Chapter: Vector Algebra.

Miscellaneous Exercise.

1. Write down a unit vector in XY-plane, making an angle of 30^0 with the positive direction of x-axis

Solution:

Unit vector is $\vec{r}=\cos\theta\hat{i}+\sin\theta\hat{j}$, where θ is angle with positive X axis

$$\vec{r}=\cos 30^0\hat{i}+\sin 30^0\hat{j}=\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}$$

2. Find the scalar components and magnitude of the vector joining the points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$

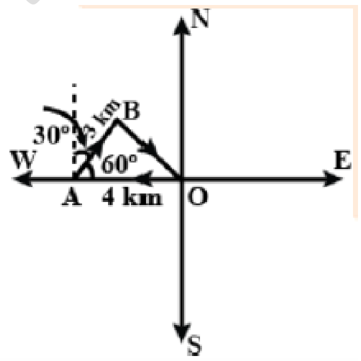
Solution:

$$\overrightarrow{PQ}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$

$$\left|\overrightarrow{PQ}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

3. A girl walks 4 km towards west, then she walks 3 km in a direction 30^0 east of north and stops. Determine the gril's displacement from her initial point of departure

Solution:



$$\overrightarrow{OA}=-4\hat{i}$$

$$\overrightarrow{AB}=\hat{i}\left|\overrightarrow{AB}\right|\cos 60^0+\hat{j}\left|\overrightarrow{AB}\right|\sin 60^0$$

$$=\hat{i}3\times\frac{1}{2}+\hat{j}3\times\frac{\sqrt{3}}{2}$$

$$=\frac{3}{2}\hat{i}+\frac{3\sqrt{3}}{2}\hat{j}$$

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$$

$$=\left(-4\hat{i}\right)+\left(\frac{3}{2}\hat{i}+\frac{3\sqrt{3}}{2}\hat{j}\right)$$

$$=\left(-4+\frac{3}{2}\right)\hat{i}+\frac{3\sqrt{3}}{2}\hat{j}$$

$$=\left(\frac{-8+3}{9}\right)\hat{i}+\frac{3\sqrt{3}}{9}\hat{j}$$

$$=\frac{-5}{2}\hat{i}+\frac{3\sqrt{3}}{9}\hat{j}$$

4. If $\vec{a}=\vec{b}+\vec{c}$, then is it true that $|\vec{a}|=|\vec{b}|+|\vec{c}|$? Justify your answer.

Solution:

In ΔABC, $\overrightarrow{CB}=\vec{a},\overrightarrow{CA}=\vec{b},\overrightarrow{AB}=\vec{c}$

$\vec{a}=\vec{b}+\vec{c}$, by triangle law of addition for vectors.

$|\vec{a}|<|\vec{b}|+|\vec{c}|$, by triangle inequality law of lengths

Hence, it's not true that $|\vec{a}|=|\vec{b}|+|\vec{c}|$

5. Find the value of x for which $x\left(\hat{i}+\hat{j}+\hat{k}\right)$ unit vectro

Solution:

$$\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$$

$$\Rightarrow \sqrt{x^2+x^2+x^2}=1$$

$$\Rightarrow \sqrt{3x^2}=1$$

$$\Rightarrow \sqrt{3}x=1$$

$$x=\pm\frac{1}{\sqrt{3}}$$

6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

Solution:

$$\vec{c}=\vec{a}+\vec{b}+\vec{b}=(2+1)\hat{i}+(3-2)\hat{j}+(-1+1)\hat{k}=3\hat{i}+\hat{j}$$

$$|\vec{c}|=\sqrt{3^2+1^2}=\sqrt{9+1}=\sqrt{10}$$

$$\therefore \vec{c}=\frac{\vec{c}}{|\vec{c}|}=\frac{\left(3\hat{i}+\hat{j}\right)}{\sqrt{10}}$$

So, vector of magnitude 5 and parallel to the resultant of $\vec{a}$ and $\vec{b}$ is

$$\pm 5(\hat{c})=\pm 5\left(\frac{1}{\sqrt{10}}\left(3\hat{i}+\hat{j}\right)\right)\pm\frac{3\sqrt{10}}{2}\hat{i}\pm\frac{\sqrt{10}}{2}\hat{j}$$

7. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$

Solution:

$$2\vec{a}-\vec{b}+3\vec{c}=2\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(2\hat{i}-\hat{j}+3\hat{k}\right)+3\left(\hat{i}-2\hat{j}+\hat{k}\right)$$

$$=2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}$$

$$=3\hat{i}-3\hat{j}+2\hat{k}$$

$$\left|2\vec{a}-\vec{b}+3\vec{c}\right|=\sqrt{3^2+(-3)^2+2^2}=\sqrt{9+9+4}=\sqrt{22}$$

Thus, required unit vector is $\dfrac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|}=\dfrac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}}=\dfrac{3}{\sqrt{22}}\hat{i}-\dfrac{3}{\sqrt{22}}\hat{j}+\dfrac{2}{\sqrt{22}}\hat{k}$

8. Show that the points $A(1,-2,-8), B(5,0,-2)$ and $C(11,3,7)$ are collinear, and find the ratio in which B divides AC

Solution:

$$\overrightarrow{AB}=(5-1)\hat{i}+(0+2)\hat{j}+(-2+8)\hat{k}=4\hat{i}+2\hat{j}+6\hat{k}$$

$$\overrightarrow{BC}=(11-5)\hat{i}+(3-0)\hat{j}+(7+2)\hat{k}=6\hat{i}+3\hat{j}+9\hat{k}$$

$$\overrightarrow{AC}=(11-1)\hat{i}+(3+2)\hat{j}+(7+8)\hat{k}=10\hat{i}+5\hat{j}+15\hat{k}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{4^2+2^2+6^2}=\sqrt{16+4+36}=\sqrt{56}=2\sqrt{14}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{6^2+3^2+9^2}=\sqrt{36+9+81}=\sqrt{126}=3\sqrt{14}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{10^2+5^2+15^2}=\sqrt{100+25+225}=\sqrt{350}=5\sqrt{14}$$

$$\therefore \left|\overrightarrow{AC}\right|=\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|$$

