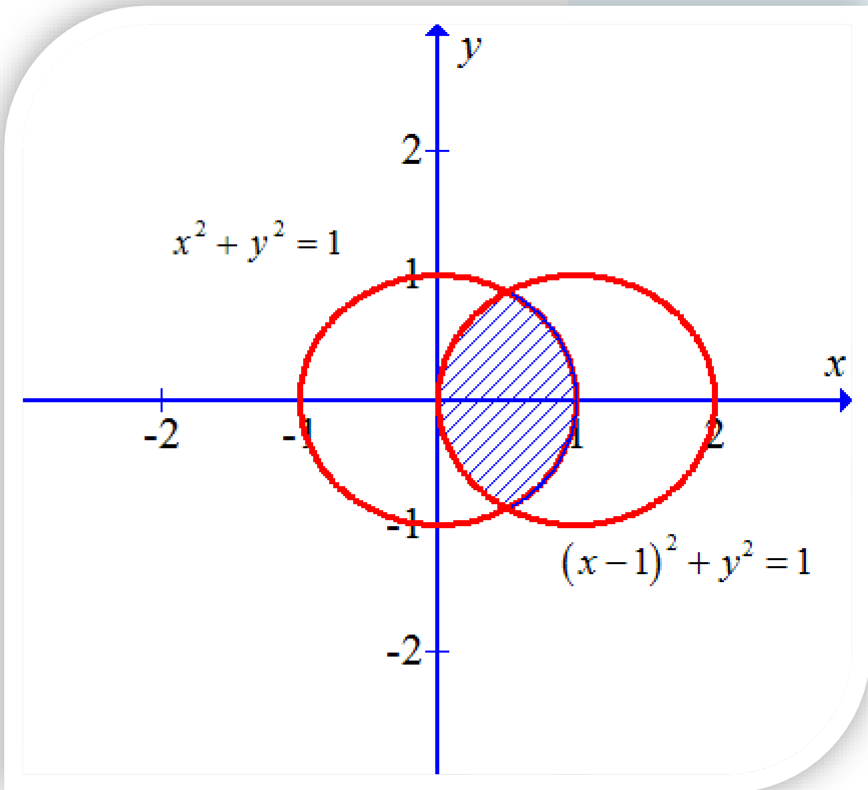
$$=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{3}$$

$$=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\frac{\sqrt{2}}{6}$$

Therefore area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$ is $\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\frac{\sqrt{2}}{6}$ square units

2. Find the area bounded by the curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

Points of intersection of the both curves

$$(x-1)^2+1-x^2=1$$
$$(x-1)^2=x^2$$
$$x-1=-x$$
$$2x=1$$
$$x=\frac{1}{2}$$

Both circles intersect at $x=\frac{1}{2}$

The area of the shaded region is sum of the following two regions

(i) Two times the area of the region bounded by $y=\sqrt{1-(x-1)^2}$, the lines $x=0, x=\frac{1}{2}$ and $x-$ axis in the first quadrant.

(ii) Two times the area of the region bounded by the $y=\sqrt{1-x^2}$, the lines $x=\frac{1}{2}, x=1$ and $x-$ axis in the first quadrant.

Hence, the required area is $A=2\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx\right]$

$$A=2\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx\right]$$

$$=2\left[\left[\frac{x-1}{2}\sqrt{1-(x-1)^2}+\frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}}+\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right]_{\frac{1}{2}}^{1}\right]$$

$$=2\left[\begin{array}{l}\left[-\frac{1}{4}\frac{\sqrt{3}}{2}-\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)-\frac{1}{2}\left(\sin^{-1}(-1)\right)\right]\\ \qquad+\left[\frac{1}{2}\sin^{-1}(1)-\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]\end{array}\right]$$

$$=4\left[-\frac{1}{4}\frac{\sqrt{3}}{2}-\frac{\pi}{12}+\frac{\pi}{4}\right]$$

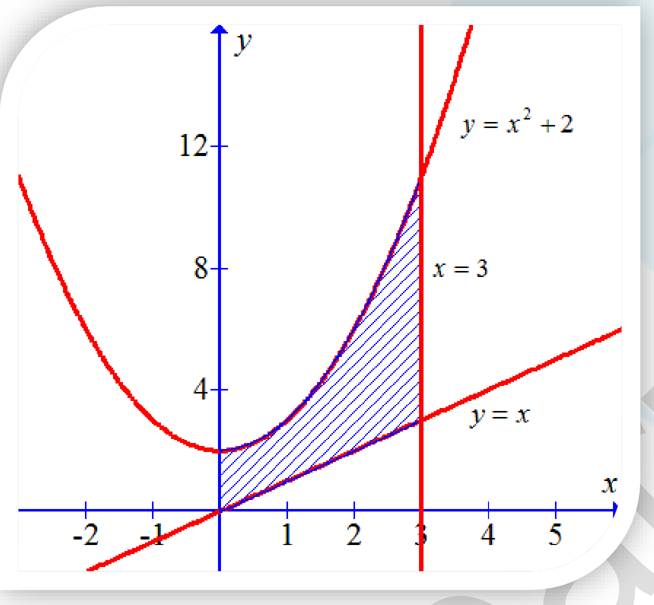
$$=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

Therefore, the area bounded by the curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$

$\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$square units

3. Find the area of the region bounded by the curve $y=x^2+2,\ y=x, x=0$ and $x=3$.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by two curves $y = f(x)$ and $y = g(x)$ the lines $x = a, x = b$ is $\int_a^b |f(x) - g(x)| dx$.

The area of the shaded region is the region bounded by the curve $y = x^2 + 2$, the lines $y = x, x = 0, x = 3$ and the $x-$ axis in the first quadrant.

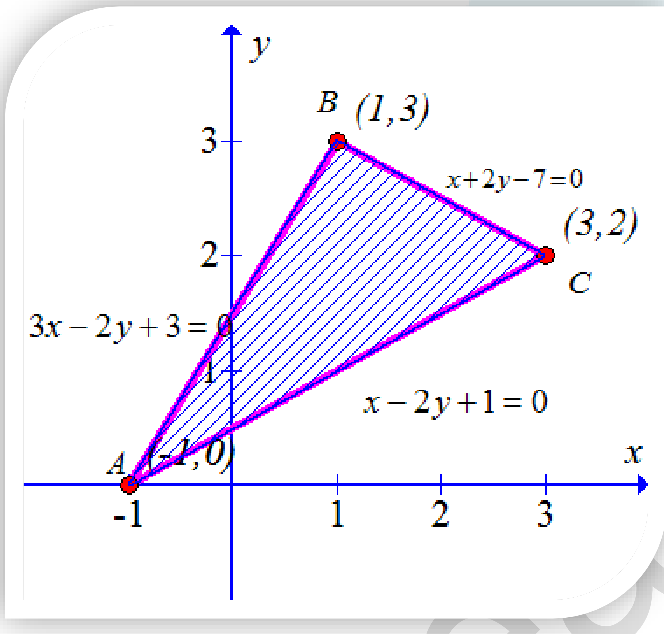
Hence, the required area is $A = \int_0^3 (x^2 + 2 - x) dx$

$$A = \int_0^3 (x^2 + 2 - x) dx$$
$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$$
$$= 9 + 6 - \frac{9}{2}$$
$$= \frac{21}{2}$$

Therefore, the area of the region bounded by the curve $y = x^2 + 2, y = x, x = 0$ and $x = 3$. $\frac{21}{2}$ square units

4. Using the integration, find the area of the region bounded by the triangle whose vertices are $(-1,0),(1,3),(3,2)$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y = f(x)$ and $y = g(x)$ between the Lines $x = a, x = b$ is $\left|\int_a^b (f(x) - g(x))dx\right|$.

