Chapter: 1. Relations and functions

Miscellaneous Exercise:

1. Let $f: R \to R$ be defined as $f(x) = 10x + 7$. Find the function $g: R \to R$ such that $gof = fog = I_R$

Solution: Given that $f: R \to R$ is defined as $f(x) = 10x + 7$

Check the function is one to one or not:

Suppose that $f(x) = f(y)$, where $x, y \in R$

It implies that

$$\begin{aligned} 10x + 7 &= 10y + 7 \\ 10x &= 10y \\ x &= y \end{aligned}$$

Therefore, $f(x)$ is a one-one function

Check the function is onto or not.

Suppose that $y \in R$, and suppose that $y = 10x + 7$

It implies that $x = \frac{y-7}{10} \in R$

For any $y \in R$, there exists $x = \frac{y-7}{10} \in R$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

Therefore, $f(x)$ is onto

Hence, f is an invertible function

The inverse of the function $f(x)$ is $\frac{x-7}{10}$

2. Let $f(x): W \to W$ be defined as $f(n) = n-1$, if is odd and $f(n) = n+1$, if n is even. Show that $f(x)$ is invertible. Find the inverse of $f(x)$. Here, W is the set of all whole numbers.

Solution: Given that $f: W \to W$ is defined as $f(n) = \begin{cases} n-1 & \text{if n is odd} \\ n+1 & \text{if n is even} \end{cases}$

Check whether the function is one to one or not.

Suppose that $f(n) = f(m)$, where n, m are whole numbers.

If n is odd and m is even, then we will have $n-1 = m+1$, it implies that $n - m = 2$

This is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored

Therefore, both n and m must be either odd or even.

Now, if both n and m are odd.

$$\begin{aligned} f(n) &= f(m) \\ n-1 &= m-1 \\ n &= m \end{aligned}$$

If both n and m are even,

$$\begin{aligned} f(n) &= f(m) \\ n+1 &= m+1 \\ n &= m \end{aligned}$$

Therefore f is one – one.

To check whether the function is onto or not.

For any odd number $2r+1$ in co domain N is the image of $2r$ in domain N and any even number

For any even number $2r$ in codomain N is the image of $2r+1$in domain N

Hence, $f(x)$ is onto.

Therefore, the function $f(x)$ is invertible

Define $g: W \to W$ as $g(m) = \begin{cases} m+1 \text{ if } m \text{ is even} \\ m-1 \text{ if } m \text{ is odd} \end{cases}$

Suppose that n is odd number

$$\begin{aligned} gof(n) &= g(f(n)) \\ &= g(n-1) \\ &= n-1+1 \\ &= n \end{aligned}$$

Suppose n is even

$$\begin{aligned} gof(n) &= g(f(n)) \\ &= g(n+1) \\ &= n+1-1 \\ &= n \end{aligned}$$

When m is odd

$$\begin{aligned} fog(m) &= f(g(m)) \\ &= f(m-1) \\ &= m-1+1 \\ &= m \end{aligned}$$

When m is even

$$\begin{aligned} fog(m) &= f(g(m)) \\ &= f(m+1) \\ &= m+1-1 \\ &= m \end{aligned}$$

Hence, $g \circ f = I$, and $fog = I$

Therefore, the function $f(x)$ is invertible and the inverse of $f(x)$ is given by $g(x)$, it is same as $f(x)$

3. If $f: R \to R$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$

Solution: It is given that $f:R\to R$ is defined as $f(x)=x^2-3x+2$

$$\begin{aligned} f(f(x)) &= f(x^2-3x+2) \\ &= (x^2-3x+2)^2-3(x^2-3x+2)+2 \\ &= (x^4+9x^2+4-6x^3-12x+4x^2)+(-3x^2+9x-6)+2 \\ &= x^4-6x^3+10x^2-3x \end{aligned}$$

Therefore, $f\circ f(x)=x^4-6x^3+10x^2-3x$

4. Show that function $f:R\to\{x\in R:-1<x<1\}$ defined by $f(x)=\dfrac{x}{1+|x|}, x\in R$ is one-one and onto function

Solution: Given that $f:R\to\{x\in R:-1<x<1\}$ is defined as $f(x)=\dfrac{x}{1+|x|}, x\in R$

Check the function $f(x)$ is one to one or not.

Suppose that $f(x)=f(y)$, where $x, y\in R$

It implies that $\dfrac{x}{1+|x|}=\dfrac{y}{1+|y|}$

If x is positive and y is negative,

$$\frac{x}{1+x}=\frac{y}{1-y}\Rightarrow x-xy=y+xy\Rightarrow 2xy=x-y$$

Since, x is positive and y is negative, $2xy$ is negative and $x-y$ is positive

Hence the condition $2xy=x-y$ is false

If x is positive, and y be negative can be ruled out.

Suppose that both x and y are positive

$$f(x)=f(y)$$
$$\frac{x}{1+x}=\frac{y}{1+y}$$
$$x+xy=y+xy$$
$$x=y$$

When x and y both are negative,

$$f(x)=f(y)$$
$$\frac{x}{1-x}=\frac{y}{1-y}$$
$$x-xy=y-xy$$
$$x=y$$

Therefore, $f(x)$ is one to one.

Check the function $f(x)$ is onto or not.

Suppose that $y \in R$ and y is negative real number

If y is negative, then, there exists $x=\frac{y}{1+y}$ such that

$$f(x)=f\left(\frac{y}{1+y}\right)=\frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|}=\frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)}=\frac{y}{1+y-y}=y$$

If y is positive, then, there exists $x=\frac{y}{1-y} \in R$ such that

$$f(x)=f\left(\frac{y}{1-y}\right)=\frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|}=\frac{\frac{y}{1-y}}{1+\left(\frac{-y}{1-y}\right)}=\frac{y}{1-y+y}=y$$

Therefore, the function $f(x)$ is onto.

5. Show that the function $f: R \rightarrow R$ given by $f(x)=x^3$ is injective

Solution: Given that the function $f: R \to R$ is defined as $f(x) = x^3$

Check the whether the function is one to one

Suppose that $f(x) = f(y)$, where $x, y \in R$

It implies that $x^3 = y^3 \Rightarrow x = y$

Hence, the function is one to one.

6. Given examples of two function $f: N \to Z$ and $g: Z \to Z$ such that gof is injective but $g(x)$ is not injective

Solution: Given that two functions $f: N \to Z$ and $g: Z \to Z$ such that $f(x) = x$ and

$g(x) = |x|$

Since the function $f(x)$ is identity function it is one to one

But the function $g(x)$ is not one to one because for both $-1,1$ has same image 1

Consider the compound function $gof(x)$ is defined as

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(x) \\ &= |x| \end{aligned}$$

Here the function $gof(x)$ is defined on the set of natural numbers

Hence the function $gof(x)$ is one to one

7. Given examples of two functions $f: N \to N$ and $g: N \to N$ such that $gof(x)$ is onto but $f(x)$ is not onto

Solution: Given $f: N \to N$ by $f(x) = x+1$ and $g: N \to N$ is defined as

$$g(x) = \begin{cases} x-1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$$

We first show that $g(x)$ is not onto.

Consider element 1 in co-domain N . this element is not an image of any of the elements in domain N .

Therefore, $f(x)$ is not onto.

Consider the function $g \circ f(x)$ is defined on the set of natural numbers

Such that

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(x+1) \\ &= x+1-1 \\ &= x \end{aligned}$$

For $y \in N$ there exists $x = y \in N$ such that $gof(x) = y$

Therefore, $gof(x)$ is onto.