So, this points are collinear

Let B divides AC in the ratio $\lambda:1$. $\overrightarrow{OB}=\dfrac{\lambda\overrightarrow{OC}+\overrightarrow{OA}}{(\lambda+1)}$

$$\Rightarrow 5\hat{i}-2\hat{k}=\frac{\lambda\left(11\hat{i}+3\hat{j}+7\hat{k}\right)+\left(\hat{i}-2\hat{j}-8\hat{k}\right)}{\lambda+1}$$

$$\Rightarrow (\lambda+1)\left(5\hat{i}-2\hat{k}\right)=11\lambda\hat{i}+3\lambda\hat{j}+7\lambda\hat{k}+\hat{i}-2\hat{j}-8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i}-2(\lambda+1)\hat{k}=(11\lambda+1)\hat{i}+(3\lambda-2)\hat{j}+(7\lambda-8)\hat{k}$$

$\Rightarrow \lambda = \frac{2}{3}$

So, the required ratio is 2:3

9. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\left(2\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-3\vec{b}\right)$ externally in the ratio 1:2 Also, show that P is the mid point of the line segment RQ

Solution:

$\overrightarrow{OP} = 2\vec{a}+\vec{b}, \overrightarrow{OQ} = \vec{a}-3\vec{b}$

$$\overrightarrow{OR} = \frac{2\left(2\vec{a}+\vec{b}\right)-\left(\vec{a}-3\vec{b}\right)}{2-1} = \frac{4\vec{a}+2\vec{b}-\vec{a}-3\vec{b}}{1} = 3\vec{a}+5\vec{b}$$

So, the position vector of R is $3\vec{a}+5\vec{b}$

Position vector of midpoint of $RQ = \frac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$= \frac{\left(a\sqrt{6}\right)+\left(3\vec{a}+5\vec{b}\right)}{2}$$

$= 2\vec{a}+\vec{b}$

$= \overrightarrow{OP}$

Thus, P is midpoint of line segment RQ

10. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution:

Diagonal of a parallelogram is $\vec{a}+\vec{b}$

$$\vec{a}+\vec{b}=(2+1)\hat{i}+(-4-2)\hat{j}+(5-3)\hat{k}=3\hat{i}-6\hat{j}+2\hat{k}$$

So, the unit vector parallel to diagonal is

$$\frac{\vec{a}+\vec{b}}{\left|\vec{a}+\vec{b}\right|}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{3^2+(-6)^2+2^2}}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4}}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{7}=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -4 & 3\\ 1 & -2 & -3\end{vmatrix}$$

$$=\hat{i}(12+10)-\hat{j}(-6-5)+\hat{k}(-4+4)$$

$$=22\hat{i}+11\hat{j}$$

$$=11\left(\hat{i}+\hat{j}\right)$$

$$\therefore\left|\vec{a}\times\vec{b}\right|=11\sqrt{2^2+1^2}=11\sqrt{5}$$

So, area of parallelogram is $11\sqrt{5}$ sq units

11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$.

Solution:

Let a vector be equally inclined to OX, OY and OZ at an angle α

So, the direction cosines of the vector are $\cos\alpha,\cos\alpha$ and $\cos\alpha$

$$\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$$

$$\Rightarrow 3\cos^2\alpha=1$$

$$\Rightarrow \cos\alpha=\frac{1}{\sqrt{3}}$$

So, the DCs of the vector are $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$

12. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d}=15$

Solution:

$$\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$$

$$\vec{d}.\vec{a}=0 \Rightarrow d_1+4d_2+2d_3=0$$

$$\vec{d}.\vec{b}=0 \Rightarrow 3d_1-2d_2+7d_3=0$$

$$\vec{c}.\vec{d}=15 \Rightarrow 2d_1-d_2+4d_3=15$$

Solving these equations, we get $d_1=\frac{160}{3}, d_2=-\frac{5}{3}, d_3=-\frac{70}{3}$

$$\therefore \vec{d}=\frac{160}{3}\hat{i}+\frac{5}{3}\hat{j}+\frac{70}{3}\hat{k}=\frac{1}{3}\left(160\hat{i}+5\hat{j}+70\hat{k}\right)$$

13. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ

Solution:

$$\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

So, unit vector along $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$ is $\left(\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\right)$

$$\left(\hat{i}+\hat{j}+\hat{k}\right).\left(\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\right)=1$$

$$\Rightarrow \frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\Rightarrow \sqrt{\lambda^2+4\lambda+44}=\lambda+6$$

$$\Rightarrow \lambda^2+4\lambda+44=(\lambda+6)^2$$

$$\Rightarrow \lambda^2+4\lambda+44=\lambda^2+12\lambda+36$$

$$\Rightarrow 8\lambda=8$$

$$\Rightarrow \lambda=1$$

14. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined $\vec{a},\vec{b}$ and $\vec{c}$

Solution:

$$\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$$

$$|\vec{a}|=|\vec{b}|=|\vec{c}|$$

Let $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b},\vec{c}$ at angles $\theta_1,\theta_2,\theta_3$ respectively.

$$\cos\theta_1=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_2=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{\vec{a}.\vec{b}+\vec{b}.\vec{b}+\vec{c}.\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_3=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{\vec{a}.\vec{c}+\vec{b}.\vec{c}+\vec{c}.\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Since, $|\vec{a}|=|\vec{b}|=|\vec{c}|\Rightarrow \cos\theta_1=\cos\theta_2=\cos\theta_3$

So, $\theta_1=\theta_2=\theta_3$

15. Prove that, $\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)=|\vec{a}|^2+\left|\vec{b}\right|^2$ if and only if $\vec{a},\vec{b}$ are perpendicular, givne $\vec{a}\neq\vec{0},\vec{b}\neq\vec{0}$