The equations of the sides of the triangle are as below

Equation of AB is $y = \frac{3}{2}(x+1)$

Equation of BC is $y = \frac{-x+7}{2}$

Equation of CA is $y = \frac{1}{2}(x+1)$

The area of the triangle is sum of the following areas

(i) The area of the region between the lines $AB, AC, x = -1, x = 1$

(ii) The area of the region between the lines $BC, AC, x = 1, x = 3$

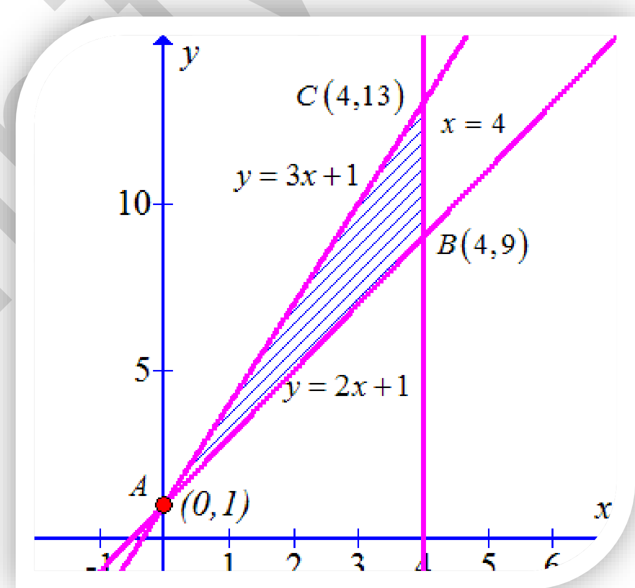
Hence, the area of the triangle is

$$A=\int_{-1}^{1}(AB-AC)\,dx+\int_{1}^{3}(BC-AC)\,dx$$
$$=\int_{-1}^{1}\frac{3}{2}(x+1)-\frac{1}{2}(x+1)\,dx+\int_{1}^{3}\left(\frac{1}{2}(-x+7)-\frac{1}{2}(x+1)\right)dx$$
$$=\int_{-1}^{1}(x+1)\,dx+\frac{1}{2}\int_{1}^{3}(-2x+6)\,dx$$
$$=\left[\frac{x^2}{2}+x\right]_{-1}^{1}+\frac{1}{2}\left[-x^2+6x\right]_{1}^{3}$$
$$=\frac{1}{2}+1-\frac{1}{2}+1+\frac{1}{2}(9+1-6)$$
$$=2+2$$
$$=4$$

Therefore, the area of the triangle formed by the points $(-1,0),(1,3),(3,2)$ is 4 square units.

5. Use the integration find the area of the triangular region whose sides have the equations $y=2x+1, y=3x+1$ and $x=4$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the Lines $x=a, x=b$ is $\left|\int_{a}^{b}(f(x)-g(x))\,dx\right|$.

The vertices of triangle are

Equation of AB is $y=2x+1$

Equation of BC is $x=4$

Equation of CA is $y=3x+1$

The area of the triangle is the area of the region between the lines $AB, AC, x=0, x=4$

Hence, the area of the triangle is

$$A=\int_0^4 (AC-AB)\,dx$$

$$=\int_0^4 (3x+1)-(2x+1)\,dx$$

$$=\int_0^4 x\,dx$$

$$=\left[\frac{x^2}{2}\right]_0^4$$

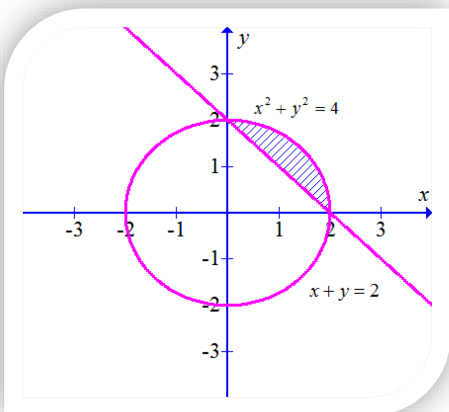
$$=8\text{square units}$$

Therefore, the area of the region bounded by the lines $y=2x+1, y=3x+1$ and $x=4$ is 8square units

6. Smaller area enclosed by the circle $x^2+y^2=4$ and the line $x+y=2$ is

1) $2(\pi-2)$ 2) $(\pi-2)$ 3) $2\pi-1$ 4) $2(\pi+2)$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the Lines $x=a, x=b$ is $\left|\int_a^b (f(x)-g(x))dx\right|$.

The area of the shaded region is area of the region bounded by the curve $y=\sqrt{4-x^2}$ and $y=2-x$, lines $x=0, x=2$ in the first quadrant.

Hence, the required area is

$$A=\int_0^2 \sqrt{4-x^2}-(2-x)dx$$

$$=\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)-2x+\frac{x^2}{2}\right]_0^2$$

$$=2\sin^{-1}(1)-4+2$$

$$=\pi-2$$

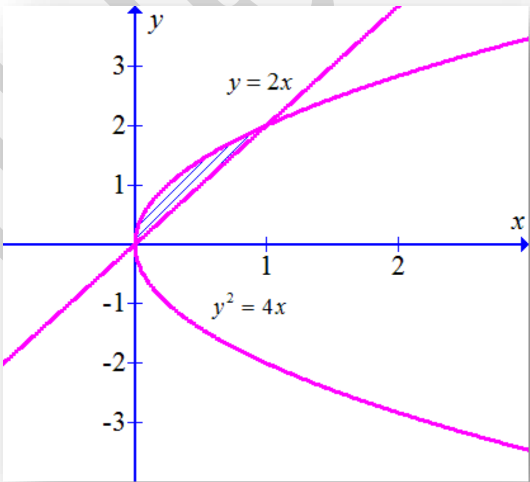
Therefore, option 2 is correct

7. Area lying between the curves $y^2=4x$ and $y=2x$ is

1) $\frac{2}{3}$ 2) $\frac{1}{3}$ 3) $\frac{1}{4}$ 4) $\frac{3}{4}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The point of intersection of the curve $y^2 = 4x$ and the line $y = 2x$

$$(2x)^2 = 4x$$
$$4x^2 = 4x$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } x = 1$$

The area of the shaded region is the area of the region bounded by the curves $y^2 = 4x$ and the line $y = 2x$, lines $x = 0, x = 1$ and $x-$ axis in the first quadrant.