8. Given a non-empty set X, consider $P(X)$ which the set of all subsets is of X .

Define the relation R in $P(X)$ is as follows:

For subsets A, B in $P(X)$, ARB if and only if A $\subset$ B. Is R an equivalence relation on $P(X)$?

Justify you answer.

Solution: For every set is a subset it self

Hence the for any set A, it is subset itself, hence ARA for all $A \in P(X)$

Therefore, the relation R is reflexive.

Suppose that $ARB \Rightarrow A \subset B$, it does not implies that $B \subset A$

Hence ARB does not implies that BRA

Therefore, the relation R is not symmetric

Suppose that ARB, BRC it implies that $A \subset B$ and $B \subset C$

It gives $A \subset C$, it means ARC

Therefore, the relation R is transitive

Hence, the relation R is not an equivalence relation as it is not symmetric

9. Find the number of all onto functions from the set {1, 2, 3, … , n) to itself.

Solution:

We know that Onto functions from the set $\{1,2,3,4,...,n\}$ to itself is simply a

permutation on n symbols

Thus, the total number of onto maps from {1, 2, … , n} to itself is the same as the total

number of permutations on n symbols 1, 2, …, n, which is n.

10. Let $S=\{a,b,c\}$ and $T=\{1,2,3\}$. Find F^{-1} of the following functions F from S to T

, if it exists.

i) $F=\{(a,3),(b,2),(c,1)\}$

ii) $F=\{(a,2),(b,1),(c,1)\}$

Solution: Let $S=\{a,b,c\}$ and $T=\{1,2,3\}$

i) Consider the function $F:S \to T$ defined as F: S → T is defined as $F=\{(a,3),(b,2),(c,1)\}$

$$F(a)=3, F(b)=2, F(c)=1$$

Therefore, $F^{-1}=\{(3,a),(2,b),(1,c)\}$

ii) Consider the function $F:S \to T$ is defined as $F=\{(a,2),(b,1),(c,1)\}$

Since $F(b)=F(c)=1$, the function F is not one – one.

Hence, F is not invertible

Therefore, F^{-1} does not exist.

11. Let $A=\{-1,0,1,2\}$ and $B=\{-4,-2,0,2\}$ are any two sets. Two functions $f(x), g(x)$ are defined from A to B as $f(x)=x^2-x, x\in A$, $g(x)=2\left|x-\frac{1}{2}\right|-1, x\in A$.

Are $f(x), g(x)$ will be equal. Justify your answer.

Solution: Given that $f(x)=x^2-x, x\in A$, $g(x)=2\left|x-\frac{1}{2}\right|-1, x\in A$

The roster form of the function $f(x)$ is $f(x)=\{(-1,2),(0,0),(1,0),(2,2)\}$

The roster form of the function $g(x)$ is $g(x)=\{(-1,2),(0,0),(1,0),(2,2)\}$

Hence the above two functions are equal

12. Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

A) 1 B) 2 C) 3 D) 4

Solution: Given that $A=\{1,2,3\}$

The smallest relation containing $(1,2)$ and $(1,3)$ which is reflexive and symmetric but not transitive relation is $R=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(3,1)\}$

Because relation R is reflexive as $(1,1),(2,2),(3,3)\in R$

Relation R is symmetric since ($(1,2),(2,1)\in R \Rightarrow (2,1),(3,1)\in R$

Relation R is not transitive as , but $(3,2)\notin R$.

If we add any two pairs $(3,2),(2,3)$ or both to relation R, then the relation becomes transitive also, so that the number of required relations is only one

This is matching with the option (A)

13. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

A) 1 B) 2 C) 3 D) 4

Solution: It is given that $A=\{1,2,3\}$.

The smallest equivalence relation containing $(1,2)$ is given by,

$$R_1=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$$

The remaining ordered pair are $(2,3),(3,2),(1,3),(3,1)$

Suppose that R_2 is another relation containing all the ordered pairs of R_1 and add $(2,3)$

To make R_2 is equivalence relation, for symmetry we must add $(3,2)$, for transitivity we have to add $(1,3)$ and $(3,1)$

Hence, there are two relations which are equivalence relations having $(1,2)$

This is matching with the option (B)

14. Let $f:R\to R$ be the signum function defined as $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<1\end{cases}$ and $g:R\to R$ be the Greatest Integer Function given by $g(x)=[x]$, where $[x]$ is greatest integer less than or equal to x. Then does $f\circ g$ and $g\circ f$ coincide in $(0,1]$?

Solution: Given that $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<1\end{cases}$

And another function $g:R\to R$ is defined as $g(x)=[x]$, where $[x]$ is the greatest integer less than or equal to x.

Let $x\in(0,1]$, then $[x]=1,$ when $x=1,$ and $[x]=0,$ when $0<x<1,$

Consider the compound functions

$$fog(x) = f(g(x))$$
$$= f([x])$$
$$= \begin{cases} f(1), & if \;\; x = 1 \\ f(0), & if \;\; x \in (0,1) \end{cases}$$
$$= \begin{cases} 1, & if \;\; x = 1 \\ 0, & if \;\; x \in (0,1) \end{cases}$$

and

$$go(x) = g(f(x))$$
$$= g(1) \qquad [as\, x > 0]$$

When $x \in (0,1)$, we have $fog(x) = 0$ and $gof(x) = 1$

It implies $f \circ g(x) \neq g \circ f(x)$

Exercise 1.1

1. Determine whether each of the following relations are reflexive, symmetric and transitive

(i) Relation R in the set $A=\{1,2,3,...14\}$ defined as $R=\{(x,y):3x-y=0\}$

(ii) Relation R in the set N as $R=\{(x,y):y=x+5 \text{ and } x<4\}$

(iii) Relation R in the set $A=\{1,2,3,4,5,6\}$ as $R=\{(x,y):y \text{ is divisible by } x\}$

(iv) Relation R in the set Z of all integers as $R=\{(x,y):x-y \text{ is an integer}\}$

(v) Relation R in the set A of human beings in a town at a particular time given by

a. $R=\{(x,y):x,y \text{ work at the same place}\}$

b. $R=\{(x,y):x,y \text{ live in the same locality}\}$

c. $R=\{(x,y):x \text{ is } 7 \text{ cm taller than } y\}$

d. $R=\{(x,y):x \text{ is wife of } y\}$

e. $R=\{(x,y):x \text{ is father of } y\}$

Solution: A relation R is defined on the set A is said to be

- Reflexive relation if $(a,a)\in R$ for all $a\in A$
- Symmetric relation if $(a,b)\in R\Rightarrow(b,a)\in R$
- Transitive relation if $(a,b)\in R,(b,c)\in R\Rightarrow(a,c)\in R$

(i) Given a relation R in the set $A=\{1,2,3,...14\}$ defined as $R=\{(x,y):3x-y=0\}$

The relation R in the roster form is $R=\{(1,3),(2,6),(3,9),(4,12)\}$,

Observing the above relation,

- It is not reflexive because $(1,1)\notin A$

- It is not symmetric because $(1,3) \in A$ and it does not implies that $(3,1) \notin A$
- If is not transitive because $(1,3),(3,9) \in A$ but $(1,9) \notin A$

Therefore, the relation R is neither Reflexive, nor symmetric nor transitive.

(ii) Relation R in the set N as $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$

The relation R in the roster form is $R = \{(1,6),(2,7),(3,8)\}$,

Observing the above relation,

- It is not reflexive because $(1,1) \notin R$
- It is not symmetric because $(1,6) \in R$ and it does not implies that $(6,1) \notin R$
- It is transitive because $(1,6),(2,7),(3,8) \in R$ but there is not image for $6,7,8$

Therefore, the relation R is neither reflexive nor symmetric, but it is transitive.