Solution:

$$\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)=|\vec{a}|^2+\left|\vec{b}\right|^2$$

$$\Rightarrow \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=|\vec{a}|^2+\left|\vec{b}\right|^2$$

$$\Rightarrow |\vec{a}|^2+2\vec{a}.\vec{b}+\left|\vec{b}\right|^2=|\vec{a}|^2+\left|\vec{b}\right|^2$$

$$\Rightarrow 2\vec{a}.\vec{b}=0$$

$$\Rightarrow \vec{a}.\vec{b}=0$$

So $\vec{a}$ and $\vec{b}$ are perpendicular

16. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\vec{b}\geq 0$ only when

A) $0<\theta<\frac{\pi}{2}$ B) $0\leq\theta\leq\frac{\pi}{2}$ C) $0<\theta<\pi$ D) $0\leq\theta\leq\pi$

Solution:

$$\therefore \vec{a}\vec{b}\geq 0$$

$$\Rightarrow |\vec{a}|\left|\vec{b}\right|\cos\theta\geq 0$$

$$\Rightarrow \cos\theta\geq 0 \quad \because \left[|\vec{a}|\geq 0 \text{ and } \left|\vec{b}\right|\geq 0\right]$$

$$0\leq\theta\leq\frac{\pi}{2}$$

$$\vec{a}.\vec{b}\geq 0 \text{ if } 0\leq\theta\leq\frac{\pi}{2}$$

So the right answer is B

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

A) $\theta = \frac{\pi}{4}$ B) $\theta = \frac{\pi}{3}$ C) $\theta = \frac{\pi}{2}$ D) $\theta = \frac{2\pi}{3}$

Solution:

$$|\vec{a}| = |\vec{b}| = 1$$

$$|\vec{a}+\vec{b}| = 1$$

$$\Rightarrow (\vec{a}+\vec{b})(\vec{a}+\vec{b}) = 1$$

$$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = 1$$

$$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1$$

$$\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$$

$$\Rightarrow 1^2 + 21.1\cos\theta + 1 = 1$$

$$\Rightarrow \cos\theta = -\frac{1}{2}$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

So, $\vec{a}+\vec{b}$ is unit vector if $\theta = \frac{2\pi}{3}$

The correct answer is D

18. The value of $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}.(\hat{i}\times\hat{j})$ is

A) 0 B) -1 C) 1 D) 3

Solution:

$$\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}.(\hat{i}\times\hat{j})$$

$=\hat{i}.\hat{\text{i}}+\hat{\text{j}}.\left(-\hat{j}\right)+\hat{k}.\hat{k}$

$=1-1+1$

$=1$

The correct answer is C

19. If θ is the angel between any two vectors $\vec{a}$ and $\vec{b}$, then $\left|\vec{a}\,\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$ when θ is equal to

A) 0 B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) n

Solution:

$\left|\vec{a}\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$

$\Rightarrow\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta$

$\Rightarrow\cos\theta=\sin\theta$

$\Rightarrow\tan\theta=1$

$\Rightarrow\theta=\frac{\pi}{4}$

The correct answer is B

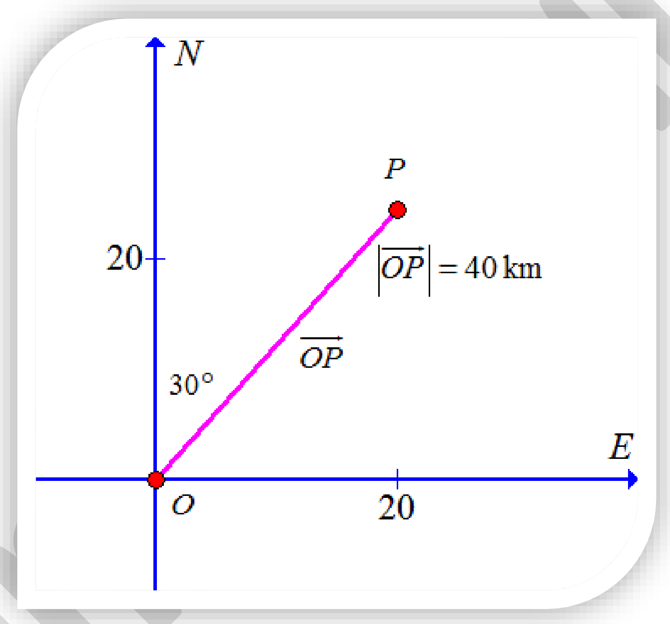
Exercise: 10.1

1. Represent graphically a displacement of $40\,\text{km}, 30^\circ$ east of north

Solution:

Given that there is a vector with magnitude 40 kilometers and it makes an angle of 30° with $y-$ axis and it makes an angle of 60° with $x-$ axis

The graph of $\overrightarrow{OP}$ is as below.



Here, the vector $\overrightarrow{OP}$ represents the displacement of $40\,\text{km}, 30^\circ$ in East of North direction.

2. Classify the following measures as scalars and vectors.

i) $10\,\text{kg}$

ii) 2 meters north-west

iii) 40°

iv) 40 watt

v) 10^{-19} coulomb

vi) $20\,\text{m/s}^2$

Solution:

i) The moss $10\,\text{kg}$ is a scalar quantity because it has only magnitude not direction

ii) 2 meters north-west is a vector, because it has magnitude and direction.

iii) $40°$ is a scalar quantity because it has only magnitude not direction.

iv) 40 watts is a scalar quantity because it has only magnitude not direction.

v) 10^{-19} Coulomb is a scalar quantity because it has only magnitude not direction.

vi) 20 m/s^2 is a vector quantity because it has both magnitude as well as direction.