Hence, the required area is

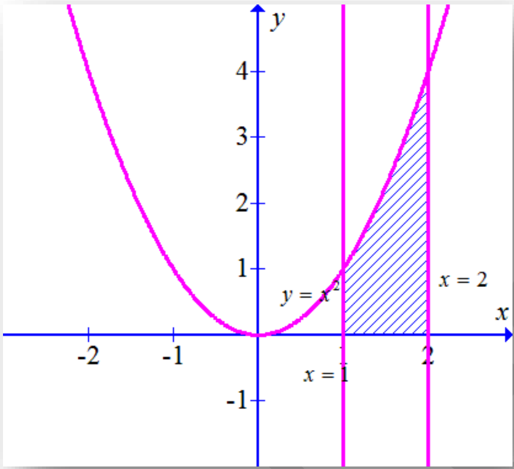
$$A = \int_0^1 \left(2\sqrt{x} - 2x\right) dx$$
$$= 2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2}\right)_0^1$$
$$= 2\left(\frac{2}{3} - \frac{1}{2}\right)$$
$$= 2\left(\frac{1}{6}\right)$$
$$= \frac{1}{3}$$

Therefore, option 2 is correct

Miscellaneous Exercise -8

1. (i) Find the area under the curve $y = x^2$, the lines $x = 1, x = 2$ and the $x-$ axis.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The area of the shaded region is the region bounded by the curve $y = x^2$, the lines $x = 1, x = 2$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_1^2 x^2 dx$

$$A = \int_1^2 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_1^2 \qquad \because \int x^n dx = \frac{x^{n+1}}{n+1}; n \neq -1$$

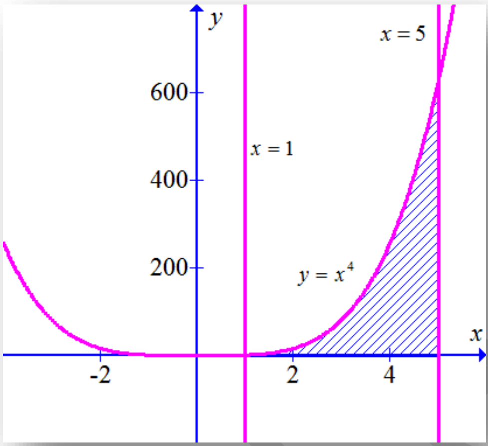
$$= \frac{1}{3}(8-1)$$

$$= \frac{7}{3}\text{square units}$$

Therefore, the area of the region bounded by the curve $y = x^2$ and the lines $x = 1, x = 2$ and the $x-$ axis in the first quadrant is $\frac{7}{3}$ square units

(ii) Find the area under the curve $y = x^4$, the lines $x = 1, x = 5$ and the $x-$ axis.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The area of the shaded region is the region bounded by the curve $y = x^4$, the lines $x = 1, x = 5$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_1^5 x^4 dx$

$$A = \int_1^5 x^4 dx$$

$$= \left[\frac{x^5}{5}\right]_1^5 \qquad \because \int x^n dx = \frac{x^{n+1}}{n+1}; n \neq -1$$

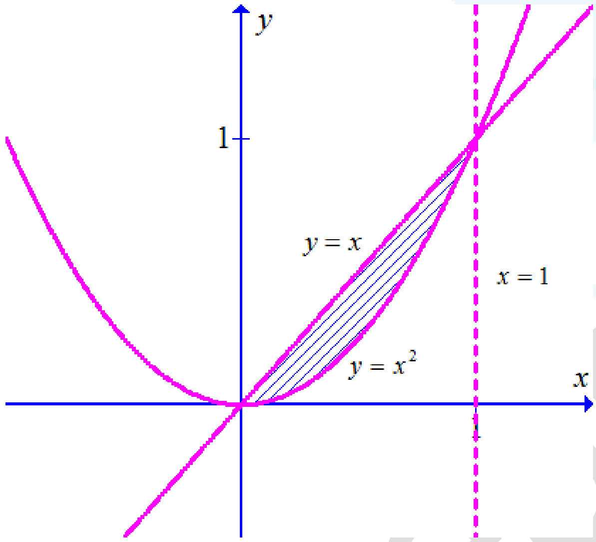
$$= \frac{1}{5}(3125 - 1)$$

$= 624.8$ square units

Therefore, the area of the region bounded by the curve $y = x^4$ and the lines $x = 1, x = 5$ and the $x-$ axis in the first quadrant is 624.8 square units

2. Find the area between the curves $y = x$ and $y = x^2$.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by two curves $y = f(x)$ and $y = g(x)$ the lines $x = a, x = b$ is $\int_a^b |f(x) - g(x)| dx$.

The point of intersection of curve $y = x^2$ and the line $y = x$

$$x = x^2$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \text{ or } x = 1$$

The area of the shaded region is the region bounded by the curve $y = x^2$, the lines $y = x, x = 0, x = 1$.

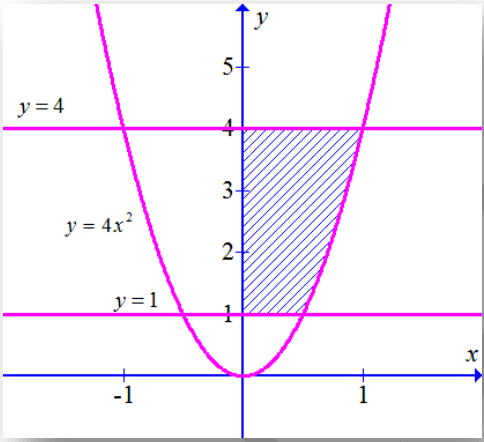
Hence, the required area is $A = \int_0^1 (x - x^2) dx$

$$A = \int_0^1 (x - x^2) dx$$
$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$
$$= \left(\frac{1}{2} - \frac{1}{3}\right)$$
$$= \frac{1}{6} \text{ square units}$$

Therefore, the area of the region bounded by the curve $y = x^2$ and the line $y = x$ is $\frac{1}{6}$ square units

3. Find the area of the region lying in the first quadrant bounded by $y = 4x^2$, lines $y = 1, y = 4$ and the $y-$axis .

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $x = g(y)$, lines $y = a, y = b$ and $y-$axis in the first quadrant is defined as $\int_a^b g(y)\,dy$

The area of the shaded region is the region bounded by the curve $y = 4x^2$, the lines $y = 1, y = 4$ and the $y-$ axis in the first quadrant.