(iii) Relation R in the set $A = \{1,2,3,4,5,6\}$ as $R = \{(x, y) : y \text{ is divisible by } x\}$

Observing the above relation,

- It is reflexive because $(x, x) \in R$, since x is divisible by x
- It is not symmetric because $(x, y) \in R$ and it does not implies that x $(y, x) \notin R$, it means x is divisible by y does not implies that y divisible by x
- If x is divisible by y and y is divisible by z it implies that x is divisible by z, For any $(x, y),(y, z) \in R$ it implies that $(x, z) \in R$. Hence, the relation R is transitive

Therefore, the relation R is reflexive and transitive but not symmetric.

(iv) Relation R in the set Z of all integers as $R = \{(x, y) : x - y \text{ is an integer}\}$

- The relation R is reflexive because 0 is an integer, so that $(x, x) \in R$

- The relation R is symmetric because for any integer a, $-a$ is also integer so that for any $(x,y) \in R$ it implies that $(y,x) \in R$.
- The relation R is transitive because for any $(x,y),(y,z) \in R$ implies that $(x,z) \in R$

  The ordered pairs $(x,y),(y,z) \in R$, implies that

  $x-y$ is an integer and $y-z$ is an integer

  Sum of the above two values $x-y+y-z=x-z$ is also an integer. Hence, $(x,y),(y,z) \in R$ implies that $(x,z) \in R$

Therefore, the relation R is reflexive, symmetric and transitive, it means that the relation R is equivalence relation.

v) Suppose that the relation R in the set A of human beings in a town at a particular time

a. Given that $R=\{(x,y): x,y \text{ work at the same place}\}$

- It is reflexive, because $(x,x) \in R$ each person, himself work at the same place
- It is Symmetric, because if (x,y) working in the same place, then (y,x) also work in the same place
- It is transitive, because if x,y working in the same place and y,z also working in the same place then x,z work in the same place
- Therefore, the relation R is reflexive, symmetric and transitive. It implies that the relation R is equivalence relation.

b. $R=\{(x,y): x,y \text{ live in the same locality}\}$

- The relation R is reflexive, because $(x,x) \in R$, each person, himself lives in the same locality

- It is Symmetric, because if (x, y) staying in the same place, then (y, x) also stay in the same place
- It is transitive, because if x, y staying in the same place and y, z also staying in the same place then x, z staying in the same place
- Therefore, the relation R is reflexive, symmetric and transitive. It implies that the relation R is equivalence relation.

c. $R=\{(x, y): x \text{ is } 7 \text{cm taller than } y\}$

- It is not reflexive, no person is 7 cm taller than him self
- If is not symmetric, if a person is 7 cm taller than other person, the other person is not 7 cm taller than first one
- It is not transitive. If $(x, y),(y, z)$ belongs to the relation means y is 7 cm taller than x and z is 7 cm taller than y it concludes that z is 14 cm taller than x

  Therefore, (x, z) is not an element of R

d. $R=\{(x, y): x \text{ is wife of } y\}$

- It is not reflexive, because no person is wife itself
- It is not symmetric, because if x is wife of y then y is not wife of x
- It is transitive, because if x is wife of y then y is male, so y is not wife of any one

e. $R=\{(x, y): x \text{ is father of } y\}$

- It is not reflexive, because no person is father to himself
- It is not symmetric, because if x is father of y then y is not father of x
- It is not transitive, because if x is father of y and y is father of z then x is not father of z

2. Show that the relation R in $\mathbb{R}$, defined as $R=\{(a,b):a\le b^2\}$ is neither reflexive nor symmetric nor transitive.

Solution: Given that the relation is defined as $R=\{(a,b):a\le b^2\}$

- Relation is not reflexive, $\frac{1}{2}$ is a real numbers, the square of $\frac{1}{2}$ is $\frac{1}{4}$

  But $\frac{1}{2}\le\frac{1}{4}$ is false, hence $\left(\frac{1}{2},\frac{1}{2}\right)\notin R$

  Hence, the relation is not reflexive.
- Relation is not symmetric, $\left(\frac{1}{2},1\right)\in R$ it implies that $\frac{1}{2}\le 1^2$ but $\left(1,\frac{1}{2}\right)$ does not belongs to R because $1\le\frac{1}{4}$ is false.
- Relation is not transitive, $(2,1)\in R$ and $(1,1.1)\in R$ but $(2,1.1)$ does not belongs to R because $2\ge 1.21$

Therefore, the relation R is neither reflexive, not symmetric nor transitive relation.

3. Check whether the relation R defined in set $\{1,2,3,4,5,6\}$ as $R=\{(a,b):b=a+1\}$ is reflexive, symmetric or transitive

Solution: Given that the relation R defined in set $\{1,2,3,4,5,6\}$ as $R=\{(a,b):b=a+1\}$

The roster form of the relation is $R=\{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

Observing the set,

- $(1,1)\notin R$, so that the relation R is not reflexive.
- $(1,2)\in R$ but $(2,1)\notin R$, so that R is not symmetric
- In the relation R, $(1,2),(2,3)\in R$, but $(1,3)\notin R$. So that the relation is not transitive.

Therefore, the relation R is neither reflexive, nor symmetric nor transitive relation.

4. Show that the relation R in $\mathbb{R}$ defined as $R=\{(a,b):a\le b\}$, is reflexive and transitive but not symmetric.

Solution:

The relation R is defined as $R=\{(a,b):a\le b\}$

- For any real number a, $a\le a$, so that $(a,a)\in R$

  Hence the relation R is reflexive.

- Suppose that $(1,2)\in R$ but $(2,1)\notin R$ because $2>1$

  So that the relation R is not symmetric

- Suppose that $(a,b),(b,c)\in R$

  Hence, $a\le b, b\le c \Rightarrow a\le c$, it implies that $(a,c)\in R$

  Hence, the relation R is transitive.

Therefore, the relation R is reflexive, not symmetric and transitive.

5. Check whether the relation R in $\mathbb{R}$ defined as $R=\{(a,b):a\le b^3\}$ is reflexive, symmetric or transitive.

Solution:

The given relation is $R=\{(a,b):a\le b^3\}$

- $\left(\frac{1}{2},\frac{1}{2}\right)\notin R$ because $\frac{1}{2}\ge\frac{1}{8}$, so that the relation is not reflexive.

- We know that $(1,2)\in R$ but $(2,1)\notin R$ because $2\le 1$ is false. So that the relation is not symmetric.

- We know that $(3,2)\in \mathrm{R},(2,1.5)\in \mathrm{R}$ but $3\le(1.5)^3$ is false. So that the relation is not transitive.

Therefore, the relation is not reflexive, not symmetric, and not transitive.

6. Show that the relation R in the set $\{1,2,3\}$ given by $R=\{(1,2),(2,1)\}$ is symmetric but neither reflexive nor transitive

Solution: Given that the relation $R=\{(1,2),(2,1)\}$ is defined on the set $A=\{1,2,3\}$

Observing the relation $R=\{(1,2),(2,1)\}$

- $(1,1),(2,2),(3,3)$ does not belongs to the relation R, so that the relation R is not reflexive relation
- $(1,2)\in R$ and $(2,1)\in R$ hence the relation is symmetric
- $(1,2)\in R$ and $(2,1)\in R$ but $(1,1)$ does not belongs to R, hence the relation is not transitive.

Therefore, the relation R is not reflexive, symmetric but not transitive

7. Show that the relation R in the set A of all the books in a library of a college, given by $R=\{(x,y): x$ and y have the same number of pages$\}$ is an equivalence relation.

