Chapter: Three-dimensional Geometry.

Exercise: Miscellaneous

1. We need to show that the line determined by the points (3,-5,1), (4,3,-1) is perpendicular to the line joining the origin to the point (2,1,1)

Solution: Let us consider the points be $B(3,5,-1)$ and $C(4,3,-1)$ and the line joining the origin $O(0,0,0)$ and $A(2,1,1)$

We can tell that the direction ratios of OA and BC will be 2, 1, 1 and $(4-3)=1,(3-5)=-2$ and $(-1+1)=0$ respectively

As we know that for lines to be perpendicular, then $a_1a_2+b_1b_2+c_1c_2=0$

Now, $a_1a_2+b_1b_2+c_1c_2=2\times1+1\times(-2)+1\times0$

$=2-2+0$

$=0$

Therefore, the lines are perpendicular

2. We needed to show that direction cosines of the perpendicular to both of the lines $m_1n_2-m_2n_1,n_1l_2-n_2l_1,l_1m_2-l_2\,\mathrm{m}_1$ when l_1,m_1,n_1 and l_2,m_2,n_3 are the direction cosines of two mutually perpendicular lines.

Solution: Let us take $l_1l_2+m_1m_2+n_1n_2=0...(1),l_1^2+m_1^2+n_1^2=1...(2),l_1^2+m_1^2+n_1^2=1.....(3)$

Let us consider l,m,n be the direction cosines of the line with direction cosines l_1,m_1,n_1 and l_2,m_2,n_2

We got, $ll_1+mm_1+nn_1=0$ and $ll_2+mm_2+nn_2=0$

$$\therefore \frac{1}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$

$$\Rightarrow \frac{l^2}{(m_1n_2-m_2n_1)^2}=\frac{m^2}{(n_1l_2-n_2l_1)^2}=\frac{n^2}{(l_1m_2-l_2m_1)^2}$$

$$=\frac{l^2+m^2+n^2}{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}......(4)$$

As we know that l, m, n are the direction cosines of the line, we get that

$l^2+m^2+n^2=1......(5)$

As we know that,

$$\left(l_1^2+m_1^2+n_1^2\right)\left(l_2^2+m_2^2+n_2^2\right)-\left(l_1l_2+m_1m_2+n_1n_2\right)$$

$$=\left(m_1n_2-m_2n_1\right)^2+\left(n_1l_2-n_2l_1\right)^2+\left(l_1m_2-l_2m_1\right)^2$$

Now, from $(1),(2),(3)$ we get

$$\Rightarrow 1.1-0=\left(m_1n_2-m_2n_1\right)^2+\left(n_1l_2-n_2l_1\right)^2+\left(l_1m_2-l_2m_1\right)^2$$

$$\therefore \left(m_1n_2-m_2n_1\right)^2+\left(n_1l_2-n_2l_1\right)^2+\left(l_1m_2-l_2m_1\right)^2=1....(6)|$$

By putting the values from equation (5) and (6) in equation (4), we get

$$\frac{l^2}{\left(m_1n_2-m_2n_1\right)^2}=\frac{m^2}{\left(n_1l_2-n_2l_1\right)^2}=\frac{n^2}{\left(l_1m_2-l_2m_1\right)^2}=1$$

$$\Rightarrow 1=m_1n_2-m_2n_1,$$

$$m=n_1l_2-n_2l_1,$$

$$n=l_1m_2-l_2m_1$$

Therefore, the direction cosines of the required line are $m_1n_2-m_2n_1, n_1l_2-n_2l_1, l_1m_2-l_2m_1$.

3. The direction ratios are a, b, c and b-c, c-a, a-b, find the angle between the lines

Solution: As we know that, for any angle $\theta,$ with direction cosines, a, b, c and b-c, c-a, a-b can be found by,

$$\cos\theta=\left|\frac{a(b-c)+b(b-c)+c(c-a)}{\sqrt{a^2+b^2+c^2+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}}\right|$$

Solving this we get, $\cos\theta=0$

$$\theta=\cos^{-1}0$$

$$\Rightarrow \theta=90^\circ$$

Therefore, the angle between the two lines will be 90°.

4. Find the equation of a line passing through the origin and line parallel to x-axis

Solution: As it is given that the line is passing through the origin and is also parallel to x-axis is x-axis,

Now,

Let us consider a point on x-axis be A

So, the coordinates of A will be (a, 0, 0)

Now, the direction ratios of OA will be,

$\Rightarrow (a-0) = a, 0, 0$

The equation of $OA \Rightarrow \frac{x-0}{a} = \frac{y-0}{0} = \frac{z-0}{0} \Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = a$

Therefore, the equation of the line passing through origin and parallel to x-axis is

$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$.

5. Find the angle between the lines AB and CD if the coordinates of the points A, B, C, D be(1, 2, 3), (4, 5, 7), (-4,3,-6) and (2, 9, 2) respectively.

Solution: It is given that coordinates A, B, C, D are (1, 2, 3), (4, 5, 7), (-4, 3, -6) and (2, 9, 2) respectively

We know that,

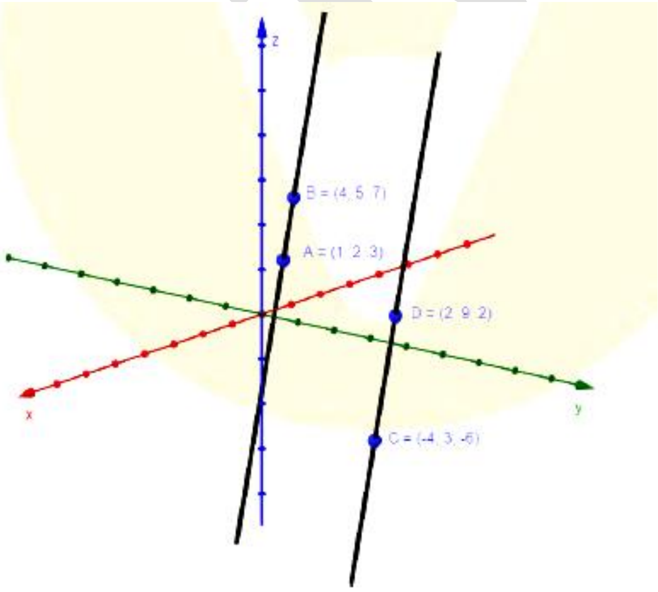
$a_1 = (4-1) = 3, b_1 = (5-2) = 3, c_1 = (7-3) = 4$

$a_2 = (2-(-2)) = 6, b_2 = (9-3) = 6, c_2 = (2-(-6)) = 8$

$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$

$\Rightarrow AB \| CD$

We get to know that the lines are parallel to each other.



Therefore, the angle between AB and CD is either $0°$ or $180°$

6. Find the value of k if the lines $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ and $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ are perpendicular.

Solution: From the given equation we can say that $a_1 = -3, b_1 = 2k, c_1 = 2$ and

$a_2 = 3k, b_2 = 1, c_2 = -5$.

We know that the two lines are perpendicular, if $a_{1=0}a_2 + b_1b_2 + c_1c_2 = 0$

$-3(3k) + 2k \times 1 + 2(-5) = 0$

$\Rightarrow -9k + 2k - 10 = 0$

$\Rightarrow 7k = -10$

$\Rightarrow k = \frac{-10}{7}$

Therefore, the value of k is $-\frac{-10}{7}$

7. Find the vector equation of the perpendicular to the plane $\vec{r}.\left(\hat{i} + 2\hat{j} - 5\hat{k}\right) + 9 = 0$ and passing through (1, 2, 3)

Solution: According to the question, we can say that we have

$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$

$\vec{N} = \hat{i} + 2\hat{j} - 5\hat{k}$

As we know we can express the equation of a line passing through a point and perpendicular to the plane in form $\vec{l} = \vec{r} + \lambda\vec{N}, \lambda \in R$

We got,

$\vec{l} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} + 2\hat{j} - 5\hat{k}\right)$

Therefore, the vector equation to the plane will be $\vec{l} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} + 2\hat{j} - 5\hat{k}\right)$.

8. Find the equation of the plane parallel to the plane $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$ and passing through (a, b, c)

Solution: According to the question, plane is parallel to plane $\vec{r}_1.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$ and it also passes through point (a, b,c)

From this we get the equation,

$\left(a\hat{i} + b\hat{j} + c\hat{k}\right).\left(\hat{i} + \hat{j} + \hat{k}\right) = \lambda$

$\Rightarrow a + b + c = \lambda$

Now, putting value in equation, we get,

$$\vec{r}_1.\left(\hat{i}+\hat{j}+\hat{k}\right)=a+b+c$$

Now we will put $\vec{r}_1.\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$ in equation, we get

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(\hat{i}+\hat{j}+\hat{k}\right)+a+b+c \Rightarrow x+y+z-a+b+c$$

Therefore, the equation of the plane will be $x+y+z=a+b+c$.

9. What is the shortest distance between these two lines $\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right)$

$$\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right)$$

Solution: According to the question, we need to find the distance between the lines,

$$\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right)$$

$$\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right)$$

As we know we can find the shortest distance by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_1-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

Now, from the equation of lines we get

$$\vec{a}_1=6\hat{i}+2\hat{j}+2\hat{k}$$

$$\vec{b}_1=\hat{i}-2\hat{j}+2\hat{k}$$

$$\vec{a}_2=-4\hat{i}-\hat{k}$$

$$\vec{b}_2=3\hat{i}-2\hat{j}-2\hat{k}$$

$$\Rightarrow \overrightarrow{a_2}-\overrightarrow{a_1}=\left(-4\hat{i}-\hat{k}\right)-\left(6\hat{i}+2\hat{j}+2\hat{k}\right)=-10\hat{i}-2\hat{j}-3\hat{k}$$

$$\Rightarrow \overrightarrow{b_1}\times\overrightarrow{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}=(4+4)\hat{i}-(-2-6)\hat{j}+(-2+6)\hat{k}$$

$$=8\hat{i}+8\hat{j}+4\hat{k}$$

$$\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_2-\vec{a}_1\right)=\left(8\hat{i}+8\hat{j}+4\hat{k}\right).\left(-10\hat{i}-2\hat{j}-3\hat{k}\right)$$

$$=-80-16-12$$

$$=-108$$

Now, putting these values in $d = \left|\frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_1 - \vec{a}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$, we get

$$d = \left|\frac{-108}{12}\right| = 9$$

Therefore, shortest distance between the above two lines is of 9 units.

10. Find the point of intersection where the line passing through (5, 1, 6) and (3, 4, 1) intersecting through the YZ plane.