Hence, the required area is $A = \frac{1}{2}\int_1^4 \sqrt{y}\,dy$

$$A = \frac{1}{2}\int_1^4 \sqrt{y}\,dy$$

$$= \frac{1}{2}\left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^4$$

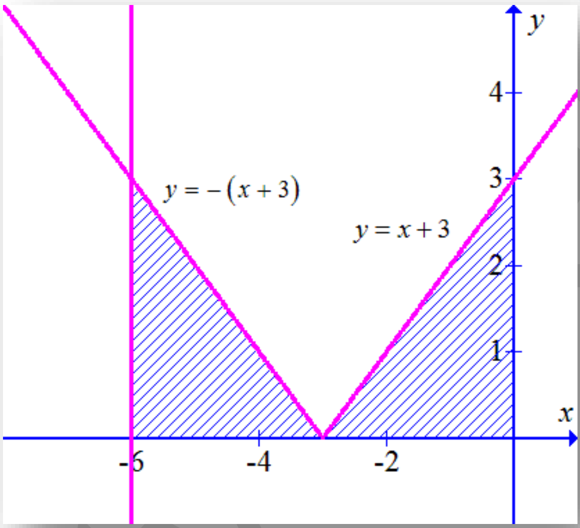
$$= \frac{1}{2}\left(\frac{2}{3}\right)(8-1)$$

$$= \frac{7}{3}$$

The area of the shaded region is the region bounded by the curve $y=4x^2$, the lines $y=1, y=4$ and the $y-$ axis in the first quadrant. $\frac{7}{3}$ square units

4. Sketch the graph of $y=|x+3|$, and evaluate $\int_{-6}^{0}|x+3|dx$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The given equation $y=|x+3|$ can be rewrite as below

$$y=\begin{cases} x+3 & x>-3 \\ -(x+3) & x<-3 \end{cases}$$

Hence,

$$\int_{-6}^{0}|x+3|dx=\left|\int_{-6}^{-3}(x+3)dx-\int_{-3}^{0}(x+3)dx\right|$$

$$=\left|\left(\frac{x^2}{2}+3x\right)_{-6}^{-3}-\left(\frac{x^2}{2}+3x\right)_{-3}^{0}\right|$$

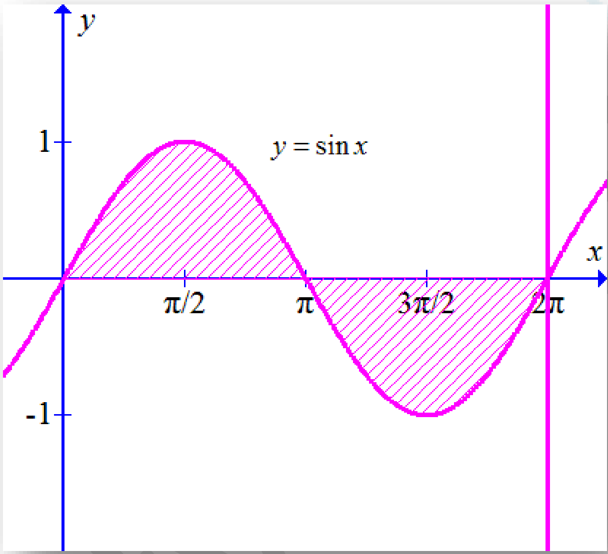
$$=\left|\frac{9}{2}-9-\frac{36}{2}+18+\frac{9}{2}-9\right|$$

$$=9$$

Therefore, the area of the region bounded by the curve $y=|x+3|$ between the lines $x=-6, x=0$ above the $x-$ axis is 9 square units

5. Find the area bounded by the curve $y=\sin x$ between the lines $x=0$ and $x=2\pi$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is 2 times the region bounded by the curve $y=\sin x$, the lines $x=0, x=\pi$ and the $x-$ axis in the first quadrant.

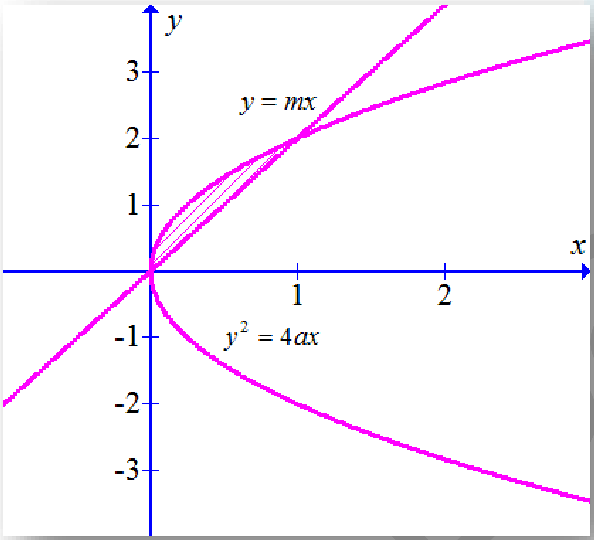
Hence, the required area is $A=2\left(\int_0^\pi \sin x dx\right)$

$$A=2\left(\int_0^\pi \sin x dx\right)$$
$$=2(-\cos x)_0^\pi$$
$$=2(1+1)$$
$$=4$$

Therefore, the area of the region bounded by the curve $y=\sin x$ between the lines $x=0, x=2\pi$ is 4 square units

6. Find the area of the region enclosed between the curves $y^2=4ax$ and the line $y=mx$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the

Lines $x=a, x=b$ is $\left|\int_a^b \left(f(x)-g(x)\right)dx\right|$.

The curve $y^2=4ax$ and the line $y=mx$ intersect:

$$(mx)^2=4ax$$

$$m^2x^2=4ax$$

$$m^2x=4a$$

$$x=\frac{4a}{m^2}$$

The area of the shaded region is the area of the region bounded between the curve $y^2=4ax$ and the line $y=mx$ is sum of the following regions

Hence, the required area is

$$A = \int_0^{\frac{4a}{m^2}} 2\sqrt{ax} - mx$$

$$= 2\sqrt{a}\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^{\frac{4a}{m^2}} - m\left(\frac{x^2}{2}\right)_0^{\frac{4a}{m^2}}$$

$$= 2\sqrt{a}\left(\frac{2}{3}\right)\left(\frac{8a^{\frac{3}{2}}}{m^3}\right) - m\left(\frac{8a^2}{m^4}\right)$$

$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$$

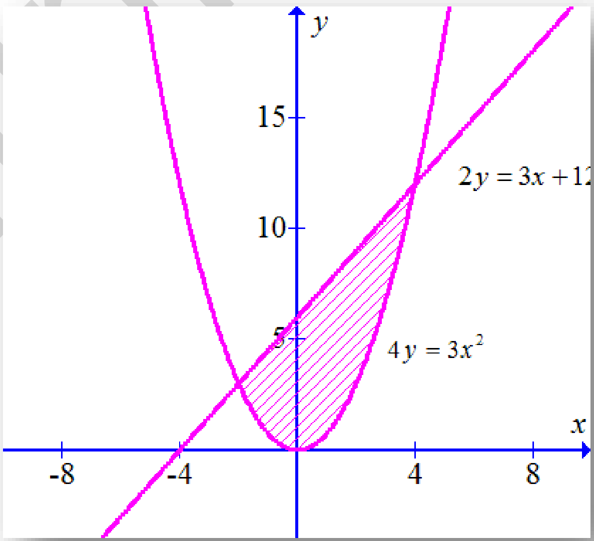
$$= \frac{8a^2}{3m^3}$$

Therefore, the enclosed between the curves $y^2 = 4ax$ and the line $y = mx$ is $\frac{8a^2}{3m^3}$ square units

7. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the Lines $x=a, x=b$ is $\left|\int_a^b (f(x)-g(x))dx\right|$.

