Chapter 4: Determinants

Exercise. Miscellaneous

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of

Solution:

Given, $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

$$= x\left(x^2 - 1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$$

$$= x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$$= x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right)$$

$$= x^3 - x + x$$

$$= x^3$$

Therefore, Δ is independent of θ.

2. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Solution:

L.H.S $= \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Applying $R_1 = aR_1, R_2 \to bR_2 \; and R_3 \to cR_3$

$$=\frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

$$=\frac{1}{abc}abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \qquad \text{[Taking out abc from } C_3\text{]}$$

$$=\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

Applying $C_1 \leftrightarrow C_3$ *and* $C_2 \leftrightarrow C_3$

$$=\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

=R.H.S

Hence proved

3. Evaluate $=\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\cos\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Solution:

Given, $\Delta=\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\cos\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along C_3

$$\Delta=-\sin\alpha\left(-\sin\alpha\sin^2\beta+\cos^2\beta\sin a\right)+\cos\alpha\left(\cos\alpha\cos^2\beta+\cos\alpha\sin^2\beta\right)$$

$$=\sin^2\alpha(1)+\cos^2\alpha(1)$$

$$=1$$

4. If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Show that either $a+b+c=0 \; or \; a=b=c$

Solution:

Given, $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1 \; and \; C_3 \to C_3 - C_1$

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1

$$\Delta = 2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right]$$

$$= 2(a+b+c)\left[-b^2 - c^2 + 2bc - bc + ba + ac - a^2\right]$$

$$= 2(a+b+c)\left[ab + bc + ca - a^2 - b^2 - c^2\right] = 0$$

$$\Rightarrow Either \; a+b+c=0, or \; ab+bc+ca-a^2-b^2-c^2=0$$

Now, $ab+bc+ca-a^2-b^2-c^2=0$ b

$$\Rightarrow -2ab-2bc-2ca+2a^3+2b^3+2c^3=0$$

$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$

$$\Rightarrow (a-b)=(b-c)^2=(c-a)^2=0 \quad \left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$$

$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$

$$\Rightarrow a=b=c$$

Therefore, if $\Delta=0$, then either $a+b+c=0$ or $a=b=c$

5. Solve the equations $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0, a\neq 0$

Solution:

Given, $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$

Applying $R_1 \to R_1+R_2+R_3$

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & a & x \\ x & x & a \end{vmatrix}=0$$

Expanding along R_1

$$\Rightarrow a^2(3x+a)=0$$

But $a \neq 0$

Therefore, we have

$$3x+a=0$$

$$\Rightarrow x=-\frac{a}{3}$$

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

Given, $\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking out a, b and c from C_1, C_2 and C_3

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

$$= 2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$

$$\Delta = 2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3

$$\Delta = 2ab^2c\left[a(c-a)+a(a+c)\right]$$

$$= 2ab^2c\left[ac-a^2+a^2+ac\right]$$

$$= 2ab^2c(2ac)$$

$$= 4a^2b^2c^2$$

Hence proved

7. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ verify that

(i) $[adjA]^{-1} = adj\left(A^{-1}\right)$

(ii) $\left(A^{-1}\right)^{-1} = A$

Solution:

Given, $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

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$$\therefore |A| = 1(15-1)+2(-10-1)+1(-2-3) = 14-22-5 = -13$$

Now, $A_{11} = 14, A_{12} = 11, A_{13} = -5$

$A_{21} = 11, A_{22} = 4, A_{23} = -3$

$A_{31} = -5, A_{32} = -3, A_{33} = -1$

$$\therefore adjA = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA)$$

$$= -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

$$(i)\ |adjA| = 14(-4-9)-11(-11-15)-5(-33+20)$$

$$= 14(-13)-11(-26)-5(-13)$$

$$= -182+286+65 = 169$$

Here, $adj(adjA) = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$

$$\therefore [adjA]^{-1} = \frac{1}{|adjA|}(adj(adjA))$$

$$= \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & 65 \end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Now, $A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$

$$=\begin{bmatrix}-\frac{14}{13} & -\frac{11}{13} & \frac{5}{13}\\ -\frac{11}{3} & -\frac{4}{13} & \frac{3}{13}\\ \frac{5}{13} & \frac{3}{13} & \frac{1}{3}\end{bmatrix}$$

$$\therefore adj\left(A^{-1}\right)=\begin{bmatrix}-\frac{4}{169}-\frac{9}{169} & -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{33}{169}+\frac{20}{169}\\ -\left(-\frac{11}{69}-\frac{15}{169}\right) & -\frac{14}{169}-\frac{25}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right)\\ -\frac{33}{169}+\frac{20}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right) & \frac{56}{169}-\frac{121}{169}\end{bmatrix}$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Therefore, $[adjA]^{-1}=adj\left(A^{-1}\right)$