Solution: Given that the relation $R=\{(x,y): x$ and y have the same number of pages$\}$

- For any $x\in A$, $(x,x)\in R$ because x and x have the same number of pages, so that the relation is reflexive.
- Suppose that $(x,y)\in R$, it implies that both x,y have the same number of pages. Hence $(y,x)\in R$. Hence the relation is symmetric.
- Suppose that $(x,y)\in R,(y,z)\in R$, it implies that both x,y have the same number of pages, and both y,z have the same number of pages Hence, both x,z have the same number of pages, hence $(x,z)\in R$. Hence the relation is transitive.

Therefore, the relation is reflexive, symmetric and transitive. So that the relation is equivalence relation

8. Show that the relation R in the set $A=\{1,2,3,4,5\}$ given by $R=\{(a,b):|a-b| \text{ is even}\}$ is an equivalence relation. Show that all the elements of $\{1,3,5\}$ are related to each

other and all the elements of $\{2,4\}$ are related to each other. But no element of $\{1,3,5\}$ is related to any element of $\{2,4\}$.

Solution: Given that the relation R in the set A is defined as d $R=\{(a,b):|a-b| \text{is even}\}$

- For any element $a \in A$, $a-a=0$ is an even number, hence $(a,a) \in R$, so that the relation R is reflexive
- Suppose that $(a,b) \in R$,

  Hence, $|a-b|=2k \Rightarrow |b-a|=2k$, It implies that $(b,a) \in R$

  Hence, the relation R is symmetric.
- Suppose that $(a,b) \in R,(b,c) \in R$, Hence, $|a-b|=2p,|b-c|=2q$

  Adding the above two equations

$$\begin{aligned} |a-b+b-c| &= 2p+2q \\ |a-c| &= 2(p+q) \\ |c-a| &= 2(p+q) \end{aligned}$$

  It implies that $(\mathrm{a},c) \in R$, Hence, the relation R is transitive.

Therefore, the relation is equivalence.

All elements of $\{1,3,5\}$ are related to each other because they are all odd. So, the modulus of the difference between any two elements is even.

Similarly, all elements of $\{2,4\}$ are related to each other because they are all even.

Because the difference of two odd numbers or the difference of two even numbers is even, so that no element of {1, 3, 5} can be related to any element of {2, 4} and vice versa.

9. Show that each of the relation R in the set $A=\{x \in z;\ 0 \le x \le 12\}$ given by

   (i) $R=\{(a,b):|a-b| \text{is multiple of } 4\}$

   (ii) $R=\{(a,b):a=b\}$ is an equivalence relation.

Find the set of all elements related to 1 in each case.

Solution: Consider the set $A=\{0,1,2,3,4,5,6,7,8,9,10,11,12\}$

(i) Relation is defined as $R=\{(a,b):|a-b| \text{is multiple of } 4\}$

- For any $a \in A$, $|a-a|=0$ is multiple of 4, so that (a,a) is an element of R is reflexive
- Suppose that $(a,b) \in R$, it means $|a-b|$ is multiple of 4, $\Rightarrow |a-b|=4k$

  It implies $|b-a|=4k \Rightarrow (b,a) \in R$

  Hence the relation R is symmetric
- Suppose that $(a,b) \in R,(b,c) \in R$

  $$\Rightarrow |a-b|=4p,|b-c|=4q$$

  Adding the above two equations

  $$\Rightarrow |a-b+b-c|=4(p+q)$$
  $$\Rightarrow |a-c|=4Q$$

  It implies that $(a,c) \in R$

  Hence, the relation R is transitive.

Therefore, the relation is equivalence relation.

The set of all elements which are related to 1 are $\{1,5,9\}$

(ii) Relation is defined as $R=\{(a,b):a=b\}$

- For any $a \in A$, $a=a$, so that (a,a) is an element of R is reflexive
- Suppose that $(a,b) \in R$, it means $a=b$ $b=a$

  It implies $(b,a) \in R$

  Hence the relation R is symmetric
- Suppose that $(a,b) \in R,(b,c) \in R$

  $$\Rightarrow a=b,b=c \Rightarrow a=c$$

  It implies that $(a,c) \in R$

  Hence, the relation R is transitive.

Therefore, the relation is equivalence relation.

The set of all elements which are related to 1 is $\{1\}$

10. Given an example of a relation. Which is

   a. Symmetric but neither reflexive nor transitive
   b. Transitive but neither reflexive nor symmetric.
   c. Reflexive and symmetric but not transitive.
   d. Reflexive and transitive but not symmetric.
   e. Symmetric and transitive but not reflexive

Solution:

(a) Suppose that $A=\{1,2,3\}$ and relation R is defined as $R=\{(1,2),(2,1)\}$

This is an example for the relation which is symmetric but not reflexive and not transitive.

(b) Suppose that $A=\{1,2,3\}$ and relation R is defined as $R=\{(1,2)\}$

This is an example for the relation which is transitive but not reflexive and not symmetric.

(c) Suppose that $A=\{1,2,3\}$ and relation R is defined as

$$R=\{(1,1),(2,2),(3,3),(1,3),(3,2),(3,1),(2,3)\}$$

This is an example for the relation which is reflexive, symmetric but not transitive.

(d) Suppose that $A=\{1,2,3\}$ and relation R is defined as

$$R=\{(1,1),(2,2),(3,3),(1,3)\}$$

This is an example for the relation which is reflexive, transitive but not symmetric.

(e) Suppose that $A=\{1,2,3\}$ and relation R is defined as

$$R=\{(3,1),(1,3),(1,1)\}$$

This is an example for the relation which is symmetric, transitive but not reflexive.

11. Show that the relation R in the set A of points in a plane given by $R=\{(P,Q):\mathrm{OP}=\mathrm{OQ}\}$, is an equivalence relation. Further, show that the set of all point related to a point $P\neq(0,0)$ is the circle passing through P with origin as centre

Solution:

Consider the set A of points in a plane

The relation R on the set A is defined as $R=\{(P,Q):\mathrm{OP}=\mathrm{OQ}\}$

- For any $P\in A$, $OP=OP$, so that (P,P) is an element of R, so that R is reflexive
- Suppose that (P,Q), it means $OP=OQ$ it gives $OQ=OP$

  It implies $(Q,P)\in R$

  Hence the relation R is symmetric
- Suppose that $(P,Q)\in R,(Q,S)\in R$

  $\Rightarrow OP=OQ,OQ=OS$
  $\Rightarrow OP=OS$

  It implies that $(P,S)\in R$

  Hence, the relation R is transitive.

Therefore, the relation is equivalence relation.

The set of all elements which are related to P is the set of all points on the circle having Centre at origin and radius OP

12. Show that the relation R defined in the set A of all triangles as $R=\{(T_1,T_2):T_1 \text{ is similar to } T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1,T_2,T_3 are related?

Solution: Consider the set A of all triangles in a plane

The relation R on the set A is defined as $R=\{(T_1,T_2):T_1 \text{ is similar to } T_2\}$

- For any $T_1 \in A$, the triangle T_1 is similar itself, so that (T_1,T_1) is an element of R, so that R is reflexive
- Suppose that (T_1,T_2), it means T_1,T_2 are similar triangles it gives T_2,T_1 Are similar triangles. It implies $(T_2,T_1)\in R$

  Hence the relation R is symmetric
- Suppose that $(T_1,T_2)\in R,(T_2,T_3)\in R$, it implies that T_1,T_2 are similar triangles and T_2,T_3 are similar triangles

  Hence, T_1,T_3 are similar triangles, it implies that (T_1,T_3) is an element of R

  It implies that $(T_1,T_3)\in R$

  Hence, the relation R is transitive.