3. Classify the following as scalar and vector quantities

i) Time period

ii) Distance

iii) Force

iv) Velocity

v) work done

Solution:

i) Time period is a scalar quantity because it has only magnitude.

ii) Distance is a scalar quantity because it has only magnitude.

iii) Force is a vector quantity because it has both magnitude as well as direction

iv) Velocity is a vector quantity because it has both magnitude as well as direction

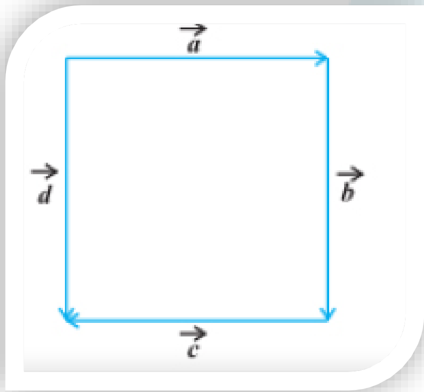
v) Work done is a scalar quantity because it has only magnitude.

4. In the following figure, identity the following

i) Co-initial

ii) Equal

iii) Collinear but not equal



Solution:

Observing the above figure, the vectors based on the opposite sides are equal

It implies that vectors $\vec{a}$ and $\vec{d}$ are co-initial, because both have same initial point. Vectors $\vec{b}$ and $\vec{d}$ are equal and vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal

5. Answer the following as true of false

i) $\vec{a}$ and $-\vec{a}$ are collinear.

ii) Two collinear vectors are always equal in magnitude.

iii) Two vectors having same magnitude are collinear.

iv) Two collinear vectors having the same magnitude are equal.

Solution:

i) Two vectors which are opposite to each other are collinear, hence the statement "$\vec{a}$ and $-\vec{a}$ are collinear" is TRUE.

ii) Collinear vectors are vectors which have same or parallel base lines, so there is no constraint on magnitude. Hence the statement " Two collinear vectors are always equal in magnitude" is FALSE

iii) Collinear vectors are vectors which have same or parallel base lines, so there is no constraint on magnitude. Hence the statement " Two vectors having same magnitude are collinear" is FALSE

iv) Two collinear vectors are may be in the same direction or in the opposite direction, hence the statement "Two collinear vectors having the same magnitude are equal" is FALSE

Exercise 10.2

1. Compute the magnitude of vectors: $\vec{a} = \hat{i} + \hat{j} + \hat{k}; \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$ and

$\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$

Solution: The magnitude of a vector $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$ is $|\vec{p}| = \sqrt{x^2 + y^2 + z^2}$

Hence, the magnitude of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ is

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2}$$
$$= \sqrt{3}$$

The magnitude of the vector $\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$ is

$$|\vec{b}| = \sqrt{(2)^2 + (-7)^2 + (-3)^2}$$
$$= \sqrt{4 + 49 + 9}$$
$$= \sqrt{62}$$

The magnitude of the vector $\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$ is

$$|\vec{c}| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2}$$
$$= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}$$
$$= 1$$

2. Write two different vectors having same magnitude:

Solution: Consider two vectors $\vec{a} = \left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$ and $\vec{b} = \left(2\hat{i} - \hat{j} + 3\hat{k}\right)$

The magnitude of a vector $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$ is $|\vec{p}| = \sqrt{x^2 + y^2 + z^2}$

Hence, the magnitudes of the above vectors are

$$\begin{aligned}|\vec{a}| &= \sqrt{1^2+(-2)^2+3^2}\\ &= \sqrt{1+4+9}\\ &= \sqrt{14}\end{aligned}$$

and

$$\begin{aligned}|\vec{b}| &= \sqrt{2^2+1^2+(-3)^2}\\ &= \sqrt{4+1+9}\\ &= \sqrt{14}\end{aligned}$$

Hence, the magnitudes are not equal.

Answer may vary. Student can give any example.

3. Write two different vectors having same direction

Solution: Suppose that two vectors are $\vec{p}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)$ and $\vec{q}=\left(2\hat{i}-4\hat{j}+6\hat{k}\right)$

The directions of two vectors are same if and only if their direction ratios are in proportion.

It means the components of vectors must be in proportion.

Here the components of the vectors are $\langle 1,-2,3\rangle,\langle 2,-4,6\rangle$, and these triads are direction ratios of the base lines of above vectors

Since the direction ratios are proportional to each other, so that the vectors $\vec{p}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)$ and $\vec{q}=\left(2\hat{i}-4\hat{j}+6\hat{k}\right)$ are in the same direction.

Observing the vectors $\vec{q}=2\vec{p}$

Hence, the above two vectors are examples which are not equal but in the same direction.

Answer may vary, student can give any other example too.

4. Find the value of x and y so that the vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ are equal

Solution: Two vectors are equal if each component of one vector must be equal to the corresponding component of the second vector.

It gives, $2\hat{i}+3\hat{j}=x\hat{i}+y\hat{j} \Rightarrow x=2, y=3$

5. Find the scalar and vector components of the vector with initial points $(2,1)$ and terminal point $(-5,7)$

Solution: Suppose that the points are $A(2,1)$ and $B(-5,7)$

Hence,

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB}-\overrightarrow{OA} \\ &= (-5-2)\hat{i}+(7-1)\hat{j} \\ &= -7i+6j \end{aligned}$$

The scalar and vector components of the vector with initial points $(2,1)$ and terminal point $(-5,7)$ are $\langle -7,6\rangle$ and $-7\hat{i},6\hat{j}$

6. Find the sum of the vectors $\vec{a}=\hat{i}-2\hat{j}+\hat{k}, \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\overline{c}=\hat{i}-6\hat{j}-7\hat{k}$.