(ii) Since, $A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$

And $adjA^{-1}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$

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Now, $\left|A^{-1}\right|=\left(\frac{1}{13}\right)^3\left[-14\times(-13)+11\times(-26)+5\times(-13)\right]=\left(\frac{1}{13}\right)^3\times(-169)=-\frac{1}{13}$

$$\therefore\left(A^{-1}\right)=\frac{adjA^{-1}}{\left|A^{-1}\right|}$$

$$=\frac{1}{\left(-\frac{1}{13}\right)}\times\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

$$=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$$

= A

$$\therefore\left(A^{-1}\right)^{-1}=A$$

8. Evaluate $\begin{vmatrix}x & y & x+y\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$

Solution:

Given, $\Delta=\begin{vmatrix}x & y & x+y\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$

Applying $R_1\to R_1+R_2+R_3$

$$\Delta=\begin{vmatrix}2(x+y) & 2(x+y) & 2(x+y)\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$$

$$=2(x+y)\begin{vmatrix}1 & 1 & 1\\ y & x+y & x\\ x+y & x & y\end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along R_1

$$\Delta = 2(x+y)\left[-x^2 + y(x-y)\right]$$

$$= -2(x+y)(x^2+y^2-yx)\text{b}$$

$$= -2(x^3+y^3)$$

9. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Solution:

Given, $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0` \\ 0 & 0 & x \end{vmatrix}$$

Expanding along C_1

$$\Delta = 1(xy - 0)$$

= xy

10. Using properties of determinants, prove that

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\alpha \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Solution:

Given, $\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\alpha \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta \\ \gamma-\alpha & \gamma^2-\alpha^2 & \alpha-\gamma \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 1 & \gamma+\alpha & -1 \end{vmatrix}$$

Appling $R_3 \to R_3 - R_2$

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 0 & \gamma-\beta & 0 \end{vmatrix}$$

Expanding along R_3

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\left[-(\gamma-\beta)(-\alpha-\beta-\gamma)\right]$$

$$= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)(\alpha+\beta+\gamma)$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

Hence proved

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11. Using properties of determinants, prove that

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

Solution:

Given, $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y-x & y^2-x^2 & p\left(y^3-x^3\right) \\ z-x & z^2-x^2 & p\left(z^3-x^3\right) \end{vmatrix}$$

$$=(y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^2 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 1 & z+x & p\left(z^2+x^2+xz\right) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\Delta = (y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix}$$

$$=(y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

Expanding along R_3

$$\Delta = (x-y)(y-z)(z-x)\left[(-1)(p)\left(xy^2+x^3+x^2y\right)+1+px^3+p(x+y+z)(xy)\right]$$

$$=(x-y)(y-z)(z-x)\left[-pxy^2-px^3-px^2y+1+px^3+px^2y+pxy^2+pxyz\right]$$

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$$=(x-y)(y-z)(z-x)(1+pxyz)$$

Hence proved

12. Using properties of determinants, prove that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+ba+ca)$$

Solution:

Given, $\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

Expanding along C_1

$$\Delta = (a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right]$$

$$=(a+b+c)\left[4bc+2ab+2ac+a^2-a^2+ac+ba-bc\right]$$

$$=(a+b+c)(3ab+3bc+3ac)$$

$=3(a+b+c)(ab+bc+ac)$

Hence proved

13. Using properties of determinants, prove this $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}=1$

Solution:

Given, $\Delta=\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

Applying $R_2 \to R_1-2R_1$ *and* $R_3 \to R_3-3R_1$

$$\Delta=\begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

Applying $R_3 \to R_3-3R_2$, we have

$$\Delta=\begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1

$$\Delta=1\begin{vmatrix} 1 & 2+p \\ 0 & 1 \end{vmatrix}$$

$=1(1-0)$

= 1

14. Using properties of determinants, prove that $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}=0$

Solution:

$$\text{Given, } \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$= \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Applying $C_1 \to +C_1 + C_3$

$$\Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \cos\alpha\cos\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \cos\beta\cos\delta & \cos\beta\cos\delta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \cos\gamma\cos\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Since, two columns C_1 *and* C_2 are identical

$\therefore \Delta = 0$

Hence proved.

15. Solve the system of the following equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} - \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Solution:

$$\text{Given, } \frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} + \frac{20}{z} = 2$$

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Let $\frac{1}{x}=p, \frac{1}{y}=q, \frac{1}{z}=r$

Then the given system of equations is as follows

$2p+3q+10r=4$

$4p-6q+5r=1$

$6p+9q+20r=1$

Let $A=\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X=\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Such that, this system can be written in the form of AX = B

Now, $|A|=2(120-45)-3(-80-30)+10(36+36)$

$=150+330+720$

$=1200$

Thus, A is non – singular

Therefore, its inverse exists.