Solution: We know that the equation of the line passing through the points is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

Now, according to the question, the line is passing through the point, (5, 1, 6) and (3, 4, 1), we get

$$\frac{x-5}{3-5} = \frac{y-1}{4-1} = \frac{z-6}{1-6} \Rightarrow \frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} = k$$

$$\Rightarrow x = 5 - 2k, y = 3k + 1, z = 6 - 5k$$

Now we know that any point on the line will be of form $(5-2k, 3k+1, 6-5k)$,

Now for YZ plane, x=0, we get

$$x = 5 - 2k = 0 \Rightarrow k = \frac{5}{2}$$

$$\Rightarrow y = 3k + 1 = 3 \times \frac{5}{2} + 1 = \frac{17}{2}$$

$$\Rightarrow z = 6 - 5k = 6 - 5 \times \frac{5}{2} = -\frac{13}{2}$$

Therefore, the required point of intersection $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$

11. Find the point of intersection where the line crosses through the ZX plane and through (5, 1, 6), (3, 4, 1)

Solution: As we know that the equation of the line passing through the point is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

According to the question, the line passing through (5, 1, 6) and (3, 4, 1), we got

$$\frac{x-5}{3-5} = \frac{y-1}{4-1} = \frac{z-6}{1-6} \Rightarrow \frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} = k$$

$$\Rightarrow x=5-2k, y=3k+1, z=6-5k$$

As we know for any point on the line, it will be in the form of (5-2k,3k+1,6-5k),

Now, for ZX plane, y=0

$$\Rightarrow y=3k+1=0 \Rightarrow k=\frac{-1}{3}$$

$$\Rightarrow x=5-2k=5-2\times\left(\frac{-1}{3}\right)=\frac{17}{3}$$

$$\Rightarrow z=6-5k=6-5\times\left(\frac{-1}{3}\right)=\frac{23}{3}$$

Therefore, the required point of intersection $\left(\frac{17}{3},0,\frac{23}{3}\right)$

12. Find the point of intersection where the line crosses through the plane $2x+y+z=7$ and through (3, -4, -5), (2,-3,1)

Solution: As we know that the equation of the line passing through the points

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1},$$

According to the question, the line passing through (3, -4, -5) and (2, -3, 1), we get,

$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5}\Rightarrow\frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=k$$

$$\Rightarrow x=3-k, y=k-4, z=6k-5$$

As we know that the point on the line will be in the form of $(3-k,k-4,6k-5)$.

As the point lies on $2x+y+z=7,$ we get

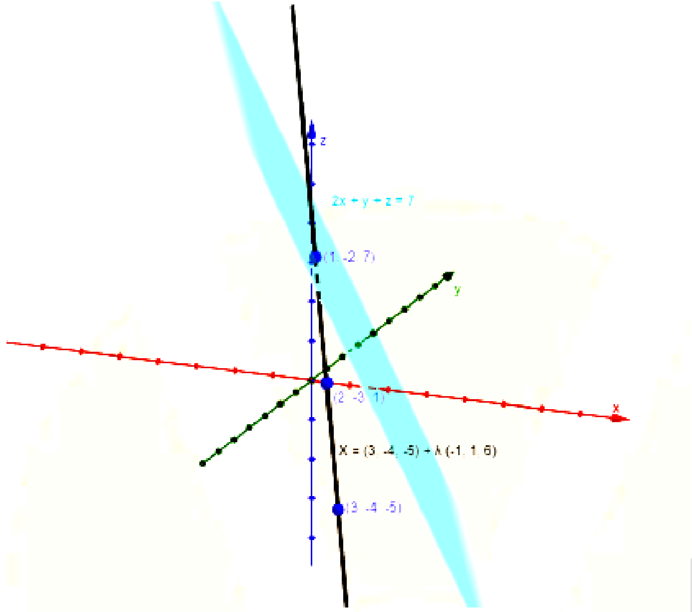
$$2(3-k)+(k-4)+(6k-5)=7$$

$$\Rightarrow 5k-3=7$$

$$\Rightarrow k=2$$

Now, by putting the value of k in equation, we get

$$(3-k,k-4,6k-5)=(3-2,2-4,6(2)-5)=(1,-2,7)$$



Therefore, the point on the required plane (1, -2, 7)

13. Find the equation of the plane passing through the points (-1, 3,2) and perpendicular to each of the planes $x+2y+3z=5$ and $3x+3y+z=0$.

Solution: As we know that the equation for plane passing through the point can be given as,

$a(x+1)+b(y-3)+c(z-2)=0$

Now that we know a, b, c are direction ratios of normal to the plane,

We know that $a_1a_2+b_1b_2+c_1c_2=0$ if the lines are perpendicular to each other,

Now, if $x+2y+3z=5$ is perpendicular then,

$a.1+b.2+c.3=0$

$\Rightarrow a+2b+3c=0$

And if $3x+3y+z=0$ is perpendicular then,

$a.3+b.3+c.1=0$

$\Rightarrow 3a+3b+c=0$

Now,

$$\frac{a}{2\times1-3\times3}=\frac{b}{3\times3-1\times1}=\frac{c}{1\times3-2\times3}$$

$$\Rightarrow \frac{a}{-7}=\frac{b}{8}=\frac{c}{-3}=k$$

$\Rightarrow a=-7k, b=8k, c=-3k$

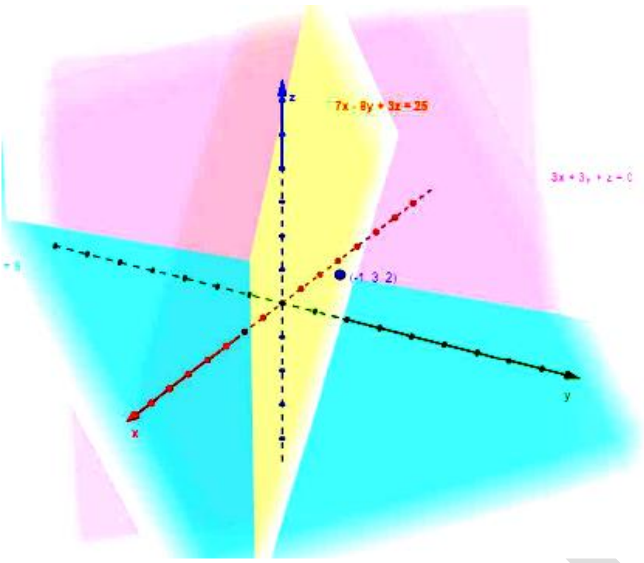
By putting values of a, b, c we get

$$-7k(x+1)+8k(y-3)-3k(z-2)=0$$

$$\Rightarrow(-7x-7)+(8y-24)-3z+6=0$$

$$\Rightarrow -7x+8y-3z-25=0$$

$$\Rightarrow 7x-8y+3z=25$$



Therefore, the equation of the plane will be $7x-8y+3z=25$

14. For $\vec{r}=\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0,$ the points (1, 1, p) and (-3,0,1) are at equal distance from the plane, then find the value of p

Solution: According to the question the vectors are,

$$\vec{a}_1=\hat{i}+\hat{j}+p\hat{k}, \vec{a}_2=-4\hat{i}+\hat{k}$$

And the plane 's equation is $\vec{r}=(3\hat{i}+4\hat{j}-12\hat{k})+13=0$

As we know that the perpendicular distance between vector and the plane can be found by $\vec{r}.\vec{N}=d,$

Now,

$$D=\left|\frac{\vec{a}.\vec{N}-d}{\vec{N}}\right|$$

$\vec{N}=3\hat{i}+4\hat{j}-12\hat{k}$ and $d=-13$

Then, the distance between the point (1, 1, p) and the given plane is

$$D_1=\left|\frac{\left(\hat{i}+\hat{j}+p\hat{k}\right).\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13}{3\hat{i}+4\hat{j}-12\hat{k}}\right|$$

$$\Rightarrow D_1 = \left|\frac{3+4-12p+13}{\sqrt{3^2+4^2+(-12)^2}}\right|$$

$$\Rightarrow D_1 = \left|\frac{20-12p}{13}\right| \;.......(i)$$

Similarly, the distance between the point (-3,0,1) and the given plane is

$$D_2 = \left|\frac{\left(-3\hat{i}+\hat{k}\right).\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13}{3\hat{i}+4\hat{j}-12\hat{k}}\right|$$

$$\Rightarrow D_2 = \left|\frac{-9-12+13}{\sqrt{3^2+4^2+(-12)^2}}\right|$$

$$\Rightarrow D_1 = \frac{8}{13} \;..............(ii)$$

Now, from the given conditions,

$$D_1 = D_2$$

$$\Rightarrow \frac{|20-12p|}{13} = \frac{8}{13}$$

$$\Rightarrow 20-12p = 8, -(20-12p) = 8$$

$$\Rightarrow 12p = 12, 12p = 28$$

$$\Rightarrow p = 1, p = \frac{7}{3}$$

Therefore, the value will be, $p = 1, p = \frac{7}{3}$

15. Find the equation of the plane parallel to x-axis and passing through the line of intersection of the planes $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$

Solution: We have been given the two planes, $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=1 \Rightarrow \vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)-1=0$ and

$$\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$$

Now, we know that the equation of line passing through the line of intersection will be,

$$\left[\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)-1\right]+\lambda\left[\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4\right]=0$$

$\vec{r}.\left[(2\lambda+1)\hat{i}+(3\lambda+1)\hat{j}+(1-\lambda)\hat{k}\right]+(4\lambda+1)=0$

$a_1=(2\lambda+1), b_1=(3\lambda+1), c_1=(1-\lambda)$.

As we know that the required plane is to be parallel to x-axis, the normal will perpendicular to x-axis,

Now, the direction ratios of x-axis will be 1, 0, and 0, which means

$a_2=1, b_2=0, c_2=0$

$1.(2\lambda+1)+0(3\lambda+1)+0(1-\lambda)=0$

$\Rightarrow 2\lambda+1=0$

$\Rightarrow \lambda=-\frac{1}{2}.$

By putting $\lambda=-\frac{1}{2}$ in (1)

$\Rightarrow \vec{r}.\left[-\frac{1}{2}\hat{j}+\frac{3}{2}\hat{k}\right]+(-3)=0 \Rightarrow \vec{r}\left(\hat{j}-3\hat{k}\right)+6=0$

Therefore, the required Cartesian equation of the plane is $y-3z+6=0$

16. If O be the origin and the coordinates of P be (1, 2, -3), then find the equation of the plane parallel to x-axis and passing though P.

Solution: From the question we know that the direction ratios of OP will be

$a=(1-0)=1, b=(2-0)=2, c=(-3-0)=-3$

Now, we know that the equation will be as,

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

As we can tell, the direction ratios of the normal are 1,2,3

Therefore, the point is P(1,2,-3)

Therefore, the equation of the plane is

$1(x-1)+2(y-2)-3(z+3)=0$

$\Rightarrow x+2y-3z-14=0$

17. Find the equation of the plane which holds the line of intersection of the planes

$\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0, \vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ and is perpendicular to the plane

$\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

Solution: According to the question, it is given that,

$\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0 \qquad ....(1)$

$\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0 \qquad ....(2)$

Now, we know that the equation of the required plane will be,

$$\left[\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4\right]+\lambda\left[\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5\right]=0$$

$$\Rightarrow \vec{r}.\left[(2\lambda+1)\hat{i}+(\lambda+2)\hat{j}+(3-\lambda)\hat{k}\right]+(5\lambda-4)=0.....(3)$$

Now according to the question, the plane is perpendicular to the plane,

So, $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

$$\therefore 5(2\lambda+1)+3(\lambda+2)-6(3-\lambda)=0$$

$$\Rightarrow 19\lambda-7=0$$

$$\Rightarrow \lambda=\frac{7}{19}$$

By putting value of $\lambda=\frac{7}{19}$ in equation(3)

$$\Rightarrow \vec{r}.\left[\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right]-\frac{41}{9}=0$$

Therefore, the required Cartesian Equation of the plane is $33x+45y+50z-41=0$

18. Find the distance of the point $(-1,5,-10)$ from the point of intersection of line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$

Solution: According to the question, it is given that line is,

$\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)....(1)$ and the plane is , $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5....(2)$

Now, we will put the value of $\vec{r}$ from (1) into (2), we get

$$\left[2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)\right].\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$

$$\Rightarrow\left[(3\lambda+2)\hat{i}+(4\lambda+1)\hat{j}+(2\lambda+2)\hat{k}\right].\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$

$$\Rightarrow(3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5$$

$$\Rightarrow \lambda=0$$

Now if we put the value in equation, we will get the equation of the line as

$$\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$$

Therefore, the required distance between both the points is,

$$d=\sqrt{(-1-2)^2+(-5+1)^2+(-10-2)^2}=\sqrt{9+16+144}=\sqrt{169}=13$$

Therefore, the distance between the |points is 13 units.