Points of intersection of the curve $4y=3x^2$ and the line $2y=3x+12$

$$3x^2=2(3x+12)$$
$$3x^2-6x-24=0$$
$$x^2-2x-8=0$$
$$x^2-4x+2x-8=0$$
$$x(x-4)+2(x-4)=0$$
$$(x+2)(x-4)=0$$

It implies that $x=-2, x=4$

The area of the shaded region is two times the area of the region bounded by the curves $y=\frac{3x^2}{4}$, the lines $y=\frac{3x+12}{2}$, $x=-2, x=4$.

Hence, the required area is

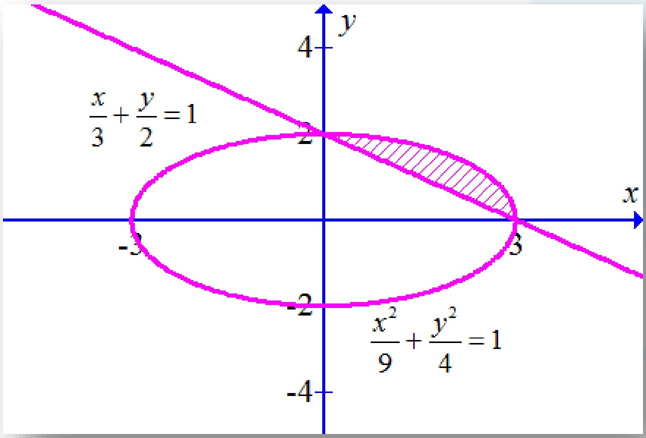
$$A=\int_{-2}^{4}\left(\frac{3x^2}{4}-\frac{3x+12}{2}\right)dx$$
$$=\frac{3}{4}\int_{-2}^{4}\left(x^2-2x-8\right)dx$$
$$=\frac{3}{4}\left(\frac{x^3}{3}-x^2-8x\right)_{-2}^{4}$$
$$=\frac{3}{4}\left(\frac{64}{3}-16-32+\frac{8}{3}+4-16\right)$$
$$=\left|\frac{3}{4}\left(\frac{72}{3}-60\right)\right|$$
$$=27$$

Therefore, the area enclosed by the parabola $4y=3x^2$ and the line $2y=3x+12$ is 27 square units

8. Find the area of the smallest region in the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the Lines $x=a, x=b$ is $\left|\int_a^b (f(x)-g(x))dx\right|$.

The area of the shaded region is the region between the curves $y=\frac{2}{3}\sqrt{9-x^2}$, $y=2\left(\frac{3-x}{3}\right)$ and the lines $x=0, x=3$

Hence, the area of the shaded region is $A=\frac{2}{3}\int_0^3\left(\sqrt{9-x^2}-(3-x)\right)dx$

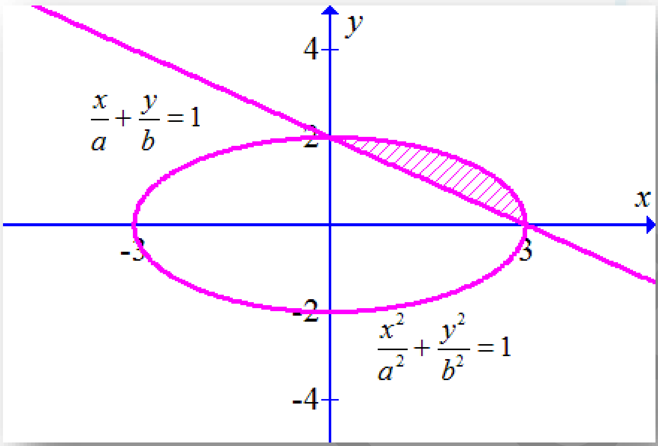
$$
\begin{aligned}
A&=\frac{2}{3}\int_0^3\left(\sqrt{9-x^2}-(3-x)\right)dx\\
&=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}-\frac{9}{2}\sin^{-1}\frac{x}{3}-3x+\frac{x^2}{2}\right]_0^3\\
&=\frac{2}{3}\left[-\frac{9}{2}\left(\frac{\pi}{2}\right)-9+\frac{9}{2}\right]\\
&=\frac{2}{3}\left(\frac{9}{2}\right)\left(\frac{\pi}{2}-1\right)\\
&=\frac{3}{2}(\pi-2)
\end{aligned}
$$

Therefore, the area of the smallest region in the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$ is $\frac{3}{2}(\pi-2)$ square units

9. Find the area of the smallest region in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the lines $x=a, x=b$ is $\left|\int_a^b (f(x)-g(x))dx\right|$.

The area of the shaded region is the region between the curves $y=\frac{b}{a}\sqrt{a^2-x^2}$ and the line $y=\frac{b}{a}(a-x)$

Hence, the area of the shaded region is $A=\frac{b}{a}\int_0^3\left(\sqrt{a^2-x^2}-(a-x)\right)dx$

$$A=\frac{b}{a}\int_{0}^{3}\left(\sqrt{a^2-x^2}-(a-x)\right)dx$$

$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}-\frac{a^2}{2}\sin^{-1}\frac{x}{a}-ax+\frac{x^2}{2}\right]_0^a$$

$$=\frac{b}{a}\left[-\frac{a^2}{2}\left(\frac{\pi}{2}\right)-a^2+\frac{a^2}{2}\right]$$

$$=\frac{b}{a}\left(\frac{a^2}{2}\right)\left(\frac{\pi}{2}-1\right)$$

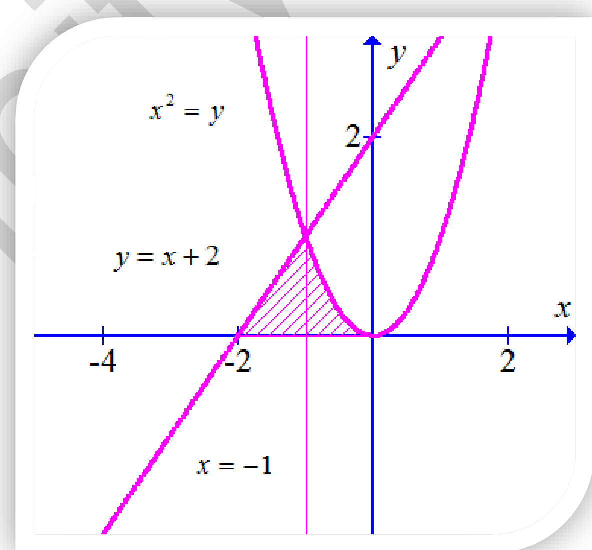
$$=\frac{ab}{4}(\pi-2)$$

Therefore, the area of the smallest region in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$ is $\frac{ab}{4}(\pi-2)$ square units

10. Find the area bounded by the curve $x^2=y$ and the line $y=x+2$ and $x-$ axis.

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the lines $x=a, x=b$ is $\left|\int_a^b\left(f(x)-g(x)\right)dx\right|$.

Points of intersection of the curve $x^2 = y$ and the line $y = x + 2$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$x^2 - 2x + x - 2 = 0$$

$$x(x-2) + 1(x-2) = 0$$

$$(x+1)(x-2) = 0$$

Hence, the points of intersection of both curve and the line are $x = -1, x = 2$

The area of the shaded region is the sum of the areas of the region bounded by the curve $y = x + 2$, lines $x = -2, x = -1$ and $x-$ axis and the area of the region bounded by the curves $x^2 = y$, the lines $x = -1, x = 0$ and $x-$ axis in the second quadrant.