Therefore, the relation is equivalence relation.

Consider three right angle triangles T_1 with sides 3, 4, 5, and T_3 with sides 6, 8, 10 are related, so that (T_1,T_3) is an element in R

13. Show that the relation R defined in the set A of all polygons as $R=\{(P_1,P_2):P_1 \text{ and } P_2 \text{ have the same number of sides}\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

Solution: Consider the set A of all polygons in a plane

The relation R on the set A is defined as

$$R=\{(P_1,P_2):P_1 \text{ and } P_2 \text{ have the same number of sides}\}$$

- For any $P_1\in A$, the number of sides of polygon P_1 is same as the number of sides of the polygon P_2, so that (P_1,P_2) is an element of R, so that R is reflexive

- Suppose that (P_1,P_2), it means P_1,P_2 have same number of sides and P_2,P_1 have the same number of sides

  It implies $(P_2,P_1)\in R$

  Hence the relation R is symmetric

- Suppose that $(P_1,P_2)\in R,(P_2,P_3)\in R$, it implies that the number of sides of the polygon P_1,P_2 are same and the number of sides of the polygon P_2,P_3 are same

  Hence, the number of sides of polygons P_1,P_3 is same, it implies that (P_1,P_3) is an element of R

  It implies that $(P_1,P_3)\in R$

  Hence, the relation R is transitive.

Therefore, the relation is equivalence relation.

The set of all elements in A related to the right angle triangle T with sides 3, 4 and 5 is the set of all triangles

14. Let L be the set of all lines in a plane and R be the relation in L defined as $R=\{(L_1,L_2): L_1 \text{ is parallel to } L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y=2x+4$

Solution: Consider the set A of all lines in a plane.

The relation R on the set A is defined as

$$R=\{(L_1,L_2): L_1 \text{ is parallel to } L_2\}$$

- For any $L_1\in A$, any line L_1 is parallel to the line itself. So that (L_1,L_1) is an element of R, so that R is reflexive
- Suppose that (L_1,L_2), it means L,L_2 are parallel, and it implies that L_2,L_1 are parallel lines It implies $(L_2,L_1)\in R$

  Hence the relation R is symmetric

- Suppose that $(L_1, L_2) \in R, (L_2, L_3) \in R$, it implies that L_1 is parallel to L_2 and L_2 is parallel to L_3

  It implies that L_1, L_3 are parallel

  Hence, (L_1, L_3) is an element of R

  It implies that $(L_1, L_3) \in R$

  Hence, the relation R is transitive.

Therefore, the relation is equivalence relation.

The set of all elements in A related to the line $y = 2x + 4$ is the set of all the lines having the equations $y = 2x + c$

15. Let R be the relation in the set $A = \{1, 2, 3, 4\}$

$$R = \{(1,1), (1,2), (2,2), (4,4), (1,3), (3,3), (3,2)\}$$

Choose the correct answer

(A) R is reflexive and symmetric but not transitive

(B) R is reflexive and transitive but not symmetric

(C) R is symmetric and transitive but not reflexive

(D) R is an equivalence relation

Solution:

Consider the given relation

$$R = \{(1,1), (1,2), (2,2), (4,4), (1,3), (3,3), (3,2)\}$$

- The relation is reflexive
- But the relation is not symmetric because $(1,2) \in R, (2,1) \notin R$
- The relation is transitive

This is matching with the option (B)

16. Let R be the relation in the set N given by $R = \{(a,b) : a = b - 2, b > 6\}$

Choose the right answer

(A) $(2,4) \in R$

(B) $(3,8) \in R$

(C) $(6,8) \in R$

(D) $(8,7) \in R$

Solution:

Given that the relation is $R = \{(a,b) : a = b-2, b > 6\}$

The roster form of the relation contains $(6,8)$

Hence the option (C) is correct.

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Exercise 1.2

1. Show that the function $f: R_* \to R_*$ defined by $f(x)=\frac{1}{x}$ is one-one and onto, where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_*?

Solution: Consider the function $f: R_* \to R_*$ defined by $f(x)=\frac{1}{x}$

The function is one to one if and only if $x, y \in R_*$ such that $f(x)=f(y)$

It implies that $\frac{1}{x}=\frac{1}{y} \Rightarrow x=y$

Hence, $f(x)$ is one to one function.

For any $y \in R_*$ there exists $x \in R_*$ such that $x=\frac{1}{y} \in R_*$ and $f(x)=\frac{1}{\left(\frac{1}{y}\right)}=y$

Therefore, the function is both one to one and onto.

If the function $g(x): N \to R$ is defined as $g(x)=\frac{1}{x}$ then it is one to one but not onto

Because there is preimage for the value $2 \in R_*$ in the set of natural numbers

2. Check the infectivity and surjectivity of the following functions

a. $f: N \to N$ Given by $f(x)=x^2$

b. $f: Z \to Z$ Given by $f(x)=x^2$

c. $f: R \to R$ Given by $f(x)=x^2$

d. $f: N \to N$ Given by $f(x)=x^3$

e. $f: Z \to Z$ Given by $f(x)=x^3$

Solution: To check the function $f(x)$ is one to one, suppose that $f(x_1)=f(x_2)$ and show that $x_1 = x_2$ for all x_1, x_2 belongs to domain, To check the function $f(x)$ is onto, show that for every element in the codomain there exists preimage in the domain

a) The given function is $f(x) = x^2$ and domain and range both are equal to set of natural numbers.

The function is one to one because $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$

The function is not onto. Because there is no preimage for 2 in the set of natural numbers.

b) The given function is $f(x) = x^2$ and domain and range both are equal to set of all integers.

The function is not one to one because $f(2) = f(-2)$ but $2 \neq -2$

The function is not onto. Because there is no preimage for 2 in the set of natural numbers.

c) The given function is $f(x) = x^2$ and domain and range both are equal to set of all Real numbers.

The function is not one to one because $f(2) = f(-2)$ but $2 \neq -2$

The function is not onto. Because there is no square root for negative real numbers.

d) The given function is $f(x) = x^3$ and domain and range both are equal to set of natural numbers.

The function is one to one because $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

The function is not onto. Because there is no preimage for 2 in the set of natural numbers.

e) The given function is $f(x) = x^3$ and domain and range both are equal to set of all integers.

The function is one to one because $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

The function is not onto. Because there is no preimage for 2 in the set of integers

3. Prove that the greatest Integer function f: R → R given by f(x) = [x], is neither one – one nor onto, where [x] denotes the greatest integer less than or equal to x

Solution: Given that $f: R \to R$ and is defined by $f(x) = [x]$

This is not one to one function because $f(1.2) = 1, f(1.3) = 1$

This is not onto function because there is no pre image to any non - integer values

Therefore, the function is neither one to one nor onto.

4. Show that the Modulus Function $f: R \to R$ given by $f(x) = |x|$, is neither one – one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Solution: The function $f: R \to R$ is defined as $f(x) = |x|$

The function is not one to one because $f(1) = f(-1)$

There is no preimage to any negative real number in the real numbers

Therefore, the function $f(x) = |x|$ is neither one to one nor onto in the set of real numbers.

5. Show that the Signum function $f: R \to R$, given by $f(x) = \begin{cases} 1, & if \quad x > 0 \\ 0, & \text{if} \quad x = 0 \\ -1, & if \quad x < 0 \end{cases}$ is neither one-one nor onto

Solution: The given function $f(x) = \begin{cases} 1, & if \quad x > 0 \\ 0, & \text{if} \quad x = 0 \\ -1, & if \quad x < 0 \end{cases}$ is many to one function

For all positive real numbers, the functional values are equal

So that the function is not one to one

The function is not onto because there is no pre image to 2

Therefore, the signum function is neither one to one nor onto.