Solution: The sum of two vectors is defined as another vector whose components are equal to the sum of the corresponding components of the vectors

The given vectors are

$$\begin{aligned} \vec{a} &= \hat{i}-2\hat{j}+\hat{k}, \\ \vec{b} &= -2\hat{i}+4\hat{j}+5\hat{k} \\ \overline{c} &= \hat{i}-6\hat{j}-7\hat{k} \end{aligned}$$

The sum of the above vectors is defined as

$$\vec{a}+\vec{b}+\vec{c}=\left(\hat{i}-2\hat{j}+\hat{k}\right)+\left(-2\hat{i}+4\hat{j}+5\hat{k}\right)+\left(\hat{i}-6\hat{j}-7\hat{k}\right)$$
$$=(1-2+1)\hat{i}+(-2+4-6)\hat{j}+(1+5-7)\hat{k}$$
$$=0\hat{i}-4\hat{j}-1\hat{k}$$
$$=-4\hat{j}-\hat{k}$$

Therefore, the required vector is $-4\hat{j}-\hat{k}$

7. Find the unit vector in the direction of the vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$

Solution: The unit vector in the direction of the vector $\vec{p}=x\hat{i}+y\hat{j}+z\hat{k}$ is $\frac{\vec{p}}{|\vec{p}|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

The magnitude of the given vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ is

$$|\vec{a}|=\sqrt{1^2+1^2+2^2}$$
$$=\sqrt{1+1+4}$$
$$=\sqrt{6}$$

The unit vector in the direction of $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ is

$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}$$
$$=\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}$$
$$=\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$$

8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$ where P and Q are two points $(1,2,3)$ and $(4,5,6)$ respectively

Solution: Given that $\overrightarrow{OP}=\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{OQ}=4\hat{i}+5\hat{j}+6\hat{k}$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$$
$$=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$
$$=3\hat{i}+3\hat{j}+3\hat{k}$$
$$=3\left(\hat{i}+\hat{j}+\hat{k}\right)$$

The unit vector in the direction of the vector $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$ is $\frac{\vec{p}}{|\vec{p}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

The magnitude of the vector is

$$\begin{aligned}|\overrightarrow{PQ}| &= \sqrt{3^2 + 3^2 + 3^2}\\ &= 3\sqrt{3}\end{aligned}$$

Therefore, the unit vector is

$$\begin{aligned}\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} &= \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}}\\ &= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\end{aligned}$$

9. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$

Solution: The given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

The sum of two vectors is

$$\begin{aligned}\vec{a} + \vec{b} &= (2-1)\hat{i} + (-1+1)\hat{j} - (2-1)\hat{k}\\ &= 1\hat{i} + 0\hat{j} + 1\hat{k}\\ &= \hat{i} + \hat{k}\end{aligned}$$

$$|\vec{a} + \vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

The unit vector in the direction of the vector $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$ is $\frac{\vec{p}}{|\vec{p}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

Therefore, the unit vector is

$$\begin{aligned}\frac{(\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|} &= \frac{\hat{i} + \hat{k}}{\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}\end{aligned}$$

10. Find a vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units

Solution: The given vector is $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$

Magnitude of the above vector is

$$\begin{aligned}|\vec{a}| &= \sqrt{5^2+(-1)^2+2^2}\\ &= \sqrt{25+1+4}\\ &= \sqrt{30}\end{aligned}$$

The vector of magnitude k units in the direction of vector $\vec{p}=x\hat{i}+y\hat{j}+z\hat{k}$ is

$$\frac{k\vec{p}}{|\vec{p}|}=k\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right)$$

Therefore, the vector with magnitude 8 units in the direction of $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$ is $8\hat{a}$

$$\begin{aligned}8\hat{a} &= \frac{8\vec{a}}{|\vec{a}|}\\ &= \frac{8\left(5\hat{i}-\hat{j}+2\hat{k}\right)}{\sqrt{30}}\\ &= \frac{40}{\sqrt{30}}\hat{i}-\frac{8}{\sqrt{30}}\hat{j}+\frac{16}{\sqrt{30}}\hat{k}\end{aligned}$$

11. Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear

Solution: Suppose that the vectors are

$$\begin{aligned}\vec{a} &= 2\hat{i}-3\hat{j}+4\hat{k},\\ \vec{b} &= -4\hat{i}+6\hat{j}-8\hat{k}\end{aligned}$$

If two vectors $\vec{p},\vec{q}$ are collinear then $\vec{p}=k\vec{q}$ where k is constant.

Consider

$$\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}=-2\left(2\hat{i}-3\hat{j}+4\hat{k}\right)=-2\vec{a}$$

Therefore, the vectors $\vec{a},\vec{b}$ are collinear

12. Find the direction cosines of the vector $\hat{i}+2\hat{j}+3\hat{k}$

Solution:

Suppose that $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

The magnitude of the vector $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ is

$$\begin{aligned}|\vec{a}| &= \sqrt{1^2+2^2+3^2}\\ &= \sqrt{1+4+9}\\ &= \sqrt{14}\end{aligned}$$

So, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$

13. Find the direction cosines of the vector joining the point $A(1,2,-3)$ and $B(-1,-2,1)$ directed from A to B

Solution: The position vectors of the given points are

$$\overrightarrow{OA}=\hat{i}+2\hat{j}-3\hat{k},\ \overrightarrow{OB}=-\hat{i}-2\hat{j}+\hat{k}$$

Hence, the vector

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB}-\overrightarrow{OA}\\ &= -\hat{i}-2\hat{j}+\hat{k}-\left(\hat{i}+2\hat{j}-3\hat{k}\right)\\ &= -2\hat{i}-4\hat{j}+4\hat{k}\end{aligned}$$

The magnitude of the vector $\overrightarrow{AB}$ is

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= \sqrt{(-2)^2+(-4)^2+4^2}\\ &= \sqrt{4+16+16}\\ &= 6\end{aligned}$$

Therefore, the direction cosines of the vector $\overrightarrow{AB}$ are $\left\langle -\frac{2}{6},-\frac{4}{6},\frac{4}{6}\right\rangle=\left\langle -\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right\rangle$

14. Show that the vectors $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axes OX, OY and OZ

Solution: The given vector is $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and its magnitude is $|\vec{a}|=\sqrt{3}$

Hence, the direction cosines of the above vector are $\left\langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}} \right\rangle$