Hence, the required area is $A = \int_{-2}^{1} (x+2)\,dx + \int_{-1}^{0} x^2 dx$

$$A = \int_{-2}^{1} (x+2)\,dx + \int_{-1}^{0} x^2 dx$$

$$= \left(\frac{x^2}{2} + 2x\right)_{-2}^{-1} + \left[\frac{x^3}{3}\right]_{-1}^{0}$$

$$= \left(\frac{1}{2} - 2 - 2 + 4 + \frac{1}{3}\right)$$

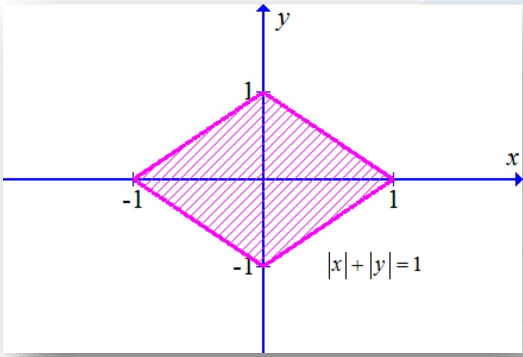
$$= \left(\frac{5}{6}\right) \text{square units}$$

Therefore, the area of the region bounded by the curve $x^2 = y$ the line $y = x + 2$ and $x-$ axis is $\frac{5}{6}$ square units

11. Using the method of integration, find the area bounded by the curve $|x| + |y| = 1$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is four times the area of the triangle in the first quadrant

Hence, the required area is $A=4\int_0^1 (x+1)\,dx$

$$A=4\int_0^1 (-x+1)\,dx$$

$$=4\left(-\frac{x^2}{2}+x\right)_0^1$$

$$=4\left(-\frac{1}{2}+1\right)$$

$$=4\left(\frac{1}{2}\right)$$

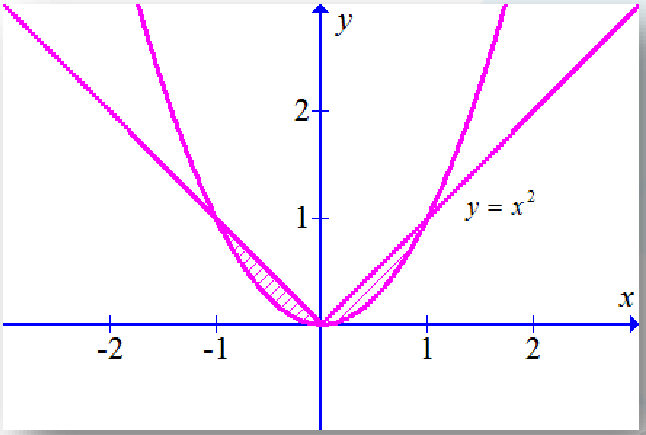
$$=2\text{ square units}$$

Therefore, the area bounded by the curve $|x|+|y|=1$ is 2 square units

12. Find the area bounded by curves $\{(x,y): y \geq x^2 \text{ and } y=|x|\}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y = f(x)$ and $y = g(x)$ between the lines $x = a, x = b$ is $\left|\int_a^b (f(x) - g(x))dx\right|$.

The area of the shaded region is two times the area of the region bounded by the line $y = x^2$ and the line $y = x$ in the first quadrant

Hence, the required area is $A = 2\int_0^1 (x^2 - x)dx$

$$A = 2\int_0^1 (x^2 - x)dx$$

$$= 2\left(\frac{x^3}{3} - \frac{x^2}{2}\right)_0^1$$

$$= 2\left(\frac{1}{3} - \frac{1}{2}\right)$$

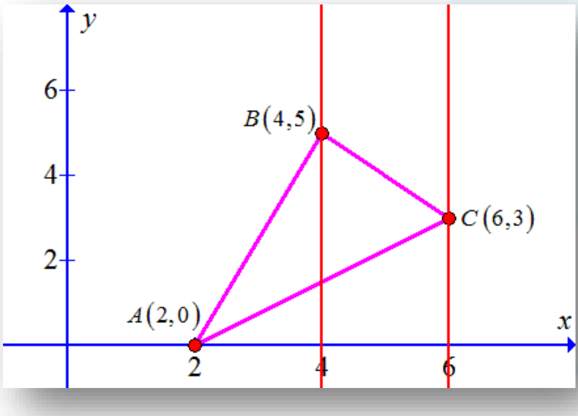
$$= \frac{1}{3}\text{square units}$$

Therefore the area bounded by curves $\{(x, y): y \geq x^2 \text{ and } y = |x|\}$ is $\frac{1}{3}$square units

13. Using the method of integration, find the area of the triangle formed by the points $A(2,0), B(4,5), C(6,3)$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y = f(x)$ and $y = g(x)$ between the Lines $x = a, x = b$ is $\left|\int_a^b \left(f(x) - g(x)\right)dx\right|$.

The equations of the sides of the triangle are as below

Equation of AB is $y = \frac{5}{2}(x-2)$

Equation of BC is $y = -x + 9$

Equation of CA is $y = \frac{3}{4}(x-2)$

The area of the triangle is sum of the following areas

(iii) The area of the region between the lines $AB, AC, x = 2, x = 4$

(iv) The area of the region between the lines $BC, AC, x = 4, x = 6$

Hence, the area of the triangle is

$$A = \int_2^4 (AB - AC)\,dx + \int_4^6 (BC - AC)\,dx$$

$$= \int_2^4 \left(\frac{5}{2}(x-2) - \frac{3}{4}(x-2)\right)dx + \int_4^6 \left((-x+9) - \frac{3}{4}(x-2)\right)dx$$

$$= \frac{5}{2}\int_2^4 (x-2)\,dx - \frac{3}{4}\int_2^6 (x-2)\,dx + \int_4^6 (-x+9)\,dx$$

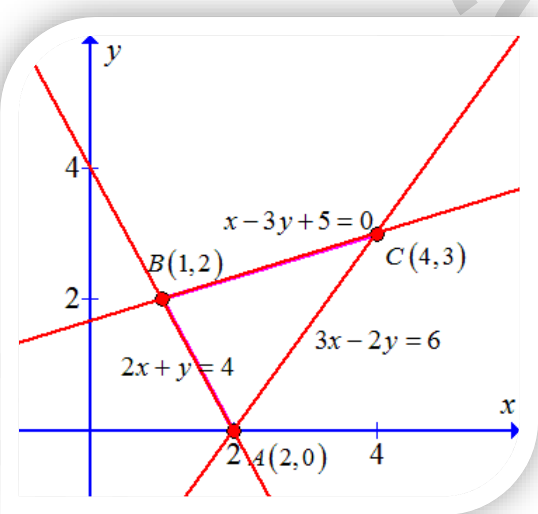
$$= \frac{5}{2}\left(\frac{x^2}{2} - 2x\right)_2^4 - \frac{3}{4}\left(\frac{x^2}{2} - 2x\right)_2^6 + \left(-\frac{x^2}{2} + 9x\right)_4^6$$

$$=\frac{5}{2}(2)-\frac{3}{4}(8)+(-18+54+8-36)$$
$$=5-6+8$$
$$=7$$

Therefore, the area of the triangle formed by the points $A(2,0),B(4,5),C(6,3)$ is 7 square units.