Now, $A_{11}=75, A_{12}=110, A_{13}=72$

$A_{21}=150, A_{22}=100, A_{23}=0$

$A_{31}=75, A_{32}=100, A_{33}=-24$

$\therefore A^{-1}=\frac{1}{|A|}(adjA)$

$=\frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

Now, $X=A^{-1}B$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$\therefore\ p = \frac{1}{2}, \text{q} = \frac{1}{3}\ and\ r = \frac{1}{5}$

Thus, x =2, y=3 and z = 5

16. Choose the correct answer.

If a,b, c are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

A) 0 B) 1 C) X D) 2X

Solution:

Given, $\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+(a+c) \\ x+4 & x+5 & x+2c \end{vmatrix}$ (Since a, b and c are in A.P., $2b = a+c$

Applying $R_1 \to R_1 - R_2$ *and* $R_3 \to R_3 - R_2$

$$\Delta = \begin{vmatrix} -1 & -1 & a-c \\ x+3 & x+4 & x+(a+c) \\ 1 & 1 & c-a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+a+c \\ 1 & 1 & c-a \end{vmatrix} = 0$$

17. Choose the correct answer.

If X, Y, Z are nonzero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

Given, $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$$\therefore |A| = x(yz-0) = xyz \neq 0$$

Now, $A_{11} = yz, A_{12} = 0, A_{13} = 0$

$A_{21} = 0, A_{22} = xz, A_{23} = 0$

$A_{31} = 0, A_{32} = 0, A_{33} = xy$

$$\therefore adjA = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA)$$

$$= \frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$= \begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xy}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

$$= \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

Thus (A) is the correct answer

18. Choose the correct answer.

Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2n$, then

A) $Det(A) = 0$ B) $Det(A) \in (2, \infty)$ C) $Det(A) \in (2,4)$

D) $Det(A) \in (2,4)$

Solution:

$$\text{Given, } A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$$

$$\therefore |A| = 1\left(1+\sin^2\theta\right)-\sin\theta\left(-\sin\theta+\sin\theta\right)+1\left(\sin^2\theta+1\right)$$

$$=1+\sin^2\theta+\sin^2\theta+1$$

$$=2+2\sin^2\theta$$

$$=2\left(1+\sin^2\theta\right)$$

$$\text{Now, } 0\le\theta\le2\pi$$

$$\Rightarrow 0\le\sin\theta\le1$$

$$\Rightarrow 0\le\sin^2\theta\le1$$

$$\Rightarrow 1\le1+\sin^2\theta\le2$$

$$\Rightarrow 2\le2\left(1++\sin^2\theta\right)\le4$$

$$\therefore Det\,(A)[2,4]$$

Thus, (D) is the correct answer

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Exercise 4.1

1. Evaluate the determinants in Exercise 1 and 2. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1) - 4(-5)$$

$$= -2 + 20$$

$$= 18$$

2. Evaluate the determinants in Exercise 1 and 2

i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta)$

$$= \cos^2\theta + \sin^2\theta = 1$$

ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

$$= (x^2 - x + 1) - (x - 1)(x + 1)$$

$$= x^3 - x^2 + x + x^2 - x + 1 - (x^2 - 1)$$

$$= x^3 + 1 - x^2 + 1$$

$$= x^3 - x^2 + 2$$

3. If $A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}$, then show that $|2A|=4|A|$

Solution:

Given, $A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}$

$\therefore 2A=2\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}=\begin{bmatrix}2 & 4\\ 8 & 4\end{bmatrix}$

$\therefore$ L.H.S $=|2A|=\begin{vmatrix}2 & 4\\ 8 & 4\end{vmatrix}$

$=2\times 4-4\times 8$

$=8-32$

$=-24$

$|A|=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}$

Now, $=1\times 2-2\times 4$

$=2-8$

$=-6$

$\therefore$ R.H.S $=4|A|=4\times(-6)=-24$

$\therefore$ L.H.S = $\therefore$ R.H.S

4. $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}$, then show that $|3A|=27|A|$

Solution:

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Given, $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}$

Expanding along the first column (C_1)

$$|A|=1\begin{vmatrix}1 & 2\\ 0 & 4\end{vmatrix}-0\begin{vmatrix}0 & 1\\ 0 & 4\end{vmatrix}+0\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix}$$

$$=1(4-0)-0+0=4$$

$$\therefore 27|A|=27(4)=108..........(i)$$

$$\text{Now, } 3A=3\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}=\begin{bmatrix}3 & 0 & 3\\ 0 & 3 & 6\\ 0 & 0 & 12\end{bmatrix}$$

$$\therefore |3A|=3\begin{vmatrix}3 & 6\\ 0 & 12\end{vmatrix}-0\begin{vmatrix}0 & 3\\ 0 & 12\end{vmatrix}+0\begin{vmatrix}0 & 3\\ 3 & 6\end{vmatrix}$$

$$=3(36-0)=3(36)=108..........(ii)$$

From equations (i) and (ii), we have

$$|3A|=27|A|$$

Hence proved

5. Evaluate the determinants

i) $\begin{vmatrix}3 & -1 & -2\\ 0 & 0 & -1\\ 3 & -5 & 0\end{vmatrix}$ ii) $\begin{vmatrix}3 & -4 & 5\\ 1 & 1 & -2\\ 2 & 3 & 1\end{vmatrix}$ iii) $\begin{vmatrix}0 & 1 & 2\\ -1 & 0 & -3\\ -2 & 3 & 0\end{vmatrix}$ iv) $\begin{vmatrix}2 & -1 & -2\\ 0 & 2 & -1\\ 3 & -5 & 0\end{vmatrix}$