Suppose that the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ makes angles α,β,γ with coordinate axes then

$$\cos\alpha=\frac{1}{\sqrt{3}},\cos\beta=\frac{1}{\sqrt{3}},\cos\gamma=\frac{1}{\sqrt{3}}$$

It implies that all the angles are equal

Therefore, the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the coordinate axes

15. Find the position vector of a point R which divides the line joining two point P and Q whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively in the ratio $2:1$

i) Internally

ii) Externally

Solutions: The position vectors are $\overrightarrow{OP}=\hat{i}+2\hat{j}-\hat{k}$ and $\overrightarrow{OQ}=-\hat{i}+\hat{j}+\hat{k}$

i) Suppose that a point A divides the line segment joining P,Q in the ratio $2:1$ internally

$$\overrightarrow{OA}=\frac{m\vec{b}+n\vec{a}}{m+n}$$
$$=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1}$$
$$=\frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-\hat{k}\right)}{3}$$
$$=\frac{-i+4j+k}{3}$$

ii) Suppose that the point B divides the line segment joining the points P and Q in the ratio $2:1$ externally.

The position vector of B is

$$\overrightarrow{OB} = \frac{m\vec{b} - n\vec{a}}{m-n}$$
$$= \frac{2\left(-\hat{i} + \hat{j} + \hat{k}\right) - 1\left(\hat{i} + 2\hat{j} - \hat{k}\right)}{2-1}$$
$$= \frac{\left(-2\hat{i} + 2\hat{j} + 2\hat{k}\right) - \left(\hat{i} + 2\hat{j} - \hat{k}\right)}{1}$$
$$= \frac{-3i + 3k}{1}$$
$$= -3(i - k)$$

16. Find the position vector of midpoint of vector joining the points $P(2,3,4)$ and $Q(4,1,-2)$

Solution: Given that the position vectors are $\overrightarrow{OP} = 2i + 3j + 4k$ and $\overrightarrow{OQ} = 4i + j - 2k$

Suppose that the point R is midpoint of PQ then the position vector of R is

$$\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}$$
$$= \frac{\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) + \left(4\hat{i} + \hat{j} - 2\hat{k}\right)}{2}$$
$$= \frac{6i + 4j + 2k}{2}$$
$$= 3i + 2j + k$$

Exercise 10.3

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$

Solution: Given that $|\vec{a}|=\sqrt{3},|\vec{b}|=2,\vec{a}\cdot\vec{b}=\sqrt{6}$

Hence, $\sqrt{6}=\sqrt{3}\times 2\times\cos\theta$

It implies that

$$\cos\theta=\frac{\sqrt{6}}{\sqrt{3}\times 2}$$
$$=\frac{1}{\sqrt{2}}$$
$$\theta=45^\circ$$

2. Find the angle between the vectors $i-2\hat{j}+3\hat{k}$ and $3i-2\hat{j}+\hat{k}$

Solution:

$$|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$|\vec{b}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{9+4+1}=\sqrt{14}$$

$$\vec{a}.\vec{b}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)\left(3\hat{i}-2\hat{j}+\hat{k}\right)$$

$$=1.3+(-2)(-2)+3.1$$

$$=3+4+3$$

$$=10$$

$$\therefore 10=\sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta=\frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

3. Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$

Ans:

$\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Projection of $\vec{a}$ and $\vec{b}$ is $\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{1+1}}\{1.1+(-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$

4. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$

Ans:

$\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ *and* $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{7^2 + (-1)^2 + 8^2}}\{1(7) + 3(-1) + 7(8)\} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

5. Show that each of the given three vectors is a unit vector, which are mutually perpendicular to each other $\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}), \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$

Ans:

$$\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

$$\vec{c}=\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)=\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}$$

$$|\vec{a}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$

$$|\vec{b}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}}=1$$

$$|\vec{c}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=1$$

So, each of the vectors is a unit vector

$$\vec{a}.\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(\frac{-6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$

$$\vec{b}.\vec{c}=\frac{3}{7}\times\frac{6}{7}+\left(\frac{-6}{7}\right)\times\frac{2}{7}+\frac{2}{7}\times\left(\frac{-3}{7}\right)=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$

$$\vec{c}.\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\left(\frac{-3}{7}\right)\times\frac{6}{7}=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$$

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)=8$ and $|\vec{a}|=8|\vec{b}|$

Ans:

$$\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)=8$$

$$\Rightarrow \vec{a}\vec{a}-\vec{a}.\vec{b}+\vec{b}.\vec{a}-\vec{b}\vec{b}=8$$

$$\Rightarrow |\vec{a}|^2-|\vec{b}|^2=8$$

$$\Rightarrow \left(8|\vec{b}|\right)^2-|\vec{b}|^2=8$$

$$\Rightarrow 64|\vec{b}|^2-|\vec{b}|^2=8$$

$$\Rightarrow 63|\vec{b}|^2 = 8$$

$$\Rightarrow |\vec{b}|^2 = \frac{8}{63}$$

$$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}}$$

$$\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}| = 8|\vec{b}| = \frac{8\times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$$

7. Evaluate the product $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$

Ans:

$$(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$$

$$=3\vec{a}.2\vec{a}+3\vec{a}.7\vec{b}-5\vec{b}.2\vec{a}-5\vec{b}.7\vec{b}$$

$$=6\vec{a}\vec{a}+21\vec{a}\vec{b}-10\vec{a}\vec{b}-35\vec{b}\vec{b}$$

$$=6|\vec{a}|^2+11\vec{a}\vec{b}-35|\vec{b}|^2$$

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60^0 and their scalar product is $\frac{1}{2}$

Ans:

Let θ be angle between $\vec{a}$ and $\vec{b}$

$$|\vec{a}| = |\vec{b}|, \vec{a}.\vec{b} = \frac{1}{2}, \text{ and } \theta = 60^0$$