6. Let $A=\{1,2,3\}$, $B=\{4,5,6,7\}$ and let $f=\{(1,4),(2,5),(3,6)\}$ be a function from A to B. Show that f is one – one.

Solution:

The function is defined as $f=\{(1,4),(2,5),(3,6)\}$

The first elements of ordered pairs in the function are different. So that the function is one to one.

7. In each of the following cases, state whether the function is one – one, onto or bijective. Justify your answer.

(i) $f:R\rightarrow R$ Defined by $f(x)=3-4x$

(ii) $f:R\rightarrow R$ Defined by $f(x)=1+x^2$

Solution:

(i) The given function is $f:R\rightarrow R$ Defined by $f(x)=3-4x$

Suppose that $f(x_1)=f(x_2)$

It implies that

$$3-4x_1=3-4x_2$$
$$-4x_1=-4x_2$$
$$x_1=x_2$$

Therefore, the function is one to one.

For any real number y in the set of all real numbers, there exists $\frac{3-y}{4}$ in the real numbers such that $f\left(\frac{3-y}{4}\right)=3-4\left(\frac{3-y}{4}\right)=y$

Hence, the function is onto.

Therefore, the function is both one to one and onto, it means it is bijective.

ii) Given that the function $f:R\rightarrow R$ is defined as $f(x)=1+x^2$

Suppose that $f(x_1)=f(x_2)$

It implies that

$$\begin{aligned}1+x_1^{\;2}&=1+x_2^{\;2}\\ x_1^{\;2}&=x_2^{\;2}\\ x_1&=\pm x_2\end{aligned}$$

Hence, the function is not one to one.

There is no preimage for the real number 2 , so that the function is not onto.

Therefore the function is neither one to one nor onto.

8. Let A and B be sets. Show that $f:A\times B\rightarrow B\times A$ such that $(a,b)=(b,a)$ is bijective function.

Solution: Given A and B be sets and the function defined as $f:A\times B\rightarrow B\times A$,

$f(a,b)=(b,a)$

Suppose that $f(a_1,b_1)=f(a_2,b_2)$, it implies that $(b_1,a_1)=(b_2,a_2)$

Hence, $(a_1,b_1)=(a_2,b_2)$, the function is one to one.

For any ordered pair $(b,a)\in B\times A$, there exists $(a,b)\in A\times B$ such that $f(a,b)=(b,a)$

Therefore, the function f is both one to one and onto.

9. Let $f:N\rightarrow N$ be defined by $f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n\in N$. State whether the function f is bijective. Justify your answer

Solution: The given function $f:N\rightarrow N$ is defined as

$$f(n)=\begin{cases}\frac{n+1}{2}, & \text{if } n \text{ is odd}\\ \frac{n}{2}, & \text{if } n \text{ is even}\end{cases} \quad \text{For all } n\in N$$

The function is not one to one because $f(1)=f(2)$

$$f(1)=\frac{1+1}{2}=1 \text{ and } f(2)=\frac{2}{2}=1$$

Hence, the function is not one to one.

Consider a natural number n in co-domain N

Suppose that n is odd, the value of n is in the form of $n=2r+1$, there exists $4r+1\in N$ such that $f(4r+1)=\frac{4r+1+1}{2}=2r+1$

Suppose that n is even, the value of n is in the form of $n=2r$, for some $r\in N$, there exists $4r\in N$ such that $f(4r)=\frac{4r}{2}=2r$

Therefore, $f(x)$ is onto.

Hence, the function is not bijective function.

10. Let $A=R-\{3\}$ and $B=R-\{1\}$. Consider the function $f:A\to B$ defined by $f(x)=\left(\frac{x-2}{x-3}\right)$. Is f ono-one and onto? Justify your answer

Solution: The given function $f:A\to B$ is defined as $f(x)=\left(\frac{x-2}{x-3}\right)$

where $A=R-\{3\},B=R-\{1\}$

Suppose that $x,y\in A$ such that $f(x)=f(y)$

Hence, $\frac{x-2}{x-3}=\frac{y-2}{y-3}$

Cross multiply

$$
\begin{aligned}
xy-3x-2y+6&=xy-2x-3y+6\\
-3x-2y&=-2x-3y\\
-x&=-y\\
x&=y
\end{aligned}
$$

Hence, $f(x)=f(y)\Rightarrow x=y$

Therefore, the function $f(x)=\left(\dfrac{x-2}{x-3}\right)$ is one to one.

Suppose that $y\in B=R-\{1\}$, it means y is a real number other than 1.

The function $f(x)$ is onto if there exists $x\in A$ such that $f(x)=y$

It implies $\dfrac{x-2}{x-3}=y$

Simplify

$$
\begin{aligned}
x-2&=xy-3y\\
xy-x&=3y-2\\
x(y-1)&=3y-2\\
x&=\frac{3y-2}{y-1}
\end{aligned}
$$

For any value of $y\neq 1$, there exists a real number $x=\dfrac{3y-2}{y-1}$ such that $f(x)=y$

Hence, the function is one to one and onto.

Therefore, the function is bijective function.

11. Let $f:R\to R$ be defined as $f(x)=x^4$. Choose the correct answer

(A) $f(x)$ is one-one onto

(B) $f(x)$ is many-one onto

(C) $f(x)$ is one-one but not onto

(D) $f(x)$ is neither one-one nor onto

Solution: The given function is $f(x)=x^4$ defined on the set of real numbers

Observing that $f(-2) = f(2)$ does not implies that $-2 = 2$

Hence, the function is not one to one

And there is no real number whose fourth root of a negative number

So that the function is not onto.

Hence, the function is neither one to one nor onto.

This is matching with the option (D)

12. Let $f : R \to R$ be defined as $f(x) = 3x$. Choose the correct answer.

A) $f(x)$ is one to one and onto

B) $f(x)$ is many to one

C) $f(x)$ is one to one but not onto

D) $f(x)$ is Neither one to one nor onto.

Solution: Consider the function $f(x) = 3x$ defined on the real numbers.

Suppose that $f(x) = f(y)$

It implies that $3x = 3y \Rightarrow x = y$

It concludes that the function $f(x) = 3x$ is one to one.

For any real number y in co-domain R, there exists $\frac{y}{3}$ in R such that

$$f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$$

Hence, the function is onto

Hence, the function is both one to one and onto

Therefore, this is matching with the option (A)

Exercise 1.3

1. Let $f:\{1,3,4\}\to\{1,2,5\}$ and the function $g:\{1,2,5\}\to\{1,3\}$ be given by $f=\{(1,2),(3,5),(4,1)\}$ and $g=\{(1,3),(2,3),(5,1)\}$ write down the $g\circ f$

Solution: Given that the functions $f:\{1,3,4\}\to\{1,2,5\}$ and $g:\{1,2,5\}\to\{1,3\}$ defined as

$$f=\{(1,2),(3,5),(4,1)\} \text{ And } g=\{(1,3),(2,3),(5,1)\}$$

The function $g\circ f$ is defined from the domain of the function f to the range of the function g

It means $g\circ f:\{1,3,4\}\to\{1,3\}$

Therefore, $g\circ f:\{(1,3),(3,1),(4,3)\}$

2. Let $f:R\to R, g:R\to R$ and $h:R\to R$ are any three functions. Show that $(f+g)\circ h=f\circ h+g\circ h$ and $(f\cdot g)\circ h=(f\circ h)\cdot(g\circ h)$