Solution:

i) Let $A=\begin{vmatrix}3 & -1 & -2\\ 0 & 0 & -1\\ 3 & -5 & 0\end{vmatrix}$

expanding along the second row

$$|A| = -0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix}$$

$$= (-15+3) = -12$$

ii) Let $A = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

expanding along the first row

$$|A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3(1+6) + 4(1+4) + 5(3-2)$$

$$= 3(7) + 4(5) + 5(1)$$

$$= 21 + 20 + 5 = 46$$

iii) Let $A = \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

expanding along the first row

$$|A| = 0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$$

$$= 0 - 1(0-6) + 2(-3-0)$$

$$= -1(-6) + 2(-3)$$

$$= 6 - 6 = 0$$

iv) let $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

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expanding along the first column

$$|A| = 2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - 0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 3\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= 2(0-5) - 0 + 3(1+4)$$

$$= -10 + 15 = 5$$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

Solution:

Given, $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$

Expanding along the first row

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= 1(-9+12) - 1(-18+15) - 2(8-5)$$

$$= 1(3) - 1(-3) - 2(3)$$

$$= 3 + 3 - 6 = 6 - 6 = 0$$

7. Find value of X, if

i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

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Solution:

i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow 2\times1-5\times4 = 2x\times x-6\times4$

$\Rightarrow 2-20 = 2x^2-24$

$\Rightarrow 2x^2 = 6$

$\Rightarrow x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$

ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow 2\times5-3\times4 = x\times5-3\times2x$

$\Rightarrow 10-12 = 5x-6x$

$\Rightarrow -2 = -x$

$\Rightarrow x = 2$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then X is equal to

A) 6 B) ±6 C) -3 D) 0

Solution:

$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

$\Rightarrow x^2-36 = 36-36$

$\Rightarrow x^2-36 = 0$

$\Rightarrow x^2 = 36$

$\Rightarrow x = \pm6$

Hence, (B) is the correct answer

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Exercise 4.2

9. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Solution:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix} = 0+0=0$$

[Since, the two coloumns of the determinants are identical]

10. Using property of determinants and without expanding, prove that

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Solution:

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$

$$\Delta = \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ -(a-c) & -(b-a) & -(c-b) \end{vmatrix}$$

$$\Delta=-\begin{vmatrix}a-c & b-a & c-b\\ b-c & c-a & a-b\\ a-c & b-a & c-b\end{vmatrix}$$

Since, the two rows R_1 and R_3 are identical

$\therefore \Delta=0$

11. Using the property of determinants and without expanding, prove that $\begin{vmatrix}2 & 7 & 65\\ 3 & 8 & 75\\ 5 & 9 & 86\end{vmatrix}=0$

Solution:

$$\begin{vmatrix}2 & 7 & 65\\ 3 & 8 & 75\\ 5 & 9 & 86\end{vmatrix}=\begin{vmatrix}2 & 7 & 63+2\\ 3 & 8 & 72+3\\ 5 & 9 & 81+5\end{vmatrix}$$

$$=\begin{vmatrix}2 & 7 & 63\\ 3 & 8 & 72\\ 5 & 9 & 81\end{vmatrix}=\begin{vmatrix}2 & 7 & 2\\ 3 & 8 & 3\\ 5 & 9 & 5\end{vmatrix}$$

$$=\begin{vmatrix}2 & 7 & 9(7)\\ 3 & 8 & 9(8)\\ 5 & 9 & 9(9)\end{vmatrix}+0$$ [Since, two coloumns are identical]

$$=9\begin{vmatrix}2 & 7 & 7\\ 3 & 8 & 8\\ 5 & 9 & 9\end{vmatrix}$$ [Since, two coloumns are identical]

12. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix}1 & bc & a(b+c)\\ 1 & ca & b(c+a)\\ 1 & ab & c(a+b)\end{vmatrix}=0$$

Solution:

$$\Delta=\begin{vmatrix}1 & bc & a(b+c)\\ 1 & ca & b(c+a)\\ 1 & ab & c(a+b)\end{vmatrix}$$

Applying $C_3 \to C_3 + C_2$

$$\Delta=\begin{vmatrix}1 & bc & ab+bc+ca\\ 1 & ca & ab+bc+ca\\ 1 & ab & ab+bc+ca\end{vmatrix}$$

Since, two columns C_1 and C_3 are proportional

$\therefore \Delta = 0$

13. Using the property of determinants and without expanding, prove that

$$\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}=2\begin{vmatrix}a & p & x\\ b & q & y\\ c & r & z\end{vmatrix}$$

Solution:

$$\Delta=\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a+b & p+q & x+y\end{vmatrix}$$

$$=\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a & p & x\end{vmatrix}+\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ b & q & y\end{vmatrix}$$