19. Find the vector equation of the line parallel to the planes $\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$ and passing through (1, 2, 3)

Solution: Let us consider that line parallel to vector $\vec{b}$ is given by, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

Now, the position vector of point (1, 2, 3) will be $\hat{a}=\hat{i}+2\hat{j}+3\hat{k}$

From this we get that the equation of line passing through (1, 2, 3) and is parallel to vector $\vec{b}$ will be,

$\vec{r}=\hat{a}+\lambda\hat{b}$

$\Rightarrow \vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$.....(1)

Therefore, the equation of the given planes are

$\vec{r}.\left(\hat{i}-\hat{j}+5\hat{k}\right)=5$ ….(2)

$\vec{r}.\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$ …..(3)

As we can tell that the line in equation (1) and plane in equation (2) are parallel, so we get that the normal to the plane of equation (2) and the given line are perpendicular

$\Rightarrow\left(\hat{i}-2\hat{j}+2\hat{k}\right).\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)=0$

$\Rightarrow(b_1-b_2+2b_3)=0$

$\Rightarrow b_1-b_2+2b_3=0$ ……(4)

Similarly,

$\left(3\hat{i}+\hat{j}+\hat{k}\right).\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)=0$

$\Rightarrow\lambda(3b_1+b_2+b_3)=0$

$\Rightarrow 3b_1+b_2+b_3=0$ .....(5)

From equation (4) and (5), we get

$$\frac{b_1}{(-1)\times1-1\times2}=\frac{b_2}{2\times3-1\times1}=\frac{b_3}{1\times1-3(-1)}$$

$$\Rightarrow \frac{b_1}{-3}=\frac{b_2}{5}=\frac{b_3}{4}$$

Now, the direction ratios of $\vec{b}$ are $-3,5,4$

Which means $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=-3\hat{i}+5\hat{j}+4\hat{k}$

Putting the value of $\vec{b}$ in equation (1)

$$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda(-3\hat{i}+5\hat{j}+4\hat{k})$$

20. Find the vector equation of the line passing through the points (1, 2, -4) and perpendicular to the two lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

Solution: According to the question, we get that $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{a}=\hat{i}+2\hat{j}-4\hat{k}$

We know that the equation of the line passing through point and also parallel to vector, we get

$$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)...(1)$$

Now, the equation of the two lines will be

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}....(2)$$

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}....(3)$$

As we know that line (1) and (2) are perpendicular to each other, we get

$$3b_1-16b_2+7b_3=0.....(4)$$

Also, we know that the line (1) and (3) are perpendicular to each other, we get

$$3b_1+18b_2-5b_3=0.....(5)$$

Now, from equation (4) and (5) we get that

$$\frac{b_1}{(-16)(-5)-8(7)}=\frac{b_2}{7(3)-3(-5)}=\frac{b_3}{3(8)-3(-16)}$$

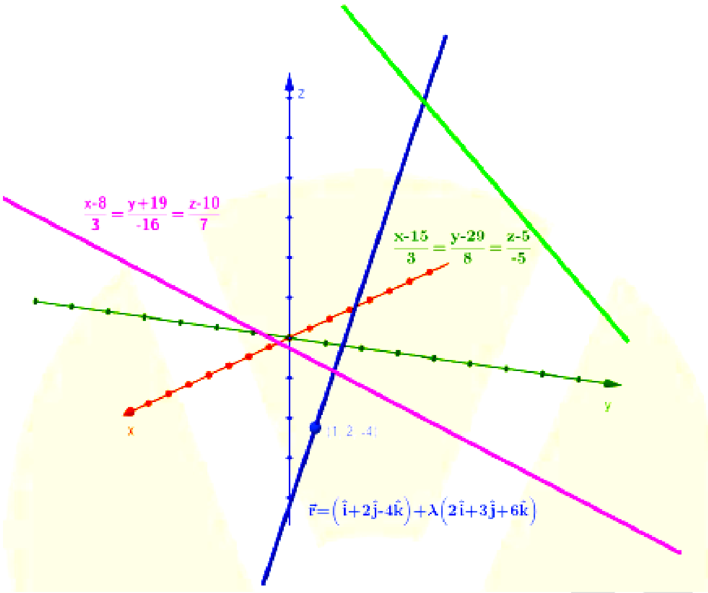
$$\Rightarrow \frac{b_1}{24}=\frac{b_2}{36}=\frac{b_3}{72}\Rightarrow \frac{b_1}{2}=\frac{b_2}{3}=\frac{b_3}{6}$$

Therefore, direction ratios of $\vec{b}$ are 2, 3, 6

Which means $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

Putting $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ in equation (1), we get

$\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$



Therefore, the vector equation will be $\vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$

21. Prove that if a plane has the intercepts a,b,c and is a distance of p units from the origin, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$.

Solution: We know that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

The distance of the plane will be,

$$p=\left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}\right| \Rightarrow p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Therefore, we have proved that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$.

22. Distance between the two planes : $2x+3y+4z=4$ and $4x+6x+8z=12$ is

(A) 2 units

(B) 4 units

(C) 8 units

(D) $\frac{2}{\sqrt{29}}$ units

Solution: According to the question, the equation of the planes are

$$2x+3y+4z=4$$

$$4x+6y+8z=12$$

We get $2x+3y+4z=6$

As we can tell, the given planes are parallel,

We know that the distance between two parallel planes, is given by,

$$D=\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right| \Rightarrow D=\left|\frac{6-4}{\sqrt{2^2+3^2+4^2}}\right|$$

$$\Rightarrow D=\frac{2}{\sqrt{29}}$$

Therefore, the distance between two parallel planes is $\frac{2}{\sqrt{29}}$ units.

Therefore, the correct answer is D.

23. The planes : $2x-y+4z=5$ and $5x-2.5y+10z=6$

(A) Perpendicular

(B) Parallel

(C) Intersect y axis

(D) Passes through $\left(0,0,\frac{5}{4}\right)$

Solution: According to the question we get,

$$2x-y+4z=5$$

$$5x-2.5y+10z=6$$

As we can see that,

$$\frac{a_1}{a_2}=\frac{2}{5},\frac{b_1}{b_2}=\frac{-1}{-2.5}=\frac{2}{5},\frac{c_1}{c_2}=\frac{4}{10}=\frac{2}{5}$$

$$\therefore \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

As we can see that the given lines are parallel,

Therefore, the correct answer is B.

Exercise: 11.1

1. Find the direction cosines if the line makes angles $90°, 135°, 45°$ with x,y and z axes respectively.

Solution: Let us consider 1,m and n be the direction cosines of line

Then,

$1 = \cos 90° = 0,$

$m = \cos 135°$

$= \cos(90° + 45°)$

$= \sin 45°$

$= -\frac{1}{\sqrt{2}}$

And, $n = \cos 45° = \frac{1}{\sqrt{2}}$

Therefore, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$

2. Find the direction cosines if the line makes equal angles with the coordinates axes.

Solution: Let us consider that the line makes an angle α with coordinates axes

Which means $1 = \cos\alpha, m = \cos\alpha, n = \cos\alpha$ Now, we know that

$1^2 + m^2 + n^2 \Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha$

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = \frac{1}{3} \Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Therefore, the direction cosines of the line are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$.

3. Find the direction cosines of a line having direction ratios -18,12,-4

Solution: We have the direction ratios as -18, 12-4,

Now, the direction cosines will be as

$$l=\frac{-18}{\sqrt{(-18)^2+(12)^2+(-4)^2}},$$

$$m=\frac{12}{\sqrt{(-18)^2+(12)^2+(-4)^2}},$$

$$n=\frac{-4}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$$

$$\frac{-18}{22},\frac{12}{22},\frac{-4}{22}\Rightarrow\frac{-9}{11},\frac{6}{11},\frac{-2}{11}$$

Therefore, direction cosines of the line are $\frac{-9}{11},\frac{6}{11}$ and $\frac{-2}{11}$.

4. Show that $(2,3,4),(-1,-2,1),(5,8,7)$ are collinear.

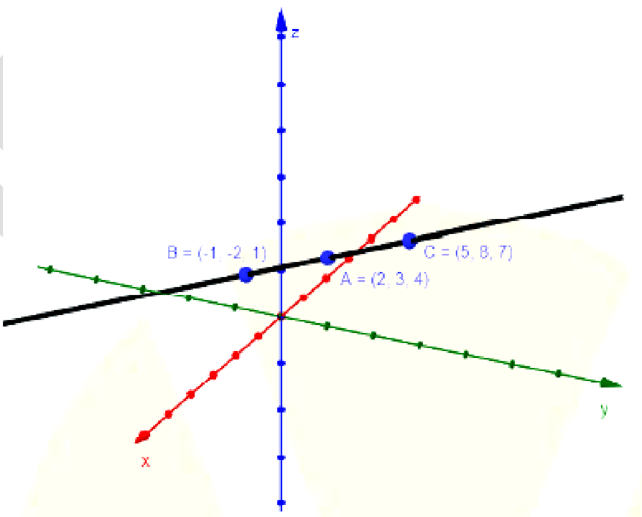
Solution: Let us consider the points be A(2, 3, 4), B(-1, -2, 1) and C(5, 8, 7).

Now, as we know that direction cosines can be found by $(x_2-x_1),(y_2-y_1),$ and (z_2-z_1)

Therefore,

Direction ratios of AB and BC be -3, -5, -3 and 6, 10, 6 respectively.

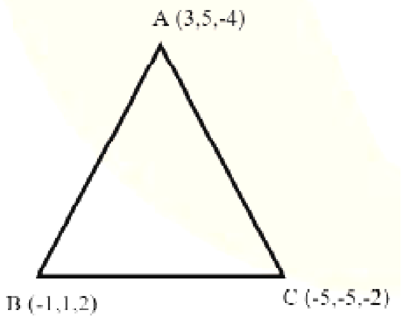
As we can see that AB and BC are proportional, we get that AB is parallel to BC.



Therefore, the points are collinear.

5. If the vertices of a triangles are $(3,5,-4),(-1,1,2),(-5,-5,-2)$, find its directions cosines.

Solution: Let us consider the points be $A(3,5,-4), B(-1,1,2)$ and $C(-5,-5,-2)$.