Solution: The compound function is defined as $f\circ g(x)=f(g(x))$

We want to prove that $(f+g)\circ h=f\circ h+g\circ h$

Consider the left hand side of the relation $(f+g)\circ h=f\circ h+g\circ h$

$$\begin{aligned}(f+g)\circ h&=(f+g)(h(x))\\&=f(h(x))+g(h(x))\\&=f\circ h(x)+g\circ h(x)\end{aligned}$$

Therefore, $(f+g)\circ h=f\circ h+g\circ h$

We want to prove that $(f\cdot g)\circ h=(f\circ h)\cdot(g\circ h)$

Consider the left hand side of the relation $(f\cdot g)\circ h=(f\circ h)\cdot(g\circ h)$

$$\begin{aligned}(f\cdot g)\circ h &= (f\cdot g)(h(x))\\ &= f(h(x))\cdot g(h(x))\\ &= f\circ h(x)\cdot g\circ h(x)\end{aligned}$$

Therefore, $(f\cdot g)\circ h=(f\circ h)\cdot(g\circ h)$

3. Find gof and fog, if

a. $f(x)=|x|$ and $g(x)=|5x-2|$

b. $f(x)=8x^3$ and $g(x)=x^{\frac{1}{3}}$

Solution:

a) The given functions are $f(x)=|x|$ and $g(x)=|5x-2|$

Finding $g\circ f(x)$

$$g\circ f(x)=g(f(x))=g(|x|)=|5|x|-2|$$

Finding $f\circ g(x)$

$$f\circ g(x)=f(g(x))=f(|5x-2|)=|5x-2|$$

Therefore, $g\circ f(x)=|5|x|-2|,\ f\circ g(x)=|5x-2|$

b) The given functions are $f(x)=8x^3$ and $g(x)=x^{\frac{1}{3}}$

Finding $g\circ f(x)$

$$g\circ f(x)=g(f(x))=g(8x^3)=(8x^3)^{\frac{1}{3}}=2x$$

Finding $f\circ g(x)$

$$f\circ g(x)=f(g(x))=f\left(x^{\frac{1}{3}}\right)=8\left(x^{\frac{1}{3}}\right)^3=8x$$

Therefore, $g\circ f(x)=2x,\ f\circ g(x)=8x$

4. If $f(x)=\frac{4x+3}{6x-4}, x\neq\frac{2}{3}$, then show that $fof(x)=x$, for all $x\neq\frac{2}{3}$. And then find the inverse of the function $f(x)$?

Solution:

Given that the function is $f(x)=\frac{4x+3}{6x-4}, x\neq\frac{2}{3}$

Consider the compound function $f\circ f(x)$

$$f\circ f(x)=f(f(x))$$
$$=f\left(\frac{4x+3}{6x-4}\right)$$

Again apply the function

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

Take the denominator common and cancel out the common denomiantor

$$=\frac{16x+12+18x-12}{24x+18-24x+16}$$
$$=\frac{34x}{34}$$
$$=x$$

Therefore, $f\circ f(x)=x$

Since $f\circ f(x)=x$, the inverse of the function $f(x)$ is it self

Therefore the inverse of the function $f(x)$ is $f(x)$

5. State with reason whether following functions have inverse

(i) A function $f:\{1,2,3,4\}\to\{10\}$ defined as $f=\{(1,10),(2,10),(3,10),(4,10)\}$

(ii) A function $g:\{5,6,7,8\}\to\{1,2,3,4\}$ defined as

$g=\{(5,4),(6,3),(7,4),(8,2)\}$

(iii) A function $h:\{2,3,4,5\}\to\{7,9,11,13\}$ defined as

$h=\{(2,7),(3,9),(4,11),(5,13)\}$

Solution:

i) The function $f:\{1,2,3,4\}\to\{10\}$ is defined as

$$f=\{(1,10),(2,10),(3,10),(4,10)\}$$

The function is not one to one, so it is not bijective

Hence the function is not invertible.

ii) A function $g:\{5,6,7,8\}\to\{1,2,3,4\}$ defined as $g=\{(5,4),(6,3),(7,4),(8,2)\}$

The function $g(x)$ is not one to one because $g(5)=g(7)=4$

Hence, the function is not invertible

iii) A function $h:\{2,3,4,5\}\to\{7,9,11,13\}$ defined as

$h=\{(2,7),(3,9),(4,11),(5,13)\}$

The function $h(x)$ is one to one because each element of the domain has unique image and all the elements of codomain of the function has preimages

Hence the function is onto

Therefore, the function is invertible.

6. Show that $f:[-1,1]\to R$, given by $f(x)=\dfrac{x}{x+2}$ is one – one. Find the inverse of the function $f:[-1,1]\to$ Range of f.

Solution: The given function $f:[-1,1]\to R$ is defined as $f(x)=\dfrac{x}{x+2}$

Suppose that $f(x)=f(y)$. It implies that $\dfrac{x}{x+2}=\dfrac{y}{y+2}$

Cross multiply

$$\begin{aligned} xy+2x&=xy+2y\\ 2x&=2y\\ x&=y \end{aligned}$$

Hence, the function is one to one.

Suppose that $f(x)=y$

$$\begin{aligned}\frac{x}{x+2}&=y\\ x&=xy+2y\\ x(1-y)&=2y\\ x&=\frac{2y}{1-y}\end{aligned}$$

Suppose that $g(x)=\frac{2x}{1-x}$

Consider $f\circ g(x)$

$$f\circ g(x)=f\left(\frac{2x}{1-x}\right)=\frac{\frac{2x}{1-x}}{\frac{2x}{1-x}+2}=\frac{\frac{2x}{1-x}}{\frac{2x+2-2x}{1-x}}=\frac{2x}{2}=x$$

And $g\circ f(x)=x$, hence $f(x),g(x)$ are inverses to each other

The inverse of the function $f(x)=\frac{x}{x+2}$ is $g(x)=\frac{2x}{1-x}$

7. Consider $f:R\to R$ given by $f(x)=4x+3$. Show that $f(x)$ is invertible. Find the inverse of $f(x)$

Solution:

The given function $f:R\to R$ is given by $f(x)=4x+3$

Suppose that $f(x)=f(y)$

$$\begin{aligned}f(x)&=f(y)\\ 4x+3&=4y+3\\ 4x&=4y\\ x&=y\end{aligned}$$

Hence, the function is one to one

Suppose that $y \in R$, and let $y = 4x + 3$

It implies $x = \frac{y-3}{4} \in R$

Hence, $f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y$

Therefore $f(x)$ is onto

Since the function is bijective function, f^{-1} exists

The function $g : \mathrm{R} \to \mathrm{R}$ defined as $g(x) = \frac{x-3}{4}$

Consider the compound function $(gof)(x)$

$$g \circ f(x) = g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = \frac{4x}{4} = x$$

And consider the compound function

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = y$$

Hence, $f \circ g(x) = g \circ f(x) = x$, so that $f(x), g(x)$ are inverse functions to each

Other

Therefore, the function $f(x)$ invertible and the inverse of function $f(x)$ is given by

$f^{-1}(x) = g(x) = \frac{x-3}{4}$

8. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that $f(x)$ is invertible with the inverse f^{-1} given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Solution:

The given function $f: R_+ \to [4, \infty)$ is defined as $f(x) = x^2 + 4$

Check the function $f(x) = x^2 + 4$ is "one to one"

Suppose that $f(x) = f(y)$

It implies that

$$x^2 + 4 = y^2 + 4 \Rightarrow x^2 = y^2$$

Function is defined on the set of all positive real numbers, hence $x^2 = y^2 \Rightarrow x = y$

Therefore, the function $f(x) = x^2 + 4$ is one to one.