$=\Delta_1+\Delta_2\ (say)$.............(1)

Now, $\Delta_1=\begin{vmatrix}b+c & q+r & y+z\\ c+a & r+p & z+x\\ a & p & x\end{vmatrix}$

Applying $R_1 \to R_1 - R_2$

$$\Delta_1=\begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_3$ and $R_2 \leftrightarrow R_3$

$$\Delta_1=(-1)^2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}=\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \text{..........}(2)$$

$$\Delta_2=\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$

$$\Delta_2=\begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$

$$\Delta_2=\begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$

$$\Delta_2=(-1)^2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}=\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \text{............}(3)$$

From (1), (2), and (3), we have

$$\Delta=2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

14. By using properties of determinants, show that $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Solution:

Given, $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$

Applying $R_1 \to cR_1$

$$\Delta = \begin{vmatrix} 0 & ac & -bc \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 - bR_2$

$$\Delta = \frac{1}{c}\begin{vmatrix} ab & ac & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$\Delta = \frac{a}{c}\begin{vmatrix} b & c & 0 \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Since, the two rows R_1 and R_3 are identical

$\therefore \Delta = 0$

15. By using properties of determinants, show that $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$=abc\begin{vmatrix}-a & b & c\\ a & -b & c\\ a & b & -c\end{vmatrix}$$ (Taking out a,b,c from R_1, R_2 and R_3)

$$=a^2b^2c^2\begin{vmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}$$ (Taking out a,b,c from C_1, C_2 and C_3)

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$

$$\Delta = a^2b^2c^2\begin{vmatrix}-1 & 1 & 1\\ 0 & 0 & 2\\ 0 & 2 & 0\end{vmatrix}$$

$$\Delta = a^2b^2c^2(-1)\begin{vmatrix}0 & 2\\ 2 & 0\end{vmatrix}$$

$$=-a^2b^2c^2(0-4)$$

$$=4a^2b^2c^2$$

16. By using properties of determinants, show that

i) $\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=(a+b)(b-c)(c-a)$

ii) $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$

Solution:

i) Let $\Delta = \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$

applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix} 0 & 1 & -a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$

$$\Delta = (b-c)(c-a)\begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Expanding along C_1 $\Delta = (a-b)(b-c)(c-a)\begin{vmatrix} 0 & -1 \\ 1 & b+c \end{vmatrix} = (a-b)(b-c)(c-a)$

ii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$=(c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ -1 & 1 & c\\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$

$$\Delta=(c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ (b^2-a^2)+(bc-ac) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$=(b-c)(c-a)(a-b)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ -(a+b+c) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ -1 & (b^2+bc+c^2) & c^3\end{vmatrix}$$

Expanding along C_1

$$\Delta=(a-b)(b-c)(c-a)(a+b+c)(-1)\begin{vmatrix}0 & 1\\ 1 & c\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)$$

17. By using properties of determinants, show that

$$\begin{vmatrix}x & x^2 & yz\\ y & y^2 & zx\\ z & z^2 & xy\end{vmatrix}=(x-y)(y-z)(z-x)(xy+yz+zx)$$

Solution:

Let $\Delta=\begin{vmatrix}x & x^2 & yz\\ y & y^2 & zx\\ z & z^2 & xy\end{vmatrix}$

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Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & zx-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix}$$

$$\Delta = \begin{vmatrix} x & x^2 & yz \\ -(x-y) & -(x-y)(x+y) & z(x-y) \\ (z-x) & (z-x)(z+x) & -(z-x) \end{vmatrix}$$

$$=(x-y)(z-x)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & z-y & z-y \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\Delta = (x-y)(z-x)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & z-y & z-y \end{vmatrix}$$

$$=(x-y)(z-x)(z-y)\begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3

$$\Delta = \left[(x-y)(z-x)(z-y)\right]\left[(-1)\begin{vmatrix} x & yz \\ -1 & z \end{vmatrix} + 1\begin{vmatrix} x & x^2 \\ -1 & -x-y \end{vmatrix}\right]$$

$$=(x-y)(z-x)(z-y)\left[(-xz-yz)+(-x^2-xy+x^2)\right]$$

$$=-(x-y)(z-x)(z-y)(xy+yz+zx)$$

$$=(x-y)(y-z)(z-x)(xy+yz+zx)$$

18. By using properties of determinants, show that

i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3x+k)$

Solution:

i) $\Delta = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

applying $R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & x+4 & 0 \\ 2x & 0 & x+4 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -x+4 & 0 \\ 2x & 0 & -x+4 \end{vmatrix}$$

$$= (5x+4)(4-x)(4-x)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2x & 0 & 1 \end{vmatrix}$$

Expanding along C_3

$$\Delta = (5x+4)(4-x^2)\begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

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$$=(5x+4)(4-x)^2$$

ii) $\Delta = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$\Delta = (3y+k)\begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = (3y+k)\begin{vmatrix} 1 & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix}$$

$$= k^2(3x+k)\begin{vmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y & 0 & 1 \end{vmatrix}$$