$\therefore \frac{1}{2} = |\vec{a}||\vec{b}|\cos 60^0$

$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$

$\Rightarrow |\vec{a}|^2 = 1$

$\Rightarrow |\vec{a}| = |\vec{b}| = 1$

9. Find $|\vec{x}|$, if for a unit vector $\vec{a}, (\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 12$

Ans:

$(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 12$

$\Rightarrow \vec{x}.\vec{x} + \vec{x}\vec{a} - \vec{a}\vec{x} - \vec{a}.\vec{a} = 12$

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$

$\Rightarrow |\vec{x}|^2 - 1 = 12$

$\Rightarrow |\vec{x}|^2 = 13$

$\therefore |\vec{x}| = \sqrt{13}$

10. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the values of λ

Ans:

$\vec{a} + \lambda\vec{b} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-\hat{i} + 2\hat{j} + \hat{k}\right) = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$

$\left(\vec{a} + \lambda\vec{b}\right).\vec{c} = 0$

$\Rightarrow \left[(2+-)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}\right].\left(3\hat{i} + \hat{j}\right) = 0$

$$\Rightarrow 3(2-\lambda)+(2+2\lambda)+0(3+\lambda)=0$$

$$\Rightarrow 6-3\lambda+2+2\lambda=0$$

$$\Rightarrow -\lambda+8=0$$

$$\Rightarrow \lambda=8$$

11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$

Ans:

$$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$$

$$|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2=0$$

12. If $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$, then what can be concluded above the vector $\vec{b}$?

Ans:

$$\vec{a}.\vec{a}=0 \Rightarrow |\vec{a}|^2=0 \Rightarrow |\vec{a}|=0$$

$\therefore \vec{a}$ is the zero vector. Thus, any vector $\vec{b}$ can satisfy $\vec{a}.\vec{b}=0$

13. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

Ans:

$$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\left(\vec{a}\vec{b}+\vec{b}\vec{c}+\vec{c}\vec{a}\right)$$

$$\Rightarrow 0=1+1+1+2\left(\vec{a}\vec{b}+\vec{b}\vec{c}+\vec{c}\vec{a}\right)$$

$$\Rightarrow \left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\frac{-3}{2}$$

14. If either vectors $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}.\vec{b}=\vec{0}$. But the converse need not be true. Justify your answer with an example

Ans:

$\vec{a} = 2\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 3\hat{j} - 6\hat{k}$

$\vec{a}.\vec{b} = 2.3 + 4.3 + 3(-6) = 6 + 12 - 18 = 0$

$|a| = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$

$\therefore \vec{a} \neq \vec{0}$

$|\vec{b}| = \sqrt{3^2 + 3^2 + (-6)^2} = 54$

$\therefore \vec{b} \neq 0$

So, it is clear from above example that the converse of the given statement need not be True.

15. If the vertices A,B,C of a triangle ABC are $(1,2,3),(-1,0,0),(0,1,2)$, respectively, then find $\angle ABC$. [$\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Ans:

$\overrightarrow{BA} = \{1-(-1)\hat{i} + (2-1)\hat{j} + (3-0)\hat{k}\} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{BC} = \{0-(-1)\hat{i} + (1-0)\hat{j} + (2-0)\hat{k}\} = \hat{i} + \hat{j} + 2\hat{k}$

$\overrightarrow{BA}.\overrightarrow{BC} = (2\hat{i} + 2\hat{j} + 3\hat{k})(\hat{i} + \hat{j} + 2\hat{k}) = 2\times1 + 2\times1 + 3\times2 = 2 + 2 + 6 = 10$

$|\overrightarrow{BA}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$

$|\overrightarrow{BC}| = \sqrt{1+1+2^2} = \sqrt{6}$

$\overrightarrow{BA}.\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos(\angle ABC)$

$\therefore 10 = \sqrt{17} \times \sqrt{6}\cos(\angle ABC)$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17}\times\sqrt{6}}$$

$$\Rightarrow (\angle ABC) = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

16. Show that the points $A(1,2,7), B(2,6,3)$ and $C(3,10,-1)$ are collinear.

Ans:

$$\overrightarrow{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = 2\sqrt{33}$$

$$\left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Hence, the given points are collinear

17. Show that the vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ from the vertices of a right angled triangle

Ans:

$$\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k}, \overrightarrow{OB} = \hat{i} - 3\hat{j} - 5\hat{k}, \overrightarrow{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\overrightarrow{CA}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$$

$$\therefore \left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{AC}\right|^2=6+35=41=\left|\overrightarrow{AB}\right|^2$$

Hence, ΔABC is a right triangle.

18. If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar. Then $\lambda\vec{a}$ is unit vector if

A) $\lambda=1$ B) $\lambda=-1$ C) $a=|\lambda|$ D) $a=\frac{1}{|\lambda|}$

Ans:

$$|\lambda\vec{a}|=1$$

$$\Rightarrow |\lambda||\vec{a}|=1$$

$$\Rightarrow |\vec{a}|=\frac{1}{|\lambda|}$$

$$\Rightarrow a=\frac{1}{|\lambda|}$$

Exercise 10.4

1. Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Solution:

We have, $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)=19\hat{j}+19\hat{k}$$

$$\therefore\left|\vec{a}\times\vec{b}\right|=\sqrt{(19)^2+(19)^2}=\sqrt{2\times(19)^2}=19\sqrt{2}$$

2. Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

Solution:

$$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k} \text{ and } \vec{b}=\hat{i}+2\hat{j}-2\hat{k}$$

$$\vec{a}+\vec{b}=4\hat{i}+4\hat{j},\vec{a}-\vec{b}=2\hat{i}+4\hat{k}$$

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\sqrt{16^2+(-16)^2+(-8)^2}$$

$$=\sqrt{2^2\times8^2+2^2\times8^2+8^2}$$

$$=8\sqrt{2^2+2^2+1}=8\sqrt{9}=8\times3=24$$

So, the unit vector is $= \pm \frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|} = \pm \frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$

$$= \pm \frac{2\hat{i}-2\hat{j}-\hat{k}}{3} = \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k}$$