Check the function $f(x) = x^2 + 4$ is "onto"

Suppose that $y \in [4, \infty)$ and suppose that $y = x^2 + 4$

It implies that $x^2 = y - 4 \geq 0 \Rightarrow x = \sqrt{y-4}$

Hence,

$$\begin{aligned} f(x) &= f\left(\sqrt{x-4}\right) \\ &= \left(\sqrt{x-4}\right)^2 + 4 \\ &= x - 4 + 4 \\ &= x \end{aligned}$$

Therefore, the function $f(x)$ is one to one and onto.

Since the function $f(x)$ is bijective, it is invertible

The function $g(x)$ is $g: [4, \infty) \to R_+$ defined as $g(x) = \sqrt{x-4}$

The composite function

$$
\begin{aligned}
g\circ f(x) &= g(f(x)) \\
&= g(x^2+4) \\
&= \sqrt{(x^2+4)-4} \\
&= \sqrt{x^2} \\
&= x
\end{aligned}
$$

The composite function

$$
\begin{aligned}
f\circ g(x) &= f(g(x)) \\
&= f\left(\sqrt{x-4}\right) \\
&= \left(\sqrt{x-4}\right)^2+4 \\
&= x-4+4 \\
&= x
\end{aligned}
$$

Therefore, $gof(x)=fog(x)=I_R$

Hence, the function $f(x)$ is invertible and its inverse is $f^{-1}(x)=\sqrt{x-4}$

9. Consider $f:R_+\to[-5,\infty)$ given by $f(x)=9x^2+6x-5$. Show that $f(x)$ is invertible with $f^{-1}(y)=\left(\dfrac{\left(\sqrt{y+6}\right)-1}{3}\right)$

Solution:

Given that the function $f:R_+\to[-5,\infty)$ is given as $f(x)=9x^2+6x-5$

Let y be an arbitrary element of $[-5,\infty)$

$$
\begin{aligned}
y &= 9x^2+6x-5 \\
y &= (3x+1)^2-1-5 \\
&= (3x+1)^2-6 \\
y+6 &= (3x+1)^2 \\
3x+1 &= \sqrt{y+6}
\end{aligned}
$$

It implies that $x=\left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$

Therefore, $f(x)$ is onto and range of the function $f(x)$ is $[-5,\infty)$

Consider the compound function

$(gof)(x)=g\left(9x^2+6x-5\right)=g\left((3x+1)^2-6\right)=\sqrt{(3x+1)^2-6+6}-1=x$

and $(fog)(y)=f(g(y))=\left(\sqrt{y+6}\right)^2-6=y+6-6=y$

Therefore, the inverse of the function $f^{-1}(y)=g(y)=\left(\frac{\sqrt{y+6}-1}{3}\right)$

10. Let $f:x\to y$ be an invertible function. Show that f has unique inverse

Solution:

Let $f:x\to y$ be an invertible function

Suppose f has two inverses $g_1(x)$ and $g_2(x)$

For all $y\in Y$

$$fog_1(y)=I_y(y)=fog_2(y)$$

It implies

$$f(g_1(y))=f(g_2(y))$$
$$g_1(y)=g_2(y)$$
$$g_1=g_2$$

11. Consider $f:\{1,2,3\}\to\{a,b,c\}$ given by $f(1)=a,\ f(2)=b\ \ f(3)=c$. Find f^{-1} and show that $\left(f^{-1}\right)^{-1}=f$

Solution: For the function $f:\{1,2,3\}\to\{a,b,c\}$ is given by $f(1)=a,\ f(2)=b$, and $f(3)=c$

If we define $g:\{a,b,c\}\to\{1,2,3\}$ as $g(a)=1, g(b)=2, g(c)=3$

We have

$$f\circ g(a)=f(g(a))=f(1)=a$$
$$f\circ g(b)=f(g(b))=f(2)=b$$
$$f\circ g(c)=f(g(c))=f(3)=\text{ c}$$

$$g\circ f(1)=g(f(1))=g(a)=1$$
$$g\circ f(2)=g(f(2))=g(b)=2$$
$$g\circ f(3)=g(f(3))=g(c)=3$$

Both compound functions are identity functions

Hence $f(x),g(x)$ are inverses to each other.

Therefore, $f^{-1}(x)=g(x)$

Suppose that the inverse of $g(x)$ is $h(x)$

Hence,

$$\begin{aligned}h(x)&=\{(1,a),(2,b),(3,c)\}\\&=f(x)\end{aligned}$$

It implies that

$$\begin{aligned}\left(f^{-1}(x)\right)^{-1}&=\left(g(x)\right)^{-1}\\&=h(x)\\&=f(x)\end{aligned}$$

Therefore, $\left(f^{-1}(x)\right)^{-1}=f(x)$

12. Let $f:X\to Y$ be an invertible function. Show that the inverse of f^{-1} is $f(x)$, it means $\left(f^{-1}(x)\right)^{-1}=f(x)$

Solution:

Let $f : X \rightarrow Y$ be an invertible function, there exists a function $g : Y \rightarrow X$ such that $g \circ f(x) = f \circ g(x) = x$.

It implies that $f^{-1}(x) = g(x)$

We know that $f^{-1} \circ f(x) = x$

It implies that the inverse of $f^{-1}(x)$ is $f(x)$

Therefore, $\left(f^{-1}(x)\right)^{-1} = f(x)$

13. If $f : R \rightarrow R$ is given by $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$, then $fof(x)$ is

A) $\dfrac{1}{x^3}$

B) x^3

C) x

D) $\left(3 - x^3\right)$

Solution:

The function $f : R \rightarrow R$ is defined as $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$

$$\begin{aligned} fof(x) &= f\left(f(x)\right) \\ &= f\left(3 - x^3\right)^{\frac{1}{3}} \\ &= \left[3 - \left(\left(3 - x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}} \\ &= \left[3 - \left(3 - x^3\right)\right]^{\frac{1}{3}} \\ &= x \end{aligned}$$

This is matching with the option (C)

14. Let $f: R-\left\{-\frac{4}{3}\right\} \rightarrow R$ be a function as $f(x)=\frac{4x}{3x+4}$. The inverse of the function $f \rightarrow R-\left\{-\frac{4}{3}\right\}$ is given by

A) $g(y)=\frac{3y}{3-4y}$

B) $g(y)=\frac{4y}{4-3y}$

C) $g(y)=\frac{4y}{4+3y}$

D) $g(y)=\frac{3y}{4-3y}$

Solution:

Consider the function $f: R-\left\{-\frac{4}{3}\right\} \rightarrow R$ defined as $f(x)=\frac{4x}{3x+4}$

Suppose that y be an arbitrary element of range of $f(x)$, there exists $x \in R-\left\{-\frac{3}{4}\right\}$ such that $y=f(x)$

It implies that

$$\begin{aligned} y &= \frac{4x}{3x+4} \\ 3xy+4y &= 4x \\ x(3y-4) &= -4y \\ x &= \frac{4y}{4-3y} \end{aligned}$$

The composite function

$$gof(x)=g(f(x))=g\left(\frac{4x}{3x+4}\right)=\frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)}=x$$

and $fog(y) = f(g(y)) = \dfrac{16}{12y + 16 - 12x} = y$

Therefore, $g \circ f(x) = x, f \circ g(y) = y$

Hence the functions $f(x), g(x)$ are invertible and $f^{-1}(x) = g(x)$

The inverse of $f(x)$ is the map $g : \text{Range of } f \to R - \left\{-\dfrac{4}{3}\right\}$, which is given by

$$g(y) = \frac{4y}{4 - 3y}$$

This is matching with the option (B)