Expanding alone C_3

$$\Delta = k^2(3y+k)\begin{vmatrix} 1 & 0 \\ y & 1 \end{vmatrix}$$

$$= k^2(3y+k)$$

19. By using properties of determinants, show that

i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

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ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

Solution:

i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^3\begin{vmatrix} 1 & 0 & 0 \\ 2b & -4 & 0 \\ 2c & 0 & -1 \end{vmatrix}$$

Expanding along C_3

$$\Delta = (a+b+c)^3(-1)(-1) = (a+b+c)^3$$

ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$

applying $C_1 \to C_2 + C_3$

$$\Delta = \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & x+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3

$$\Delta = 2(x+y+z)^3(1)(1-0)$$

$$= 2(x+y+z)^3$$

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20. By using properties of determinants, show that $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

Solution:

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = \left(1+x+x^2\right) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right)(1-x)(1-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)(1-x) \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

Expanding along R_1

$$\Delta = \left(1-x^3\right)(1-x)(1) \begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)(1-x)\left(1+x+x^2\right)$$

$$= \left(1-x^3\right)\left(1-x^3\right)$$

$$= \left(1-x^3\right)^2$$

21. By using properties of determinants, show that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$

Solution:

$$\Delta=\begin{vmatrix}1+a^2-b^2 & 2ab & -2b\\ 2ab & 1-a^2+b^2 & 2a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

Applying $R_1 \to R_1+bR_3$ and $R_2 \to R_2-aR_3$

$$\Delta=\begin{vmatrix}1+a^2+b^2 & 0 & -b\left(1+a^2+b^2\right)\\ 0 & 1+a^2+b^2 & a\left(1+a^2+b^2\right)\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

$$=\left(1+a^2+b^2\right)\begin{vmatrix}1 & 0 & -b\\ 0 & 1 & a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

Expanding along R_1

$$\Delta=\left(1+a^2+b^2\right)^2\left[(1)\begin{vmatrix}1 & a\\ -2a & 1-a^2-b^2\end{vmatrix}-b\begin{vmatrix}0 & 1\\ 2b & -2a\end{vmatrix}\right]$$

$$=\left(1+a^2+b^2\right)^2\left[1-a^2-b^2+2a^2-b(-2b)\right]$$

$$=\left(1+a^2+b^2\right)^2\left(1+a^2+b^2\right)$$

$$=\left(1+a^2+b^2\right)^3$$

22. By using properties of determinants, show that $\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}=1+a^2+b^2+c^2$

Solution:

$$\Delta=\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}$$

Taking out a, b and c from R_1, R_2 and R_3 respectively

$$\Delta = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

Applying $C_1 \to aC_1, C_2 \to bC_2$ and $C_3 \to cC_3$

$$\Delta = abc \times \frac{1}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expanding along R_3

$$\Delta = -1\begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} a^2+1 & b^2 \\ -1 & 1 \end{vmatrix}$$

$$= -1\left(-c^2\right) + \left(a^2+1+b^2\right) = 1 + a^2 + b^2 + c^2$$

23. Choose the correct answer

Let A be a square matrix of order 3×3, then $|kA|$ is equal to

A) $k^2|A|$ B) $k^3|A|$ C) $3k|A|$

Solution:

Since, A is a square matrix of order 3×3

Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Then, $kA=\begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}$

$\therefore |kA|=\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$

$k^3=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$=k^3|A|$

$\therefore |kA|=k^3|A|$

24. Which of the following is correct?

A) Determinant is a square matrix

B) Determinant is a number associated to a matrix

C) Determinant is a number associated to a square matrix

D) None of these

Solution:

Since, to every square matrix, $A=[aij]$ of order n. We can associate a number called the determinant of square matrix A, where $aij=(i,j)^{th}$ element of A.

Thus, the determinant is a number associated to a square matrix

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Exercise 4.3

1. Find area of the triangle with vertices at the point given in each of the following

i) $(1,0),(6,0),(4,3)$ ii) $(2,7),(1,1),(10,8)$ iii) $(-2,-3),(3,2),(-1,-8)$

Solution:

i) The area of the triangle with vertices $(1,0),(6,0),(4,3)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ 6 & 0 & 1\\ 4 & 3 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[1(0-3)-0(6-4)+1(18-0)\right]$$

$$=\frac{1}{2}[-3+18]$$

$$=\frac{15}{2}\,square\,units$$

ii) The area of the triangle with vertices $(2,7),(1,1),(10,8)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}2 & 7 & 1\\ 1 & 1 & 1\\ 10 & 8 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[2(1-8)-7(1-10)+1(8-10)\right]$$

$$=\frac{1}{2}\left[2(-7)-7(-9)+1(-2)\right]$$

$$=\frac{1}{2}[-14+63-2]$$

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$=\frac{1}{2}[-16+63]$

$=\frac{47}{2}$ Square units

iii) The area of the triangle with vertices $(-2,-3),(3,2),(-1,-8)$ is given by the relation

$$\Delta=\frac{1}{2}\begin{vmatrix}-2 & -3 & 1\\ 3 & 2 & 1\\ -1 & -8 & 1\end{vmatrix}$$