Now, the direction ratios of AB will be -4, -4 and 6, we get

$$\sqrt{(-4)^2+(-4)^2+(6)^2}=\sqrt{68}\Rightarrow 2\sqrt{17}$$

Now,

$$1=\frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, m=\frac{-4}{\sqrt{(-4)^2(-4)^2+(6)^2}}, n=\frac{6}{\sqrt{(-4)^2+(-4)^2+(6)^2}}$$

$$\Rightarrow 1=\frac{-2}{\sqrt{17}}, m=\frac{-2}{\sqrt{17}}, n=\frac{3}{\sqrt{17}}$$

Therefore, the direction cosines of AB are $\frac{-2}{\sqrt{17}},\frac{-2}{\sqrt{17}},\frac{3}{\sqrt{17}}$

Similarly, the direction ratios of side BC will be -4, -6 and -4.

Now,

$$1=\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, m=\frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, n=\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$$

$$1=\frac{-4}{2\sqrt{17}}, m=\frac{-6}{2\sqrt{17}}, n=\frac{-4}{2\sqrt{17}}$$

Therefore, the direction cosines of BC is $\frac{-2}{\sqrt{17}},\frac{-3}{\sqrt{17}},\frac{-2}{\sqrt{17}}$

Similarly, the direction ratios of CA will be -8, -10 and 2.

Now,

$$1=\frac{-8}{\sqrt{(-8)^2+(10)^2+(2)^2}}, m=\frac{10}{\sqrt{(-8)^2+(10)^2+(2)^2}}, n=\frac{2}{\sqrt{(-8)^2+(10)^2+(2)^2}}$$

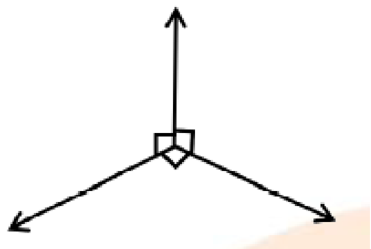
$$1=\frac{-8}{2\sqrt{42}}, m=\frac{-10}{2\sqrt{42}}, n=\frac{2}{2\sqrt{42}}.$$

Therefore, the direction cosines of CA is $\frac{-4}{\sqrt{42}},\frac{-5}{\sqrt{42}},\frac{1}{\sqrt{42}}$

Exercise: 11.2

1. Show that the three lines are mutually perpendicular if they have direction cosines be

$$\frac{12}{13},\frac{-3}{13},\frac{-4}{13};\frac{4}{13},\frac{12}{13},\frac{3}{13};\frac{3}{13},\frac{-4}{13},\frac{12}{13}$$



Solution: As we know, if $l_1l_2+m_1m_2+n_1n_2=0,$ the lines are perpendicular

i. Now, from direction cosines $\frac{12}{13},\frac{-3}{13},\frac{-4}{13}$ and $\frac{4}{13},\frac{12}{13},\frac{3}{13},$ we get

$$l_1l_2+m_1m_2+n_1n_2=\frac{12}{13}\times\frac{4}{13}+\left(\frac{-3}{13}\right)\times\frac{12}{13}+\left(\frac{-4}{13}\right)\times\frac{3}{13}$$

$$\Rightarrow\frac{48}{169}-\frac{36}{169}-\frac{12}{169}$$

$$\Rightarrow 0$$

Therefore, the lines are perpendicular.

ii. Similarly, if we take $\frac{4}{13},\frac{12}{13},\frac{3}{13}$ and $\frac{12}{13},\frac{-3}{13},\frac{-4}{13},$ we get

$$l_1l_2+m_1m_2+n_1n_2=\frac{4}{13}\times\frac{3}{13}+\left(\frac{12}{13}\right)\times\left(\frac{-4}{13}\right)+\frac{3}{13}\times\left(\frac{12}{13}\right)$$

$$\Rightarrow\frac{12}{169}-\frac{48}{169}-\frac{36}{169}=0$$

Therefore, the lines are perpendicular.

iii. Again, if we consider $\frac{-3}{13},\frac{-4}{13},\frac{12}{13}$ and $\frac{12}{13},\frac{-3}{13},\frac{-4}{13},$ we get

$$l_1l_2+m_1m_2+n_1n_2=\frac{3}{13}\times\frac{12}{13}+\left(\frac{-4}{13}\right)\times\left(\frac{-4}{13}\right)+\frac{12}{13}\times\left(\frac{-4}{13}\right)$$

$$\Rightarrow \frac{36}{169}-\frac{12}{169}-\frac{48}{169}=0$$

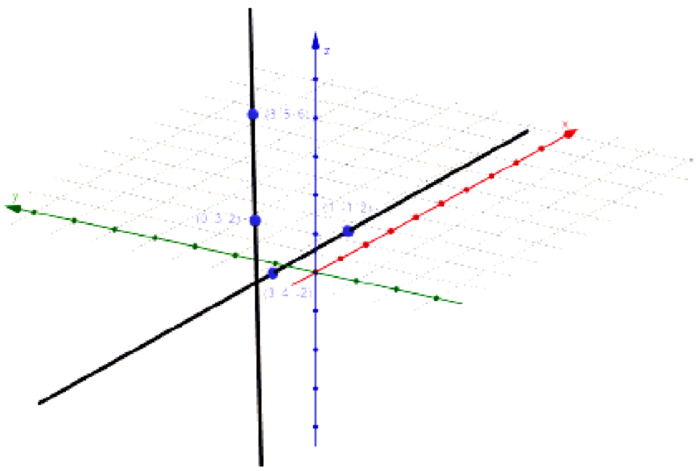
Therefore, the lines are perpendicular.

Therefore, we can say that all the lines are mutually perpendicular.

2. How can you show that the line passing through the points $(1,-1,2)(3,4,-2)$ is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6) ?

Solution: Let us consider that AB and CD are the lines that pass through the points,

$(1,-1,2),(3,4,-2)$ and $(0,3,2),(3,5,6),$ respectively,



Now, we have $a_1=(2),b_1=(5),c_1=(-4)$ and $a_2=(3),b_2=(2),c_2=(4)$

As we know that if $AB\perp CD$ then $a_1a_2+b_1b_2+c_1c_2=0$

Now,

$$a_1a_2+b_1b_2+c_1c_2=2\times3+5\times2+(-4)\times4$$

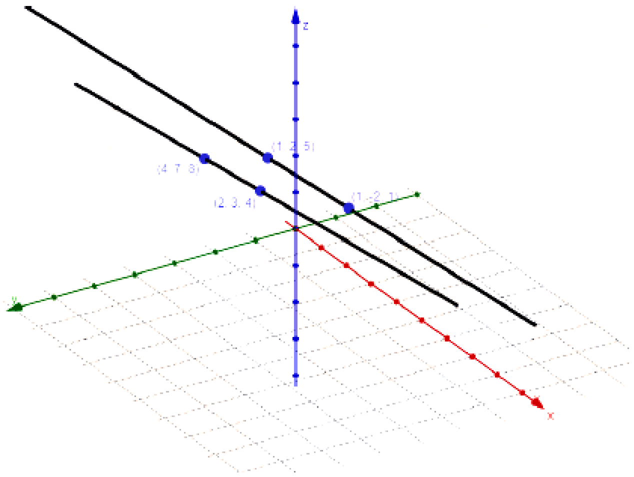
$$\Rightarrow 2\times3+5\times2-4\times4=6+10-16$$

$$\Rightarrow 0$$

Therefore, AB and CD are perpendicular to each other.

3. Show that the line through the points $(4,7,8)(2,3,4)$ is parallel to the line through the points $(1,-2,1)(1,2,5)$.

Solution: Let us consider the lines AB and CD that pass through points $(4,7,8),(2,3,4),$ and $(-1,-2,1),(1,2,5)$ respectively.



Now, we get

$a_1 = (2-4), b_1 = (3-7), c_1 = (4-8)$ and $a_2 = (1+1), b_2 = (2+2), c_2 = (5-1)$

$a_1 = (-2), b_1 = (-4), c_1 = (-4)$ and $a_2 = (2), b_2 = (4), c_2 = (4)$

Now, we know that if $AB \parallel CD$ then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$,

Now,

$\frac{a_1}{a_2} = \frac{-2}{2} \Rightarrow -1, \frac{b_1}{b_2} = \frac{-4}{4} \Rightarrow -1, \frac{c_1}{c_2} = \frac{-4}{4} = -1$

We got $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Therefore, AB is parallel to CD.

4. Find the equation of the line if it is parallel to vector $3i + 2j - 2k$ and which passes through point (1, 2, 3).

Solution: Now, let us consider the position vector A be $a = i + 2j + 3k$ and let $\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$

Now, we know that the line passes through A and is parallel to $\vec{b}$,

As we know $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a constant

$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$

Therefore, the equation of the line is $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$

5. If the line passes through the point with positive vector $2\hat{i}-\hat{j}-4\hat{k}$ and is in the direction $\hat{i}+2\hat{j}-\hat{k}$. Find the equation of the line in vector and in Cartesian form.

Solution: We know that the line passes through the point with positive vector

Now, let us consider $\vec{a}=2\hat{i}-\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$

Now, line passes through point A and parallel to $\vec{b}$, we get

$\vec{r}=2\hat{i}-\hat{j}+4\hat{k}+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$

Therefore, the equation of the line in vector form is $\vec{r}=2\hat{i}-\hat{j}+4\hat{k}+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$.

Now, we know

$\vec{r}=x\hat{i}-y\hat{j}+z\hat{k} \Rightarrow x\hat{i}-y\hat{j}+z\hat{k}=(\lambda+2)\hat{i}+(2\lambda-1)\hat{j}+(-\lambda+4)\hat{k}$

Therefore, the equation of the line in Cartesian form will be $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{-1}$.

6. If the line passes through the point (-2, 4, -5) and parallel to the line given by $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$, find the Cartesian equation of the line.

Solution: We know that the line passes through point (-2, 4, -5) and also parallel to

$\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

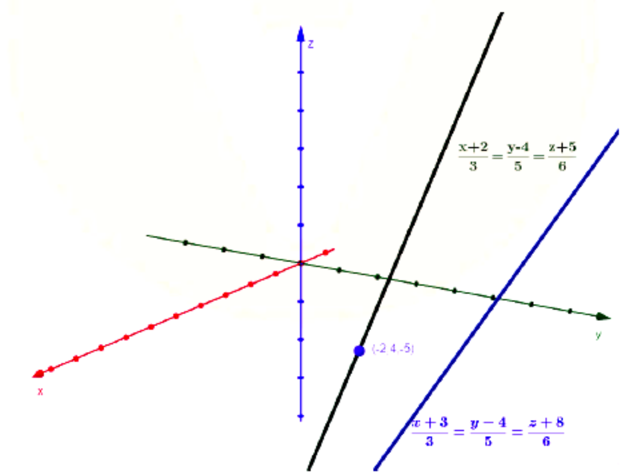
Now, as we can see the direction ratios of the line are 3,5 and 6.

As we know the required line is parallel to $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

Therefore, the direction ratios will be 3k, 5k and 6k

As we know that the equation of the line through the point and with direction ratio is shown in form $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Therefore, the equation of the line $\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$.



