Chapter 2: Relations and Functions

Miscellaneous Exercises

Question 1. The relation f is defined by $f(x)=\begin{cases}x^2, 0\le x\le 3\\ 3x, 3\le x\le 10\end{cases}$

The relation g is defined by $g(x)=\begin{cases}x^2, 0\le x\le 2\\ 3x, 2\le x\le 10\end{cases}$. Show that f is a function and g is not a function.

Solution : Here, the relation f is defined as $f(x)=\begin{cases}x^2, 0\le x\le 3\\ 3x, 3\le x\le 10\end{cases}$.

It is observed that for,

$0\le x<3,\quad f(x)=x^2$
$3<x\le 10,\quad f(x)=3x$

Also, at $x=3,\ f(x)=3^2=9$

Or $f(x)=3\times 3=9$ that is, at $x=3, f(x)=9$.

Thus, for $0\le x\le 10$, the images of $f(x)$ are unique.

Therefore, the given relation is a function.

The relation g is defined as $g(x)=\begin{cases}x^2, 0\le x\le 2\\ 3x, 2\le x\le 10\end{cases}$.

It can be observed that for, $x=2,\ g(x)=2^2=4$ and $g(x)=3\times 2=6$.

Thus, element 2 of the domain of the relation g corresponds to two different images that is, 4 and 6.

Therefore, this relation is not a function.

Question 2. If $f(x)=x^2$, find. $\dfrac{f(1.1)-f(1)}{(1.1-1)}$

Solution : Given that, $f(x)=x^2$.

Now, $\dfrac{f(1.1)-f(1)}{(1.1-1)}=\dfrac{(1.1)^2-(1)^2}{(1.1-1)}$

Simplify the squares,

$= \frac{1.21-1}{0.1}$

Subtract the numbers,

$= \frac{0.21}{0.1}$

Divide the numbers,

$= 2.1$

Therefore, $\frac{f(1.1)-f(1)}{(1.1-1)} = 2.1$.

Question 3. Find the domain of the function $f(x) = \frac{x^2+2x+1}{x^2-8x+12}$.

Solution : Here, the function is $f(x) = \frac{x^2+2x+1}{x^2-8x+12}$.

Simplify $x^2-8x+12$,

$$f(x) = \frac{x^2+2x+1}{x^2-8x+12} = \frac{x^2+2x+1}{(x-6)(x-2)}$$

It can be observed that function f is defined for all real numbers except at $x = 6$ and $x = 2$.

Therefore, the domain of f is $\mathbf{R}-\{2,6\}$.

Question 4. Find the domain and the range of the real function f defined by $f(x) = \sqrt{(x-1)}$

Solution : Here, the real function is $f(x) = \sqrt{(x-1)}$.

It can be observed that $\sqrt{(x-1)}$ is defined for $x \geq 1$.

Thus, the domain of f is the set of all real numbers greater than 0 equal to 1.

That is, the domain of $f = [1, \infty)$.

As $x \geq 1 \Rightarrow (x-1) \geq 0 \Rightarrow \sqrt{(x-1)} \geq 0$

Therefore, the range of f is the set of all real numbers greater than or equal to 0 . Therefore, the range of $f = [0, \infty)$.

Question 5. Find the domain and the range of the real function f defined by $f(x) = |x-1|$.

Solution : Here, the real function is $f(x)=|x-1|$.

It is clear that $|x-1|$ is defined for all real numbers.

Thus, Domain of $f=\mathbf{R}$.

And for $x\in\mathbf{R}, |x-1|$ assumes all real numbers.

Therefore, the range of f is the set of all non-negative real numbers.

Question 6. Let $f=\left\{\left(x,\frac{x^2}{1+x^2}\right): x\in\mathbf{R}\right\}$ be a function from $\mathbf{R}$ into $\mathbf{R}$. Determine the range of f.

Solution : Here,

$$f=\left\{\left(x,\frac{x^2}{1+x^2}\right): x\in\mathbf{R}\right\}=\left\{(0,0),\left(\pm0.5,\frac{1}{5}\right),\left(\pm1,\frac{1}{2}\right),\left(\pm1.5,\frac{9}{13}\right),\left(\pm2,\frac{4}{5}\right),\left(3,\frac{9}{10}\right),\left(4,\frac{16}{17}\right),\ldots\right\}$$

The range of f is the set of all second elements.

It can be seen that all these elements are greater than or equal to 0 but less than 1. Therefore, range of $f=[0,1)$.

Question 7. Let $f,g:\mathbf{R}\to\mathbf{R}$ be defined, respectively by $f(x)=x+1, g(x)=2x-3$. Find $f+g, f-g$ and $\frac{f}{g}$.

Solution : Here, $f,g:\mathbf{R}\to\mathbf{R}$ is defined as $f(x)=x+1, g(x)=2x-3$.

$(f+g)(x)=f(x)+g(x)=(x+1)+(2x-3)=3x-2$

Thus, $(f+g)(x)=3x-2$

$(f-g)(x)=f(x)-g(x)=(x+1)-(2x-3)=x+1-2x+3=-x+4$

Thus, $(f-g)(x)=-x+4$

Now, $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}, g(x)\neq 0, x\in\mathbf{R}$

$\left(\frac{f}{g}\right)(x)=\frac{x+1}{2x-3}, 2x-3\neq 0$ or $2x\neq 3$

Therefore, $\left(\frac{f}{g}\right)(x)=\frac{x+1}{2x-3}, x\neq\frac{3}{2}$

Question 8. Let $f=\{(1,1),(2,3),(0,-1),(-1,-3)\}$ be a function from $\mathbf{Z}$ to $\mathbf{Z}$ defined by $f(x)=ax+b$, for some integers a,b. Determine a,b.