14. Using the method of integration find the area of the region bounded by the lines $2x+y=4,3x-2y=6,x-3y+5=0$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the Lines $x=a,x=b$ is $\left|\int_a^b (f(x)-g(x))dx\right|$.

The vertices of triangle are

Equation of AB is $y=-2x+4$

Equation of BC is $y=\frac{x+5}{3}$

Equation of CA is $y=\frac{3x-6}{2}$

The area of the triangle is sum of the following areas

(i) The area of the region between the lines $AB, BC, x=1, x=2$

(ii) The area of the region between the lines $BC, AC, x=2, x=4$

Hence, the area of the triangle is

$$A=\int_1^2 (BC-AB)\,dx+\int_2^4 (BC-AC)\,dx$$

$$=\int_1^2\left(\frac{1}{3}(x+5)-(-2x+4)\right)dx+\int_2^4\left(\frac{1}{3}(x+5)-\left(\frac{3x-6}{2}\right)\right)dx$$

$$=\frac{1}{3}\int_1^4 (x+5)\,dx+\int_1^2 (2x-4)\,dx-\frac{3}{2}\int_2^4 (x-2)\,dx$$

$$=\frac{1}{3}\left(\frac{x^2}{2}+5x\right)_1^4+\left(x^2-4x\right)_1^2-\frac{3}{2}\left(\frac{x^2}{2}-2x\right)_2^4$$

$$=\frac{1}{3}\left(28-\frac{11}{2}\right)+(-4+3)-\frac{3}{2}(2)$$

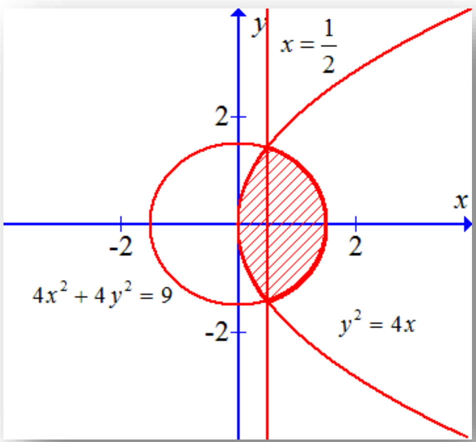
$$=\frac{15}{2}-1-3$$

$$=\frac{7}{2}$$

Therefore, the area of the region bounded by the lines $2x+y=4, 3x-2y=6$ and $x-3y+5=0$ is $\frac{7}{2}$ square units

15. Find the area of the region $\{(x,y): y^2 \le 4x \text{ and } 4x^2+4y^2 \le 9\}$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y = f(x)$ and $y = g(x)$ between the Lines $x = a, x = b$ is $\left|\int_a^b \left(f(x) - g(x)\right)dx\right|$.

Point of intersection of $y^2 = 4x, 4x^2 + 4y^2 = 9$

$$4x^2 + 4(4x) = 9$$
$$4x^2 + 16x - 9 = 0$$
$$4x^2 + 18x - 2x - 9 = 0$$
$$2x(2x+9) - 1(2x+9) = 0$$
$$(2x-1)(2x+9) = 0$$

Observing the figure both curves intersect at $x = \frac{1}{2}$

The area of the shaded region is sum of the following areas

(i) Two times the area of the region bounded by the curve $y^2 = 4x$, lines $x = 0, x = \frac{1}{2}$ and $x-$ axis in the first quadrant.

(ii) Two times the area of the region bounded by the curve $4x^2 + 4y^2 = 9$, lines $x = \frac{1}{2}, x = \frac{3}{2}$ and $x-$ axis in the first quadrant.

Hence, the area of the shaded region is $A = 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{9-4x^2}dx\right]$

$$A=2\left[\int_0^{\frac{1}{2}}2\sqrt{x}dx+\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{1}{2}\sqrt{9-4x^2}dx\right]$$

$$=2\left[\int_0^{\frac{1}{2}}2x^{\frac{1}{2}}dx+\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{1}{2}\sqrt{3^2-(2x)^2}dx\right]$$

$$=2\left[2\left(\frac{2}{3}x^{\frac{3}{2}}\right)_0^{\frac{1}{2}}+\frac{1}{4}\left(\frac{2x}{2}\sqrt{9-4x^2}+\frac{9}{2}\sin^{-1}\left(\frac{2x}{3}\right)\right)_{\frac{1}{2}}^{\frac{3}{2}}\right]$$

$$=2\left[2\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{4}\left(\frac{9}{2}\left(\frac{\pi}{2}\right)-\frac{1}{2}2\sqrt{2}-\frac{9}{2}\sin^{-1}\frac{1}{3}\right)\right]$$

$$=2\left(\frac{2\sqrt{2}}{3}+\frac{9\pi}{16}-\sqrt{2}-\frac{9}{8}\sin^{-}\frac{1}{3}\right)$$

$$=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{1}{3\sqrt{2}}$$

Therefore, the area of the region $\{(x,y): y^2\le 4x \text{ and } 4x^2+4y^2\le 9\}$ is

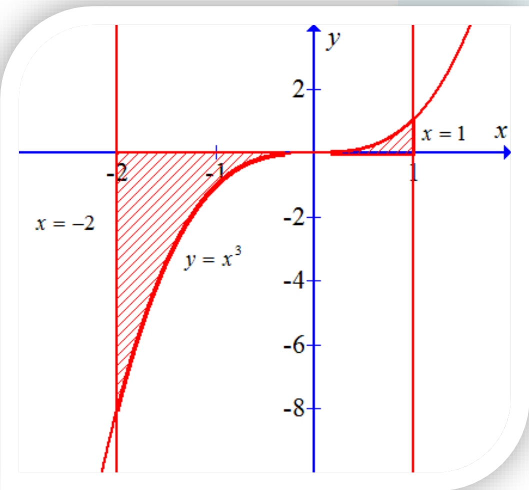
$\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{1}{3\sqrt{2}}$

16. Area bounded by the curve $y=x^3$, the $x-$ axis and ordinates $x=-2$ and $x=1$ is in square units

2) -9 2) $-\frac{15}{4}$ 3) $\frac{15}{4}$ 4) $\frac{17}{4}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is sum of areas of the regions bounded by the curve $y = x^3$, the line $x = 0, x = 1$ and $x-$ axis in the first quadrant and bounded by the curve $y = -x^3$, the line $x = 0, x = -2$ and $x-$ axis in the third quadrant