7. Write the vector form of the line if the Cartesian equation of a line is
$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$.

Solution: As we can see the Cartesian equation of the line, we can tell that the line is passing through $(5,4,-6)$, and he direction ratios are 3, 7 and 2.

Now, we got the position vector $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

From this we got the direction of the vector be $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

Therefore, the vector form of the line will be $\vec{r}=5\hat{i}-4\hat{j}+6\hat{k}+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$

8. If the line passes through the origin and $(5,-2,3)$, find the vector and the Cartesian equation of the line.

Solution: According to the question, line passes through the origin,

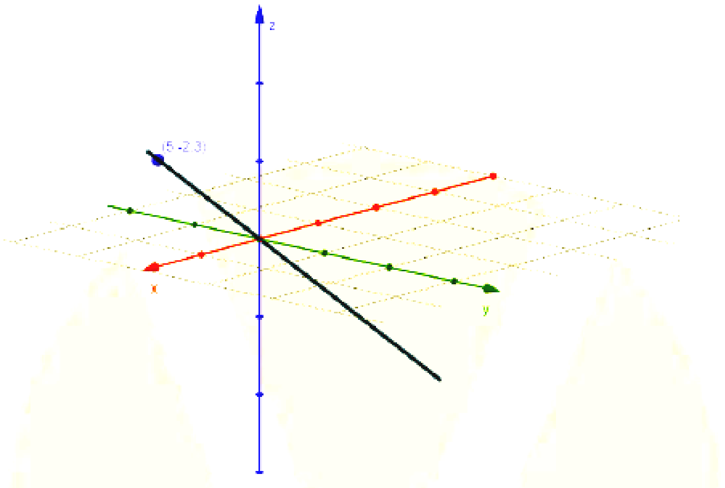
Now, the position vector will be $\vec{a}=0$

As the line pass through the point (5, 2, 3), the direction ratios of the line through origin will be 5, 2, 3

As the line is parallel to the vector $\vec{b}=5\hat{i}-2\hat{j}+3\hat{k}$

We can say that the equation of the line in vector form will be $\vec{r}=\lambda\left(5\hat{i}-2\hat{j}+3\hat{k}\right)$

And, the equation of the line in the Cartesian form will be $\frac{x}{5}=\frac{y}{2}=\frac{z}{3}$.



9. If the line passes through the point $(3,-2,-5),(3,-2,6)$, find the vector and the Cartesian equation of the line.

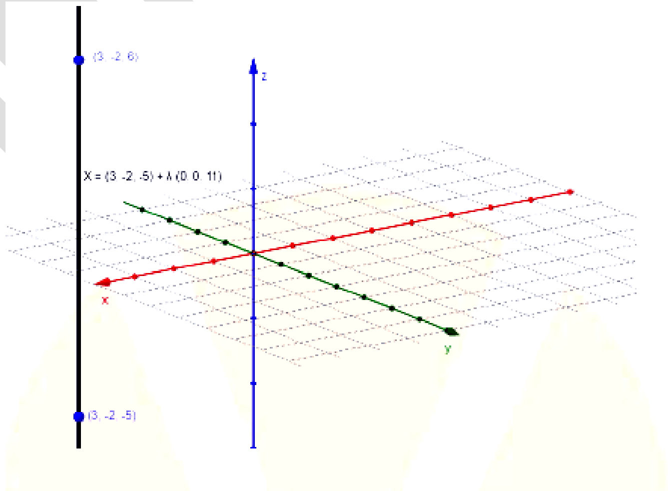
Solution: Let us consider the points be $P(3,-2,-5)$ and $Q(3,-2,6)$, so the line passing through the point will be PQ.

Therefore, the position vector will be $\vec{a}=3\hat{i}-2\hat{j}-5\hat{k}$ and the direction ratios will be $(3-3)=0,(-2+2)=0,(6+5)=11$

As the equation of the vector in the same direction as PQ, we get

$\vec{b}=0\hat{i}-0\hat{j}+11\hat{k}=11k$

Therefore, the equation of the line in vector form will be $\vec{r}=3\hat{i}-2\hat{j}-5\hat{k}+11k\lambda$ and in Cartesian form it will be $\dfrac{x-3}{11}=\dfrac{y+2}{11}=\dfrac{z+5}{11}$



10. Find the angle between the lines

(i) $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda\left(3\hat{i}-2\hat{j}+6\hat{k}\right)$ and $\vec{r}=7\hat{i}-6\hat{k}+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

Solution: Let us consider the angle be θ,

As we know that the angle between the lines can be found by $\cos\theta=\left|\frac{\vec{b}_1\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

As the line are parallel to $\vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$ and $\vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$, we got

$|\vec{b}_1|=\sqrt{3^2+2^2+6^2}=7, |\vec{b}_2|=\sqrt{1^2+2^2+2^2}=3$ and

$\vec{b}_1\vec{b}_2=\left(3\hat{i}+2\hat{j}+6\hat{k}\right)\left(\hat{i}+2\hat{j}+2\hat{k}\right)=19$

Therefore, the angle between the lines will be

$$\cos\theta=\frac{19}{7\times 3}$$

$$\Rightarrow \theta=\cos^{-1}\frac{19}{21}$$

(ii) $\vec{r}=3\hat{i}+\hat{j}-2\hat{k}+\lambda\left(\hat{i}-\hat{j}-2\hat{k}\right)$ **and** $\vec{r}=3\hat{i}-\hat{j}-56\hat{k}+\mu\left(3\hat{i}-5\hat{j}-4\hat{k}\right)$

Solution: As the lines are parallel to the vectors $\vec{b}_1=\hat{i}-\hat{j}-2\hat{k}$ and $\vec{b}_2=3\hat{i}-5\hat{j}+-4\hat{k}$,

we get $|\vec{b}_1|=\sqrt{1^2+(-1)^2+(-2)^2}=\sqrt{6}, |\vec{b}_2|=\sqrt{3^2+(-5)^2+(-2)^2}=5\sqrt{2}$ and

$\vec{b}_1\vec{b}_2=\left(\hat{i}-\hat{j}-2\hat{k}\right)\left(3\hat{i}-5\hat{j}+-4\hat{k}\right)=16$

Therefore, the angle between them will be,

$$\cos\theta=\frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos\theta=\frac{8}{5\sqrt{3}}$$

$$\Rightarrow \theta=\cos^{-1}\frac{8}{5\sqrt{3}}$$

11. Find the angle between the lines

(i) $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$

Solution: Let us take $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the lines, we get

$\vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b}_2 = \hat{i} + 8\hat{j} + 4\hat{k}$

Now $\left|\vec{b}_1\right| = \sqrt{2^2 + 5^2 + (-3)^2} = \sqrt{38}, \left|\vec{b}_2\right| = \sqrt{(-1)^2 + 8^2 + 4^2} = 9$

And,

$\vec{b}_1\vec{b}_2 = \left(2\hat{i} + 5\hat{j} - 3\hat{k}\right)\left(-\hat{i} + 8\hat{j} + 4\hat{k}\right)$

$= 2(-1) + 5(8) + 4(-3)$

=26

We can find the angle by using $\cos\theta = \left|\dfrac{\vec{b}_1\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

Therefore,

$\cos\theta = \dfrac{26}{9\sqrt{38}}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{26}{9\sqrt{38}}\right)$

Therefore, the angle will be $\cos^{-1}\left(\dfrac{26}{9\sqrt{38}}\right)$.

ii. $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z}{1}$ and $\dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$

Solution: Similarly let us consider $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to lines, we get

$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$

Now, $\left|\vec{b}_1\right| = \sqrt{2^2 + 2^2 + (1)^2} = 3, \left|\vec{b}_2\right| = \sqrt{4^2 + 1^2 + 8^2} = 9$ and

$\vec{b}_1\vec{b}_2 = \left(2\hat{i} + 2\hat{j} + 1\hat{k}\right).\left(4\hat{i} + \hat{j} + 8\hat{k}\right)$

$= 2(4) + 2(1) + 1(8)$

$= 18$

As we know the angle can be found by $\cos\theta = \left|\dfrac{\vec{b}_1\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

Therefore,

$\cos\theta = \dfrac{18}{17} = \dfrac{2}{3}$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Therefore, the angle is $\cos^{-1}\left(\frac{2}{3}\right)$

12. We needed to find the values of p so the line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{2}$ are at right angles.

Solution: As we know that the correct form of the equation is as follows,

$$\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2} \text{ and } \frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

From this we get the direction ratios as

$$a_1 = -3, b_1 = \frac{2p}{7}, c_1 = 2 \text{ and } a_2 = \frac{-3p}{7}, b_2 = 1, c_2 = -5$$

As we know the lines are perpendicular, we get

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} = 10$$

$$\Rightarrow 11p = 70$$

$$\Rightarrow p = \frac{70}{11}$$

Therefore, the value of p is $\frac{70}{11}$.

13. We needed to show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

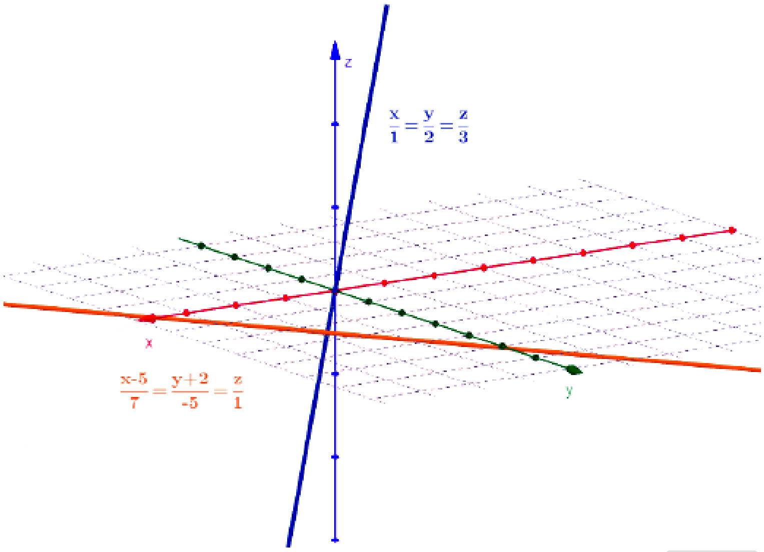
Solution: From the given equation, we get the direction ratios as,

$$a_1 = 7, b_1 = -5, c_1 = 1, a_2 = 1, b_2 = 2, c_2 = 3$$

As we know, if $a_1a_2 + b_1b_2 + c_1c_2 = 0,$ the lines are perpendicular to each other

Now,

$$7(1) + (-5)2 + 1(3) \Rightarrow 7 - 10 + 3 = 0$$



Therefore, the lines are perpendicular.

14. If the lines are $\vec{r}=\hat{i}+2\hat{j}+\hat{k}+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$, find the shortest distance between them.