Solution : Given that, $f=\{(1,1),(2,3),(0,-1),(-1,-3)\}$ and $f(x)=ax+b$.

Consider the pairs $(1,1)$ and $(0,-1)$,

$(1,1)\in f\Rightarrow f(1)=1\Rightarrow a\times 1+b=1\ \Rightarrow a+b=1$

$(0,-1)\in f\Rightarrow f(0)=-1\Rightarrow a\times 0+b=-1\Rightarrow b=-1$

Substitute $b=-1$ in $a+b=1$,

That is, $a+(-1)=1\Rightarrow a=1+1=2$.

Therefore, the respective values of a and b are 2 and -1.

Question 9. Let R be a relation from $\mathbf{N}$ to $\mathbf{N}$ defined by $R=\{(a,b):a,b\in\mathbf{N}$ and $a=b^2\}$. Are the following true?

(i) $(a,a)\in R$, for all $a\in\mathbf{N}$

(ii) $(a,b)\in R$, implies $(b,a)\in R$

(iii) $(a,b)\in R,(b,c)\in R$ implies $(a,c)\in R$.

Justify your answer in each case.

Solution : Given that, $R=\{(a,b):a,b\in\mathbf{N}$ and $a=b^2\}$.

(i) It can be observed that $2\in\mathbf{N}$; but $2\neq 2^2=4$.

Thus, the statement "$(a,a)\in R$, for all $a\in\mathbf{N}$" is not true.

(ii) It can be observed that $(9,3)\in\mathbf{N}$ because $9,3\in\mathbf{N}$ and $9=3^2$.

Then, $3\neq 9^2=81$

Thus, $(3,9)\notin\mathbf{N}$

Therefore, the statement " $(a,b)\in R$, implies $(b,a)\in R$" is not true.

(iii) It can be observed that $(9,3)\in R,(16,4)\in R$ because $9,3,16,4\in\mathbf{N}$ and $9=3^2$ also $16=4^2$.

Now, $9\neq 4^2=16$; therefore, $(9,4)\notin\mathbf{N}$

Therefore, the statement " $(a,b)\in R,(b,c)\in R$ implies $(a,c)\in R$" is not true.

Question 10. Let $A=\{1,2,3,4\},B=\{1,5,9,11,15,16\}$ and $f=\{(1,5),(2,9),(3,1),(4,5),(2,11)\}$ Are the following true?

(i) f is a relation from A to B.

(ii) f is a function from A to B.

Justify your answer in each case.

Solution : Here, $A=\{1,2,3,4\}$ and $B=\{1,5,9,11,15,16\}$.

Now,

$$A\times B=\{(1,1),(1,5),(1,9),(1,11),(1,15),(1,16),(2,1),(2,5),(2,9),(2,11),$$
$$(2,15),(2,16),(3,1),(3,5),(3,9),(3,11),(3,15),(3,16),(4,1),(4,5),(4,9),,$$
$$(4,11),(4,15),(4,16)\}$$

It is given that $f=\{(1,5),(2,9),(3,1),(4,5),(2,11)\}$.

(i) A relation from a non-empty set A to a non-empty set B is a subset of the Cartesian product $A\times B$.

It is seen that f is a subset of $A\times B$.

Therefore, f is a relation from A to B.

(ii) Because the same first element that is, 2 corresponds to two different images 9 and 11, relation f is not a function.

Question 11. Let f be the subset of $\mathbf{Z}\times\mathbf{Z}$ defined by $f=\{(ab,a+b):a,b\in\mathbf{Z}\}$. Is f a function from $\mathbf{Z}$ to $\mathbf{Z}$? justify your answer.

Solution : Here, the relation f is defined as $f=\{(ab,a+b):a,b\in\mathbf{Z}\}$.

It is known that a relation f from a set A to B is said to be a function if every element of set A has unique images in set B.

Since $2,6,-2,-6\in\mathbf{Z},(2\times 6,2+6),(-2\times -6,-2+\ (-6))\in f$

That is, $(12,8),(12,-8)\in f$

It can be observed that the same first element 12 corresponds to two different images 8 and -8.

Therefore, relation f is not a function.

Question 12. Let $A=\{9,10,11,12,13\}$ and let $f:A\to\mathbf{N}$ be defined by $f(n)=$ the highest prime factor of n. Find the range of f.

Solution : Here, $A = \{9, 10, 11, 12, 13\}$ and $f : A \to \mathbf{N}$ is defined as $f(n) =$ The highest prime factor of n.

Determine the prime factor of each number,

Prime factor of $9 = 3$

Prime factors of $10 = 2, 5$

Prime factor of $11 = 11$

Prime factors of $12 = 2, 3$

Prime factor of $13 = 13$

Determine the highest prime factor of each number,

$f(9) =$ The highest prime factor of $9 = 3$

$f(10) =$ The highest prime factor of $10 = 5$

$f(11) =$ The highest prime factor of $11 = 11$

$f(12) =$ The highest prime factor of $12 = 3$

$f(13) =$ The highest prime factor of $13 = 13$

The range of f is the set of all $f(n)$, where $n \in A$.

Therefore, Range of $f = \{3, 5, 11, 13\}$.