Hence, the required area is $A = \int_{-2}^{0} -x^3 dx + \int_0^1 x^3 dx$

$$A = \int_{-2}^{0} -x^3 dx + \int_0^1 x^3 dx$$

$$= \left(-\frac{x^4}{4}\right)_{-2}^{0} + \left(\frac{x^4}{4}\right)_0^1$$

$$= \frac{16}{4} + \frac{1}{4}$$

$$= \frac{17}{4} \text{square units}$$

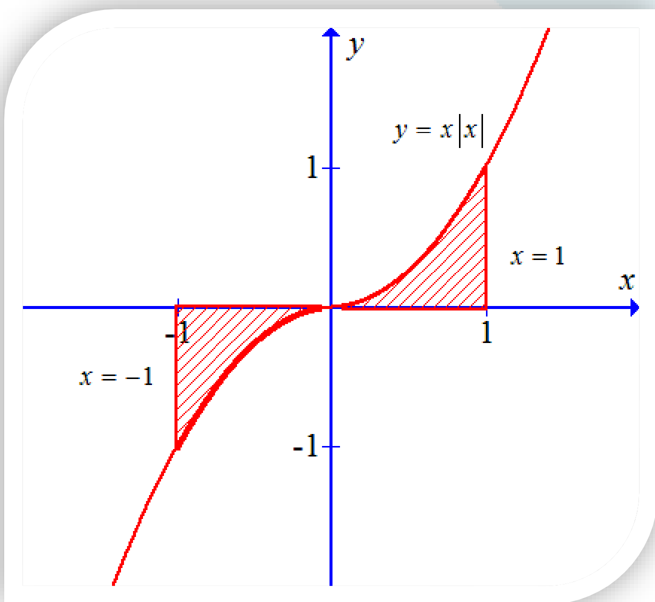
Therefore, the option 4 is correct

17. Area bounded by the curve $y = x|x|$, $x-$ axis and the ordinates $x = -1$ and $x = 1$ is given by in square units.

2) 0 2) $\frac{1}{3}$ 3) $\frac{2}{3}$ 4) $\frac{4}{3}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The given equation of the curve $y = x|x|$ can be rewrite as

$$y = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

The area of the shaded region is two times the area of the region bounded by the curve $y = x^2$, the line $x = 0, x = 1$ and $x-$ axis.

Hence, the required area is $A = 2\int_0^1 x^2 dx$

$$A = 2\int_0^1 x^2 dx$$

$$= 2\left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{2}{3} \text{square units}$$

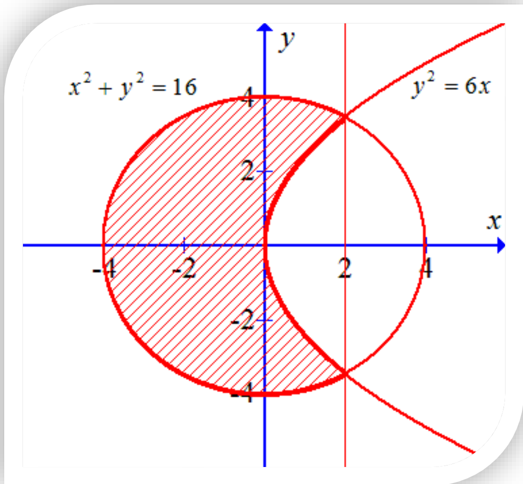
Therefore, the option 3 is correct.

18. The area of the circle $x^2+y^2=16$ exterior to the parabola $y^2=6x$

1) $\frac{4}{3}\left(4\pi-\sqrt{3}\right)$ 2) $\frac{4}{3}\left(4\pi+\sqrt{3}\right)$ 3) $\frac{4}{3}\left(8\pi-\sqrt{3}\right)$ 4) $\frac{4}{3}\left(8\pi+\sqrt{3}\right)$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the Lines $x=a, x=b$ is $\left|\int_a^b \left(f(x)-g(x)\right)dx\right|$

The points of intersection of the circle $x^2+y^2=16$ and the parabola $y^2=6x$

$$x^2+6x-16=0$$

$$x^2+8x-2x-16=0$$

$$x(x+8)-2(x+8)=0$$

$$(x-2)(x+8)=0$$

It implies that $x=2$ or $x=-8$

Observing the graph, the point of intersection of circle and parabola is $x=2$

The area of the shaded region is sum of the area of the semicircle $x^2+y^2=16$ and two times the area of the region bounded by the circle $y=\sqrt{16-x^2}$, parabola $y=\sqrt{6x}$, the lines $x=0, x=2$, $x-$ axis in the first quadrant.

Hence, the required area is $A=8\pi+2\int_0^2 \sqrt{16-x^2}-\sqrt{6x}dx$

$$A=8\pi+2\int_0^2 \sqrt{16-x^2}-\sqrt{6x}dx$$

$$=8\pi+2\left(\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)-\sqrt{6}\left(\frac{2}{3}\right)x^{\frac{3}{2}}\right)_0^2$$

$$=8\pi+2\left(\sqrt{12}+8\cdot\frac{\pi}{6}-\sqrt{6}\left(\frac{2}{3}\right)2\sqrt{2}\right)$$

$$=8\pi+2\left(2\sqrt{3}+\frac{4\pi}{3}-\frac{8\sqrt{3}}{3}\right)$$

$$=2\left(\frac{16\pi}{3}-\frac{2\sqrt{3}}{3}\right)$$

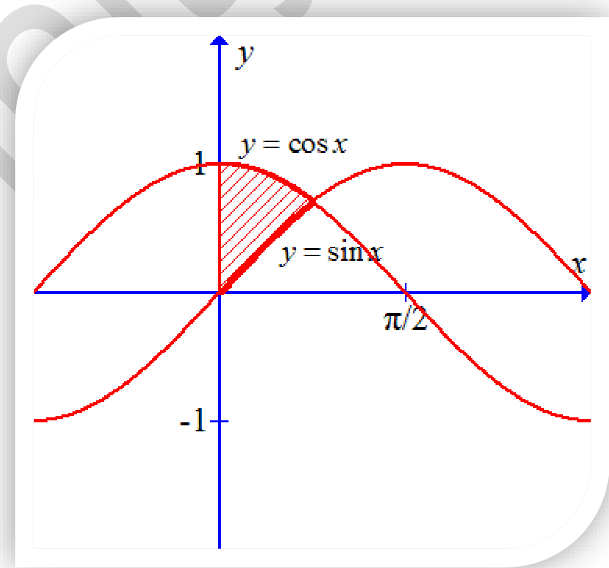
$$=\frac{4}{3}\left(8\pi-\sqrt{3}\right)$$

Therefore, the option 3 is correct.

19. The area bounded by the $y-$ axis , $y=\cos x, y=\sin x$ when $0\le x\le\frac{\pi}{2}$

1) $2\left(\sqrt{2}-1\right)$ 2) $\left(\sqrt{2}-1\right)$ 3) $\left(\sqrt{2}+1\right)$ 4) $\sqrt{2}$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the Lines $x=a, x=b$ is $\left|\int_a^b (f(x)-g(x))dx\right|$.

The area of the shaded region is the area of the region between the two curves $y=\sin x$ and $y=\cos x$, lines $x=0, x=\frac{\pi}{4}$ in the first quadrant.

Hence, the required area is $A=\int_0^{\frac{\pi}{4}} (\cos x-\sin x)dx$

$$A=\int_0^{\frac{\pi}{4}} (\cos x-\sin x)dx$$

$$=(\sin x+\cos x)_0^{\frac{\pi}{4}}$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1$$

$$=(\sqrt{2}-1)\text{square units}$$

Therefore, the option 2 is correct.