$=\frac{1}{2}[-2(2+8)+3(3+1)+1(-24+2)]$

$=\frac{1}{2}[-2(10)+3(4)+1(-22)]$

$=\frac{1}{2}[-20+12-22]$

$=-\frac{30}{2}$

$=-15$

Hence, the area of the triangle is $|-15|=15$ square units

2. Show that points $A(a,b+c),B(b,c+a),C(c,a+b)$ are collinear

Solution:

Area of ΔABC is given by $\Delta=\frac{1}{2}\begin{vmatrix}a & b+c & 1\\ b & c+a & 1\\ c & a+b & 1\end{vmatrix}$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$

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$$=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b-a & a-b & 0 \\ c-a & a-c & 0 \end{vmatrix}$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$=0$ (since, all elements of R_3 are 0)

Thus, the area of the triangle formed by points A, B and C is zero

Hence, the points are collinear

3. Find values of k if area of triangle is 4 square units and vertices are

i) $(k,0),(4,0),(0,2)$ ii) $(-2,0),(0,4),(0,k)$

Solution:

i) The area of the triangle with vertices $(k,0),(4,0),(0,2)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[k(0-2)-0(4-0)+1(8-0)\right]$$

$$=\frac{1}{2}[-2k+8]=k+4$$

$\therefore -k+4=\pm 4$

When $-k+4=-4, k=8$

When $-k+4=+4, k=0$

Therefore, $k=0,8$

ii) The area of the triangle with vertices $(-2,0),(0,4),(0,k)$ is given by the relation

$$\Delta=\frac{1}{2}\begin{vmatrix}-2 & 0 & 1\\ 0 & 4 & 1\\ 0 & k & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[-2(4-k)\right]$$

$$=k-4$$

$$\therefore k-4=\pm 4$$

When $k-4=-4, k=0$

When $k-4=4, k=8$

Therefore, $k=0,8$

4. i) Find equation of line joining (1,2) and (3,6) using determinates

ii) Find equation of line joining (3,1) and (9,3) using determinants

Solution:

i) Let $P(x,y)$ be any point on the line joining points $A(1,2)$ and $B(3,6)$.

Then the points A, B and P are collinear

Therefore, the area of triangle ABP will be zero

$$\therefore \frac{1}{2}\begin{vmatrix}1 & 2 & 1\\ 3 & 6 & 1\\ x & y & 1\end{vmatrix}=0$$

$$\Rightarrow \frac{1}{2}\left[1(6-y)-2(3-x)+1(3y-6x)\right]=0$$

$\Rightarrow 6-y-6+2x+3y-6x=0$

$\Rightarrow 2y-4x=0$

$\Rightarrow y=2x$

Therefore, the equation of the line joining the given points is $y=2x$

ii) Let $P(x,y)$ be any point on the line joining points $A(3,1)$ and $B(9,3)$

then, the points are collinear

thus, the area of the triangle ABP will be zero

$$\therefore \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix}=0$$

$$\Rightarrow \frac{1}{2}\left[3(3-y)-1(9-x)+1(9y-3x)\right]=0$$

$\Rightarrow 9-3y-9+x+9y-3x=0$

$\Rightarrow 6y-2x=0$

$\Rightarrow x-3y=0$

Therefore, the equation of the line joining the given points is $x-3y=0$

5. If the area of triangle is 35 square units with vertices $(2,-6),(5,4)$ and $(k,4)$. Then k is

A) 12 B) -2 C) -12, -2 D) 12, -2

Solution:

The area of the triangle with vertices $(2,-6),(5,4)$ and $(k,4)$ is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[2(4-4)+6(5-k)+1(20-4k)\right]$$

$$=\frac{1}{2}[30-6k+20-4k]$$

$$=\frac{1}{2}[50-10k]$$

$$=25-5k$$

Given, the area of the triangle is ± 35

Thus, we have

$$\Rightarrow 25-5k=\pm 35$$

$$\Rightarrow 5(5-k)=\pm 35$$

$$\Rightarrow (5-k)=\pm 7$$

When $5-k=-7, k=5+7=12$

When $5-k=7, k=5-7=-2$

Therefore, $k=12,-2$

Exercise 4.4

1. Write Minors and Cofactors of the elements of following determinants

i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Solution:

i) Given, determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

minor of element a_{ij} is M_{ij}

$\therefore M_{11} = 3$

$M_{12} = 0$

$M_{21} = -4$

$M_{22} = 2$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

ii) Given, determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

minor of element a_{ij} is M_{ij}

$\therefore M_{11} = d$

$M_{12} = b$

$M_{21} = c$

$M_{22} = a$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = b$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$