Solution: We have been given lines, $\vec{r}=\hat{i}+2\hat{j}+\hat{k}+\lambda\left(\hat{i}-\hat{j}-\hat{k}\right)$ and

$$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

As we know that the shortest distance can be found as $d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$

Now, from the given lines we get that

$$\vec{a}_1=\hat{i}+2\hat{j}+\hat{k},$$

$$\vec{b}_1=\hat{i}-\hat{j}-\hat{k}$$

$$\vec{a}_2=2\hat{i}-\hat{j}-\hat{k},$$

$$\vec{b}_2=2\hat{i}+\hat{j}+2\hat{k}$$

$$\vec{a}_2-\vec{a}_1=\left(2\hat{i}-\hat{j}-k\right)-\left(\hat{i}+2\hat{j}+\hat{k}\right)$$

$$=\hat{i}-3\hat{j}-2\hat{k},$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & -1 & 2\end{vmatrix}$$

$\Rightarrow \vec{b}_1 \times \vec{b}_2 = -3\hat{i} + 3\hat{k}$

Then, $\left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$

Now, if we put all the values in theirs places, we get

$$d = \left|\frac{\left(-3\hat{i} + 3\hat{k}\right)\left(\hat{i} - 3\hat{j} - 2\hat{k}\right)}{3\sqrt{2}}\right| \Rightarrow d = \left|\frac{-3(1) + 3(2)}{3\sqrt{2}}\right|$$

$$d = \left|\frac{-9}{3\sqrt{2}}\right| \Rightarrow d = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the lines is $\frac{3\sqrt{2}}{2}$ units.

15. Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Solution: As we know that the shortest distance can be found by,

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

Now, from the given lines we got that

$x_1 = -1, y_1 = -1, z_1 = -1, a_1 = 7, b_1 = -6, c_1 = 1$

$x_2 = 3, y_2 = 5, z_2 = 7, a_2 = 1, b_2 = -2, c_2 = 1$.

And,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$= 4(-6+2) - 6(1+7) + 8(-14+6)$

$= -16 - 36 - 64$

$= -116$

And,

$$\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}=\sqrt{(-6+2)^2+(1+7)^2+(-14+6)^2}$$

$$\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}=2\sqrt{29}$$

Putting all the values, we get

$$d=\frac{-116}{2\sqrt{29}}$$

$$d=\frac{-58}{\sqrt{29}}\Rightarrow\frac{-58\sqrt{29}}{29}$$

$$d=\frac{-58}{\sqrt{29}}\Rightarrow|d|=2\sqrt{29}$$

Therefore, the distance the distance between the lines is $2\sqrt{29}$ units.

16. Find the shortest distance between the lines whose vector equations are $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$ and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+,\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$

Solution: We have been given lines $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$ and

$$\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$$

As we know that the shortest distance between the lines can be found by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

Now, from the given lines, we got

$$\vec{a}_1=\hat{i}+2\hat{j}+3\hat{k},\vec{b}_1=\hat{i}-3\hat{j}+2\hat{k}$$

$$\vec{a}_2=4\hat{i}+5\hat{j}+6\hat{k},\vec{b}_2=2\hat{i}+3\hat{j}+\hat{k}$$

$$\vec{a}_2-\vec{a}_1=\left(4\hat{i}+5\hat{j}+6\hat{k}\right),\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$

$$=3\hat{i}+3\hat{j}+3\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-3&2\\2&3&1\end{vmatrix}$$

$$\Rightarrow\vec{b}_1\times\vec{b}_2=-9\hat{i}+3\hat{j}+9\hat{k}$$

$$\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(-9)^2+3^2+9^2}=3\sqrt{19}$$

Now, putting all the values, we get

$$d=\left|\frac{9}{3\sqrt{19}}\right|=\frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the lines is $\frac{3}{\sqrt{19}}$ units.

17. We needed to find the shortest distance between the lines whose vector equations are $\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and $\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$.

Solution: We have been given lines $\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and

$$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$$

$\Rightarrow \vec{r}=\hat{i}-2\hat{j}+3\hat{k}+t\left(-\hat{i}+\hat{j}-2\hat{k}\right)$ **and** $\vec{r}=\hat{i}-\hat{j}+\hat{k}+s\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

Now, the shortest distance can be found by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

Now, from the given lines we got,

$$\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k},\vec{b}_1=-\hat{i}+\hat{j}-2\hat{k},$$

$$\vec{a}_2=\hat{i}-\hat{j}-\hat{k},\vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$$

$$\vec{a}_2-\vec{a}_1=\left(\hat{i}-\hat{j}-\hat{k}\right),\left(\hat{i}+2\hat{j}+3\hat{k}\right)=\hat{j}-4\hat{k},$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & -2\\ 1 & 2 & -2\end{vmatrix}$$

$$\Rightarrow \vec{b}_1\times\vec{b}_2=2\hat{i}-4\hat{j}-3\hat{k},$$

$$\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(2)^2+(-4)^2+(-3)^2}=\sqrt{29}$$

$$\left(\vec{b}_1\times\vec{b}_2\right)\times\left(\vec{a}_2-\vec{a}_1\right)=\left(2\hat{i}-4\hat{j}-3\hat{k}\right)\left(\hat{j}-4\hat{k}\right)$$

$$=-4+12$$

$$=8$$

Putting all the values, we get

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Exercise: 11.3

1. Determine the direction cosines of the normal to the plane and the distance from the origin.

(a) z=2

Solution: It is given that equation of the plane is z=2

Now we can tell that the direction ratios are 0,0,1.

Which means $\sqrt{0+0+1^2}=1$

Now we will divide both sides of equation by 1, we get

$$0+0+\frac{z}{1}=2$$

Therefore, the direction cosines and distance of the plane is $(0,0,1)$ and 2 units respectively.

(b) $x+y+z=1$

Ans : $x+y+z=1$ is the equation of the normal

Now, from the equation given we can say that the direction ratios of normal are 1, 1 and 1.

Which means $\sqrt{1^2+1^2+1^2}=\sqrt{3}$

Now we will divide the equation by $\sqrt{3},$ we get

$$\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=\frac{1}{\sqrt{3}}$$

Therefore, the direction cosines and distance from the origin $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

And $\frac{1}{\sqrt{3}}$ units respectively.

(c). $2x+3y-z=5$

Solution: $2x+3y-z=5$ is the equation of the normal

Now, from the given equation we get the direction ratios of normal as 2, 3,-1.

Which means $\sqrt{2^2+3^2+(-1)^2}=\sqrt{14}$

Now, we will divide the equation by $\sqrt{14}$, we get

$$\frac{2}{\sqrt{14}}x+\frac{3}{\sqrt{14}}y-\frac{1}{\sqrt{14}}z=\frac{5}{\sqrt{14}}$$

Therefore, the direction cosines and the distance from the origin of the normal are $\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}}$ **and** $\frac{5}{\sqrt{14}}$ units respectively.

(d) $5y+8=0$.

Solution: $5y+8=0 \Rightarrow 0x+5y+0z=-8$ is the given equation

Now, from the equation we can tell that the direction ratios of normal are 0,-5 and 0.

Which means $\sqrt{0^2+(-5)^2+0^2}=5$

Now, we will divide the equation by 5, we get

$$-y=\frac{8}{5}$$

Therefore, the direction cosines and the distance from the origin of the normal are 0,-1,0 and $\frac{8}{5}$ units respectively.

2. Find the vector equation of plane which is at the distance of 7 units from the origin and the normal vector $3\hat{i}+5\hat{j}-6\hat{k}$

Solution: Let us consider the normal vector be $\vec{a}=3\hat{i}+5\hat{j}-6\hat{k}$

We know that $\hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{3^2+5^2+6^2}}=\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}$

As we know the equation of the plane with position vector is shown in form $\vec{r}.\hat{n}=d$, Therefore,

$$\Rightarrow \vec{r}.\left(\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\right)=7$$

Therefore, the vector equation is in the form $\vec{r}.\left(\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\right)=7$.

3. Find the Cartesian equation of planes

(a) $\vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=2$

Solution: We have been given equation of the plane as $\vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=2$

As we know, the position vector is $\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$

Putting the values of $\vec{r}$ in equation, we get

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right).\left(\hat{i}+\hat{j}-\hat{k}\right)=2$$

$$\Rightarrow x+y-z=2$$

Therefore, the cartesian equation will be $x+y-z=2$

(b) $\vec{r}.\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$

Solution: We have been given equation of the plane as $\vec{r}.\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$

As we know the position vector $\vec{r}$ is given by,

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Putting the values of $\vec{r}$ in equation, we get

$$(x\hat{i}+y\hat{j}-z\hat{k}).\left(2i+3\hat{j}-4\hat{k}\right)=1$$

$$\Rightarrow 2x+3y-4z=1$$

Therefore, the Cartesian equation will be $2x+3y-4z=1, x+y-z=2$

(c) $\vec{r}.\left(\left(s-2t\right)\hat{i}+\left(3-t\right)\hat{j}+\left(2s+t\right)\hat{k}\right)=15$

Solution: We have been given equation of the plane as

$$\vec{r}.\left((s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right)=15$$

As the position vector is in form as $\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$

Putting the values of $\vec{r}$ in equation, we get

$$(x\hat{i}+y\hat{j}-z\hat{k}).\left((s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right)=15$$

$$\Rightarrow (s-2t)x+(3-t)y+(2s+t)z=1$$

Therefore, the cartesian equation will be $(s-2t)x+(3-t)y+(2s+t)z=1$

4. Find the coordinates of the foot of the perpendicular drawn from the origin.

a) $2x+3y+4z-12=0$

Solution: Let us consider the coordinates of the foot be (x_1, y_1, z_1)

Now, we have been given equation as $2x+3y+4z-12=0$

As we can tell the direction ratios will be 2, 3 and 4.

Which means, $\sqrt{2^2+3^2+4^2}=\sqrt{29}$

Now, we will divide the equation by $\sqrt{29}$, we get

$$\frac{2}{\sqrt{29}}x+\frac{3}{\sqrt{29}}y+\frac{4}{\sqrt{29}}z=\frac{12}{\sqrt{29}}$$

The coordinates of the foot of the perpendicular will be,

$$\left(\frac{2}{\sqrt{29}}.\frac{12}{\sqrt{29}},\frac{3}{\sqrt{29}}.\frac{12}{\sqrt{29}},\frac{4}{\sqrt{29}}.\frac{12}{\sqrt{29}}\right)$$

$$\Rightarrow\left(\frac{24}{29},\frac{36}{49},\frac{48}{29}\right)$$

Therefore, the coordinates of the foot of the perpendicular will be $\left(\frac{24}{29},\frac{36}{49},\frac{48}{29}\right)$

(b) $3y+4z-6=0$

Solution: Let us take the coordinates of the foot of perpendicular be (x_1, y_1, z_1)

Now, we have been given equation as $3y+4z-6=0$

As we can tell the direction ratios will be 0,3 and 4.

Which means, $\sqrt{0+3^2+4^2}=5$

Now, we will divide the equation by 5, we get

$$0x+\frac{3}{5}y+\frac{4}{5}z=\frac{6}{5}$$

Now, the coordinates of the foot of the perpendicular will be,

$$\left(0,\frac{3}{5}.\frac{6}{5},\frac{4}{5}.\frac{6}{5}\right)$$

$$\Rightarrow\left(0,\frac{18}{25},\frac{24}{25}\right)$$

Therefore, the coordinates of the foot of the perpendicular will be $\left(0,\frac{18}{25},\frac{24}{25}\right)$

(c) $x+y+z=1$

Solution: Let us consider the coordinates of the foot of perpendicular be (x_1,y_1,z_1)

Now, we have been given equation $x+y+z=1$

As we can tell the direction ratios will be 1, 1 and 1.