3. If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the component of $\vec{a}$

Solution:

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ Put

$$|\vec{a}| = 1.\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{2} = a_1$$

$$\cos\frac{\pi}{4} = \frac{a_2}{|a_1|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = a_2$$

$$\cos\theta = \frac{a_3}{|\vec{a}|}$$

$$\Rightarrow a_3 = \cos\theta$$

$$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\Rightarrow \frac{3}{4}+\cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1-\frac{3}{4}=\frac{1}{4}$$

$$\Rightarrow \cos\theta=\frac{1}{2}\Rightarrow \theta=\frac{\pi}{3}$$

$$\therefore a_3=\cos\frac{\pi}{3}=\frac{1}{2}$$

So, $\theta=\frac{\pi}{3}$ and components of $\vec{a}$ are $\left(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2}\right)$

4. Show that $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$

Solution:

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$$

$$=(a-b)\times\vec{a}+(a-b)\times b$$

$$=a\times\vec{a}-\vec{b}\times\vec{d}+a\times\vec{b}-\vec{b}\times\vec{b}$$

$$=0+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}=0$$

$$=2\left(\vec{a}\times\vec{b}\right)$$

5. Find λ and μ if $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$

Solution:

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$$

$$\Rightarrow \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 27\\ 1 & \lambda & \mu\end{vmatrix}=0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow \hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)=0\hat{i}+0\hat{j}+0\hat{k}$$

$$6\mu-27\lambda=0$$

$$2\mu-27=0$$

$$2\lambda-6=0$$

$$\lambda=3$$

$$\mu=\frac{27}{3}$$

6. Given that $\vec{a}.\vec{b}=0$ and $\vec{a}\times\vec{b}=0$. What can you conclude about $\vec{a}$ and $\vec{b}$?

Solution:

$$\vec{a}.\vec{b}=0$$

i) $|\vec{a}|=0\,or\,|\vec{b}|=0\,or\,\vec{a}\perp\vec{b}$ (if $|\vec{a}|\neq 0$ and $|\vec{b}|\neq 0$)

$$\vec{a}\times\vec{b}=0$$

ii) $|\vec{a}|=0\ or\,|\vec{b}|=0\,or\,\vec{a}\,||\,\vec{b}$ (if $|\vec{a}|\neq 0$ and $|\vec{b}|\neq 0$)

But $\vec{a}$ and $\vec{b}$ cannot be parallel and perpendicular at same time.

So, $|\vec{a}|=0$ or $|\vec{b}|=0$

7. Let the vectors $\vec{a},\vec{b},\vec{c}$ gives as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, b_1\hat{i}+b_2\hat{j}+b_3\hat{k}, c_1\hat{i}+c_2\hat{j}, c_3\hat{k}$. Then show that $=\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Solution:

$$\left(\vec{b}+\vec{c}\right)=\left(b_1+c_1\right)\hat{i}+\left(b_2+c_2\right)\hat{j}+\left(b_3+c_3\right)\hat{k}$$

Now, $\vec{a}\times\left(\vec{b}+\vec{c}\right)\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$

$$=i\left[a_2\left(b_3+c_3\right)-a_3\left(b_2+c_2\right)\right]-\hat{j}\left[a_1\left(b_3+c_3\right)-a_3\left(b_1+c_1\right)\right]+\hat{k}\left[a_1\left(b_2+c_2\right)-a_2\left(b_1+c_1\right)\right]$$

$$=i\left[a_2b_3-a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3+a_1c_3+a_3b_1+a_3c_1\right]+\hat{k}[a_1b_2+a_1c_2-a_2b_1-a_2c_1$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=i\left[a_2b_3-a_3b_2\right]+\hat{j}\left[b_1a_3-a_2b_3\right]+k\left[a_1b_2-a_2b_1\right]$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[ab_2-a_2b\right]$$

$$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[b_1a_3+a_3c_1-a_1b_3-a_i\}3\right]$$

$$+\hat{k}\left[a,b+a_1c,-a,b_1-a,c_1\right]$$

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence, proved

8. If either $\vec{a}=0$ or $b=0$, then $\vec{a}\times b=0$. Is the converse true? Justify your answer with an example

Solution:

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k},\vec{b}=4\hat{i}+6\hat{j}+8\hat{k},\vec{a}\times\vec{b}=\vec{0}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}=\hat{i}\left(24-24\right)-\hat{j}\left(16-16\right)+\hat{k}\left(12-12\right)=0\hat{i}+0\hat{j}+0\hat{k}$$

$$|\vec{a}|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

$\therefore \vec{a} \neq 0$

$$\left|\vec{b}\right| = \sqrt{4^2+6^2+8^2} = \sqrt{116}$$

$\therefore \vec{b} \neq 0$

Hence, converse of the statement need not be ture

9. Find the area of the triangle with vertices $A(1,1,2), B(2,3,5)$ and $C(1,5,5)$

Solution:

$$\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$$

$$\text{Area } = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

So, area of ΔABC is $\frac{\sqrt{61}}{2}$ sq units

10. Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$\therefore \left|\vec{a} \times \vec{b}\right| = \sqrt{20^2 + 5^2 + 5^2} = \sqrt{400+25+25} = 15\sqrt{2}$$

So, area of parallelogram is $15\sqrt{2}$ sq units

11. Let the vector $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$

Solution:

$$\left|\vec{a}\times\vec{b}\right|=1$$

$$\Rightarrow|\vec{a}||\vec{b}||\sin\theta|=1$$

$$\Rightarrow|\vec{a}||\vec{b}||\sin\theta|=1$$

$$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$$

$$\Rightarrow\sin\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow\theta=\frac{\pi}{4}$$

12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$ respectively is

A) $\frac{1}{2}$ B) 1 C) 2 D) 4

Solution:

$$\overrightarrow{AB}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{BC}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \hat{k}(-2) = -2\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-2)^2} = 2$$

So, area of the required rectangle is 2 square units