Which means, $\sqrt{1^2+1^2+1^2}=\sqrt{3}$

Now, we will divide the equation by $\sqrt{3},$ we get

$$\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=\frac{1}{\sqrt{3}}$$

Now, the foot of the perpendicular will be,

$$\left(\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

Therefore, the coordinates of the foot of the perpendicular will be $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

(d) $\Rightarrow 5y+8=0\backslash$

Solution: Let us consider the coordinates of the foot of perpendicular be (x_1,y_1,z_1)

Now, we have been given equation as $5y+8=0\Rightarrow 0x-5y+0z=8$

Now, we can tell that the direction ratios will be 0,5 and 0.

Which means, $\sqrt{0+5^2+0}=5$

Now, we will divide the equation by 5, we get

$$-y=\frac{8}{5}$$

Therefore, the foot of the perpendicular will be $\left(0,(-1).\frac{8}{5},0\right)$

Therefore, the coordinates of the foot of the perpendicular is $\left(0,-\left(\frac{8}{5}\right),0\right)$.

5. Find the vector and Cartesian equation of the planes

(a) That passes through the point $(1,0,-2)$ and the normal to the plane is $\hat{i}+\hat{j}-\hat{k}$

Solution: Now, according to the question, the position vector of point $(1,0,-2)$ be $\vec{a}=\hat{i}-2\hat{k}$

Now, the normal vector $\vec{N}$ perpendicular to the plane will be $\vec{N}=\hat{i}+\hat{j}-\hat{k}$

Now, the vector equation of the plane will be in form $(\vec{r}-\vec{a}).\vec{N}=0$

$$\left[\vec{r}-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$

As, $\vec{r}$ is the positive vector of any point p(x,y,z), we get

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

Now,

$$\left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0\Rightarrow\left[(x-1)\hat{i}+y\hat{j}+(z+2)\hat{k}\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$

$$\Rightarrow(x-1)+y-(z-2)=0$$

$$\Rightarrow x+y-z=3$$

Therefore, the equation will be $x+y-z=3$.

(b) That passes through the point (1,4,6) and the normal to the plane is $\hat{i}-2\hat{j}+\hat{k}$

Solution: Now, according to question, the position vector of point vector of point (1, 4, 6) will be

$$\vec{a}=\hat{i}+4\hat{j}+6\hat{k}$$

As we know that the normal vector $\vec{N}$ perpendicular to the plane, $\vec{N}=\hat{i}-2\hat{j}+\hat{k}$

So, the vector equation of the plane is will be in form,

$$\left[\vec{r}-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$

As, $\vec{r}$ is the positive vector of any point p(x,y,z) in the plane,

Now,

$$\left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(i-2\hat{j}+\hat{k}\right)=0\Rightarrow\left[(x-1)\hat{i}+(y-4)\hat{j}+(z+6)\hat{k}\right].\left(\hat{i}-2\hat{j}=\hat{k}\right)=0$$

$$\Rightarrow(x-1)-2(y-4)+(z-6)=0$$

$$\Rightarrow(x-1)-2(y-4)+(z-6)=0$$

$$\Rightarrow x-2y+z+1=0$$

Therefore, the equation of the plane will be x-2y+z+1=0.

6. If the plane passes through the given points, Find the equations of the plane.

(a) $(1,1,-1),(6,4,-5),(-4,-2,3)$

Solution: Now, let us consider the points be $A(1,1,-2),B(6,4,-5),C(-4,-2,3)$

Now, $\begin{vmatrix}1 & 1 & -1\\ 6 & 4 & -5\\ -4 & -2 & 3\end{vmatrix}=(12-10)-(18-20)-(12+16)=2+2-4=0$

Therefore, A,B,C are collinear points,

The number of planes passing through will be infinite

(b) (1,1,0), (1, 2, 1), (-2,2,-1)

Solution: Now, let consider the points be $A(1,1,0)), B(1,2,1), C(-2,2,1)$

Now, $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = (-2-2)-(2+2) = -8 \neq 0$

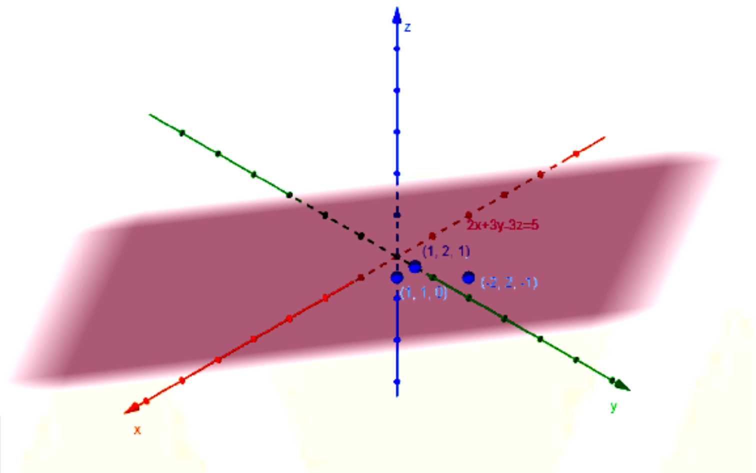
From this we get to know that a plane will pass through the points A, B, C

Now, the equation of the plane through the points will be,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (-2)(x-1)-3(y-1)+3z = 0 \Rightarrow -2x-3y+3z+2+3=0$$

$$\Rightarrow -2x-3y+3z=-5 \Rightarrow 2x+3y-3z=5$$



Therefore, the equation of the plane is $2x+3y-3z=5$.

7. Find the intercepts cut off by the plane $2x+y-z=5$

Solution: Now, the equation of the plane is given as $2x+y-z=5$

Now we will divide both sides by 5, we get intercepts,

$$\frac{2x}{5}+\frac{y}{5}-\frac{z}{5}=1 \Rightarrow \frac{x}{\frac{5}{2}}+\frac{y}{5}+\frac{z}{-5}=1$$

Now, as we know the equation of a plane in intercepts form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$ As

We can see that we got $a=\frac{5}{2}, b=5, c=-5$ as the intercepts of the plane.

Therefore, the intercepts of the plane will be $\frac{5}{2}, 5$ and 5.

8. Find the equation of the plane parallel to ZOX plane and having intercept 3 on the y-axis.

Solution: We have been given plane ZOX with intercept 3

As we know, if the plane is parallel to the equation, it will be in the form y=a

Since the y-intercept of the plane is 3, we get

A=3

Therefore, the equation of the required plane is y=3.

9. Find the equation of the plane through the point (2,2,1) and the intersection of the plane $3x-y+2z-4=0$ and $x+y+z-2=0$

Solution: As it's given that the equation of the plane pass through the intersection of the planes $3x-y+2z-4=0$ and $x+y+z-2=0$, and passes through the point (2, 2, 1)

We know that $(3x-y+2z-4)+\alpha(x+y+z-2)=0, \alpha \in R$

Therefore, $(3\times2-2+2\times1-4)+\alpha(2+2+1-2)=0$

$\Rightarrow 2+3\alpha=0 \Rightarrow \alpha=-\frac{2}{3}$

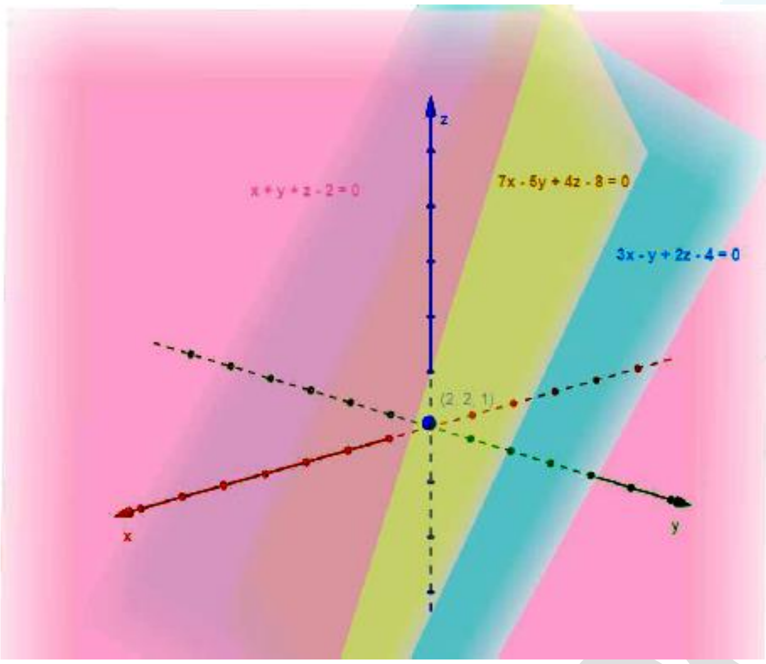
Now, putting $\alpha=-\frac{2}{3}$, we get

$(3x-y+2z-4)-\frac{2}{3}(x+y+z-2)=0$

$\Rightarrow 3(3x-y+2z-4)-2(x+y+z-2)=0$

$\Rightarrow (6x-3y+6z-12)-2(x+y+z-2)=0$

$\Rightarrow 7x-5y+4z-8=0$



Therefore, the equation of the plane will be $7x-5y+4z-8=0$

10. Find the vector equation of the plane passing through the point (2, 1, 3) and the intersection of the planes $\vec{r}.\left(2\hat{i}+\hat{j}2-3\hat{k}\right)=7, \vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$

Solution: It is given that the equations for planes passes through (2, 1, 3) and the intersection of the given planes,

$$\Rightarrow \vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7=0 \text{ and } \vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9=0$$

Now, the equation of the required plane will be

$$\left[\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7\right]+\lambda\left[\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9\right]=0, \lambda\in R$$

$$\vec{r}.\left[\left(2\hat{i}+2\hat{j}-3\hat{k}\right)+\lambda\left(2\hat{i}+5\hat{j}+3\hat{k}\right)\right]=9\lambda+7$$

$$\vec{r}.\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}\right]=9\lambda+7$$

As plane passes through (2, 1, 3), the position vector will be,

$$\vec{r}=2\hat{i}+2\hat{j}+-3\hat{k}$$

Putting this in equation $\vec{r}.\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}\right]=9\lambda+7$ we get,

$$\left(2\hat{i}+2\hat{j}+-3\hat{k}\right).\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}\right]=9\lambda+7$$

$$\Rightarrow(2+2\lambda)+(2+5\lambda)+(3\lambda-3)=9\lambda+7$$

$\Rightarrow 18\lambda - 3 = 9\lambda + 7$

$\Rightarrow 9\lambda = 10 \Rightarrow \lambda = \frac{10}{9}$

After putting this value in equation $\vec{r}.\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}\right]=9\lambda+7$ we

Will get,

$\vec{r}.\left[\frac{38}{9}\hat{i}+\frac{68}{9}\hat{j}+\frac{3}{9}\hat{k}\right]=17$

$\Rightarrow \vec{r}.\left[38\hat{i}+68\hat{j}+3\hat{k}\right]=153$

Therefore, the vector equation will be $\vec{r}.\left[38\hat{i}+68\hat{j}+3\hat{k}\right]=153$

11. Find the equation of the plane perpendicular to the plane $x-y+z=0$.

And through the line of intersection of the plane $x+y+z=1$ and $2x+3y+4z=5$

Solution: It is given that the equation of the plane is perpendicular to $x-y+z=0$ and pass through the intersection of planes, we got

$(x+y+z-1)+\lambda(2x+3y+4z-5)=0$

$\Rightarrow (2\lambda+1)x+(3\lambda+1)y+(4\lambda+1)z-(5\lambda+1)=0$

From this we can tell that $a_1=(2\lambda+1), b_1(3\lambda+1), c_1=(4\lambda+1)$

Now, according to the question the plane is perpendicular to $x-y+z=0$

We got, $a_2=1, b_2=-1, c_2=1$

We know that if planes are perpendicular, then,

$a_1a_2+b_1b_2+c_1c_2=0$

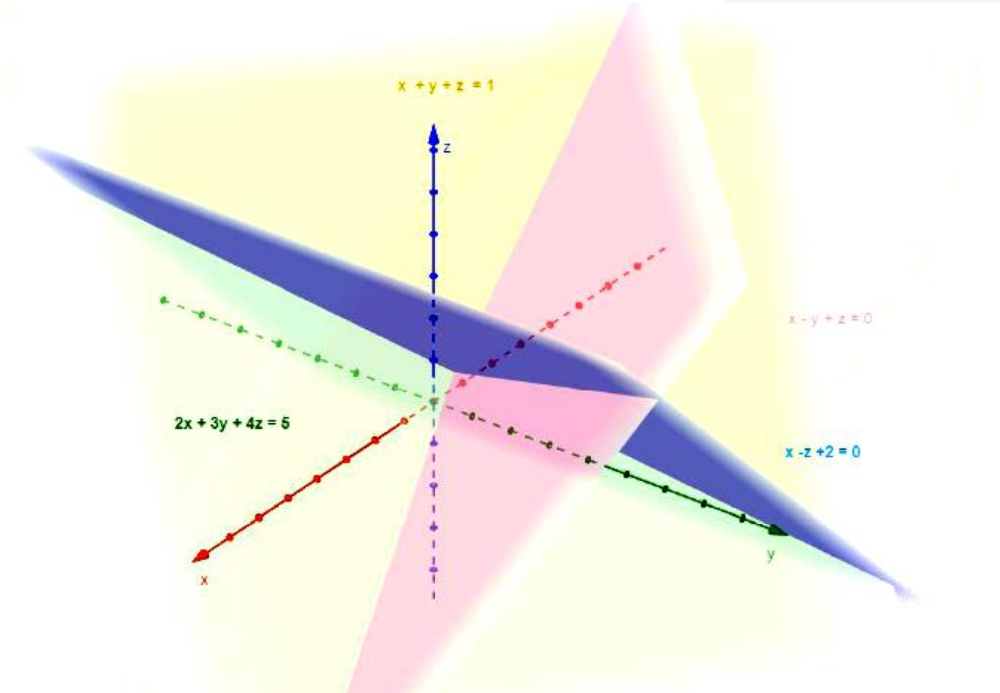
$\Rightarrow (2\lambda+1)-(3\lambda+1)+(4\lambda+1)=0$

$\Rightarrow 3\lambda+1=0 \Rightarrow \lambda=-\frac{1}{3}$

By putting the value in equation $(2\lambda+1)x+(3\lambda+1)y+(4\lambda+1)z-(5\lambda+1)=0,$ we got

$\frac{1}{3}x+\frac{1}{3}z+\frac{2}{3}=0$

$\Rightarrow x-z+2=0$



Therefore, the equation of the plane will be $x-z+2=0$

12. For these vectors equations of planes $\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$, find the angle between them.

Solution: According to the question we have been given two equation of planes,

$\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$

Now, we know that if $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes, then,

$\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$,

As we know, we can find the angle by $\cos\theta=\left|\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$

We got,

$\vec{n}_1=2\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{n}_2=3\hat{i}-3\hat{j}+5\hat{k}$

$\therefore \vec{n}_1.\vec{n}_2=\left(2\hat{i}+2\hat{j}-3\hat{k}\right)\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=2.3+2(-3)+(-3)5=-15,$

$|\vec{n}_1|=\sqrt{2^2+2^2+(-3)^2}=\sqrt{17}$

$$|\vec{n}_2| = \sqrt{3^2 + (-3)^2 + (5)^2} = \sqrt{43}$$

Now, by putting all these values in equation $\cos\theta = \left|\dfrac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$

$$\cos\theta = \left|\frac{-15}{\sqrt{17}\sqrt{43}}\right|$$

$$\Rightarrow \cos\theta = \frac{15}{\sqrt{731}} \Rightarrow \theta = \cos^{-1}\left[\frac{15}{\sqrt{731}}\right]$$

Therefore, the angle between them is $\theta = \cos^{-1}\left[\dfrac{15}{\sqrt{731}}\right]$.

13. Determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

Solution: We know that the direction ratios of normal to the plane are a_1, b_1, c_1 and a_2, b_2, c_2,

We know that if lines are parallel then, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ and if lines are

Perpendicular then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

The angle between the planes can be found by, $\theta = \cos^{-1}\left|\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

Solution: The equations are given as $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

From the equations we got $a_1 = 7, b_1 = 5, c_1 = 6$ and $a_2 = 3, b_2 = -1, c_2 = -10$

Now we will check whether the planes are perpendicular or parallel, then

Now, $a_1a_2 + b_1b_2 + c_1c_2 = 7\times3 + 5\times(-1) + 6\times(-10) = -44 \neq 0$

Therefore, the planes are not perpendicular.

Now, $\frac{a_1}{a_2}=\frac{7}{3}, \frac{b_1}{b_2}=\frac{5}{-1}, \frac{c_1}{c_2}=\frac{-3}{5}$

It can be seen that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

The angle between them is given by

$$\theta=\cos^{-1}\left|\frac{7\times 3+5\times(-1)+6\times(-10)}{\sqrt{7^2+5^2+6^2}\sqrt{3^2+(-1)^2+(-10)^2}}\right|$$

$$\theta=\cos^{-1}\frac{44}{110}=\cos^{-1}=\frac{2}{5}$$

(b) 2x+y+3z-2=0 and x-2y+5=0

Solution: The equations are given as $2x+y+3z-2=0$ and $x-2y+5=0$

From this we got, $a_1=2, b_1=1, c_1=3$ and $a_2=1, b_2=2, c_2=0$

Now, $a_1a_2+b_1b_2+c_1c_2=2\times 1+1\times(-2)+3\times 0\neq 0$

Therefore, the planes are not perpendicular to each other.

Now, $\frac{a_1}{a_2}=\frac{2}{1}, \frac{b_1}{b_2}=\frac{1}{2}, \frac{c_1}{c_2}=\frac{-3}{0}$

It can be seen that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Now, the angle between them will be

$$\theta=\cos^{-1}\left|\frac{2\times 1+1\times(2)+3\times(0)}{\sqrt{2^2+1^2+3^2}\sqrt{1^2+(2)^2+0^2}}\right|$$

$$\theta=\cos^{-1}\frac{4}{\sqrt{70}}=\cos^{-1}=\frac{2}{5}$$

(c) $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$

Solution: The equations are given as $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$

From this we got, $a_1=2, b_1=-2, c_1=4$ and $a_2=3, b_2=-3, c_2=6$

Now, $a_1a_2+b_1b_2+c_1c_2=2\times3+(-2)\times(-3)+4\times(6)=36\neq0$

Therefore, the given planes are not perpendicular.

Now, $\frac{a_1}{a_2}=\frac{2}{3},\frac{b_1}{b_2}=\frac{2}{3},\frac{c_1}{c_2}=\frac{2}{3}$

$\therefore \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Therefore, the given planes are parallel to each other.

(d) $2x-y+3z-1=0$ and $2x-y+3z+3=0$

Solution: The equations are given as $2x-y+3z-1=0$ and $2x-y+3x+3=0$

From this we got, $a_1=2,b_1=-1,c_1=3$ and $a_2=2,b_2=-1,c_2=3$

Now, as we can see $\frac{a_1}{a_2}=\frac{2}{2}=1,\frac{b_1}{b_2}=\frac{-1}{-1}=1,\frac{c_1}{c_2}=\frac{-1}{1-}=1$

$\therefore \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Therefore, the given planes are parallel to each other.

(e) $4x+8y+z-8=0$ and $y+z-4=0$

Solution: The equations are given as $4x+8y+z-8=0$ and $y+z-4=0$

From this we get, $a_1=4,b_1=8,c_1=1$ and $a_2=0,b_2=1,c_2=1$

Now, $a_1a_2+b_1b_2+c_1c_2=4\times0+8\times(1)+1=9\neq0$

Therefore, the given planes are not perpendicular.

Now, $\frac{a_1}{a_2}=\frac{4}{0},\frac{b_1}{b_2}=\frac{8}{1}=8,\frac{c_1}{c_2}=\frac{1}{1}=1$

It can be seen that $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

Therefore, the angle between them will be,

$$\theta=\cos^{-1}\left|\frac{4\times0+8\times(1)+1\times1}{\sqrt{4^2+8^2+1^2}\sqrt{0+(1)^2+(1)^2}}\right|$$

$$\theta=\cos^{-1}\frac{9}{9\sqrt{2}}=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=45^0$$

Therefore, the angle between them is $45°$.

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

Solution: The distance between a point and a plane is given by,

$$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|$$

(a) $(0,0,0)3x-4y+12z=3$

Ans: The given point is (0, 0, 0) and the plane is $3x-4y+12z=3$

Now the distance will be,

$$d=\left|\frac{3\times0-4\times0+12\times0-3}{\sqrt{(-3)^2+(-4)^2+12^2}}\right|=\frac{3}{\sqrt{169}}=\frac{3}{13}$$

Therefore, the distance will be $\frac{3}{13}$.

(b) $(3,-2,1)2x-y+2z+3=0$

Ans: The given point is $(3,-2,1)$ and the plane is $2x-y+2z+3=0$

Now the distance will be $d=\left|\frac{2\times3-(-2)2\times1+3}{\sqrt{(2)^2+(-1)^2+2^2}}\right|=\left|\frac{13}{3}\right|=\frac{13}{3}$

Therefore, the distance will be $\frac{3}{13}$.

(c) $(2,3,-5)x+2y-2z=9$

Ans: The given point is (2, 3, -5) and the plane is x+2y-2z=9

Now the distance will be,

$$d=\left|\frac{2+2\times 3-2\times(-5)-3}{\sqrt{(1)^2+(2)^2+(-2)^2}}\right|=\frac{9}{3}=3$$

Therefore, the distance will be 3.

(d) (-6,0,0) 2x-3y+6z-2=0)

Ans: The given point is (-6, 0, 0) and the plane is 2x-3y+6z-2=0

Now, the distance will be,

$$d=\left|\frac{2\times(-6)-3\times 0+6\times 0-2}{\sqrt{(2)^2+(-3)^2+6^2}}\right|=\left|\frac{-14}{\sqrt{49}}\right|=\frac{14}{7}=2$$

Therefore, the distance will be $\frac{3}{13}$.