2. i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Solution:

i) Given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

minor of element a_{ij} is M_{ij}

$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

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$$M_{21}=\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}=0$$

$$M_{22}=\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}=1$$

$$M_{23}=\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}=0$$

$$M_{31}=\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix}=0$$

$$M_{32}=\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}=0$$

$$M_{33}=\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}=1$$

Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$

$$A_{11}=(-1)^{1+1}M_{11}=1$$

$$A_{12}=(-1)^{1+2}M_{12}=0$$

$$A_{13}=(-1)^{1+3}M_{13}=0$$

$$A_{21}=(-1)^{2+1}M_{21}=0$$

$$A_{22}=(-1)^{2+2}M_{22}=1$$

$$A_{23}=(-1)^{2+3}M_{23}=0$$

$$A_{31}=(-1)^{3+1}M_{31}=0$$

$$A_{32}=(-1)^{3+2}M_{32}=0$$

$$A_{33}=(-1)^{3+3}M_{33}=1$$

ii) The given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Minor of element a_{ij} is M_{ij}

$$M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11$$

$$M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$$

$$M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 4 \end{vmatrix} = 3 - 0 = 3$$

$$M_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$$

$$M_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

$$M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$M_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$$

$$M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$$

$$M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$$A_{11} = (-1)^{1+1} M_{11} = 11$$

$$A_{12} = (-1)^{1+2} M_{12} = -6$$

$A_{13}=(-1)^{1+3}M_{13}=3$

$A_{21}=(-1)^{2+1}M_{21}=4$

$A_{22}=(-1)^{2+2}M_{22}=2$

$A_{23}=(-1)^{2+3}M_{23}=-1$

$A_{31}=(-1)^{3+1}M_{31}=-20$

$A_{32}=(-1)^{3+2}M_{32}=13$

$A_{33}=(-1)^{3+3}M_{33}=5$

3. Using Cofactors of elements of second row, evaluate $\Delta=\begin{vmatrix}5 & 3 & 8\\ 2 & 0 & 1\\ 1 & 2 & 3\end{vmatrix}$

Solution:

Given determinant is $\Delta=\begin{vmatrix}5 & 3 & 8\\ 2 & 0 & 1\\ 1 & 2 & 3\end{vmatrix}$

Using definition of minors and cofactors

$M_{21}=\begin{vmatrix}3 & 8\\ 2 & 3\end{vmatrix}=9-16=-7$

$\therefore A_{21}=(-1)^{2+1}M_{21}=7$

$M_{22}=\begin{vmatrix}5 & 8\\ 1 & 3\end{vmatrix}=15-8=7$

$\therefore A_{22}=(-1)^{2+2}M_{22}=7$

$$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$$

$$\therefore A_{23} = (-1)^{2+3} M_{23} = -7$$

Since, Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors

$$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{33}A_{33}$$

$$= 2(7) + 0(7) + 1(-7)$$

$$= 14 - 7 = 7$$

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & xz \\ 1 & z & xy \end{vmatrix}$

Solution:

Given determinant is $\begin{vmatrix} 1 & x & yz \\ 1 & y & xz \\ 1 & z & xy \end{vmatrix}$

Using definition of minors and cofactors

$$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$$

$$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$$

$$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$$

$$\therefore A_{13} = (-1)^{1+3} M_{13} = (z - y)$$

$$A_{23} = (-1)^{2+3} M_{23} = -(z - x) = (x - z)$$

$A_{33} = (-1)^{3+3} M_{33} = (y - x)$

Since, Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors

$\therefore \Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$= yz(z-y) + zx(x-z) + xy(y-x)$

$= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$

$= (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y)$

$= z(x^2 - y^2) + z^2(y-x) + xy(y-x)$

$= (x-y)[zx + zy - z^2 - xy]$

$= (x-y)[z(x-z) + y(z-x)]$

$= (x-y)(y-z)(z-x)$

Thus, $\Delta = (x-y)(y-z)(z-x)$

5. For the matrices A and B, verify that $(AB)' = B'A'$ where

i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

Solution:

i) $AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$

$$\therefore (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Now $A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}, B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$\therefore B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Hence, proved that $(AB)' = B'A'$

ii) $AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$

$$\therefore (AB)' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

Now $A' = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, B' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$

$$\therefore B'A' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

Hence, proved that $(AB)' = B'A'$

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Exercise 4.5

1. Find adjoint of each of the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Since, Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

Then, $A_{11} 4, A_{12} = -3, A_{13} = -2, A_{22} = 1$

$$\therefore adjA = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

2. Find adjoint of each of the matrices $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Since, cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2+10) = -12$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0+6 = 6$$

$$A_{21} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1+4 = 5$$

$$A_{23} = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix} = -5-6 = -11$$

$$A_{32} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5$$

Thus, $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

3. Verify $A(adjA) = (adjA)A = |A|_I \cdot \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Here,

$|A| = -12 - (-12)$

$= -12 + 12 = 0$

$\therefore |A| I = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Since, cofactor of a_{ij} is $A_{ij} = (-1)^{i+i} M_{ij}$

Then,

$A_{11} = -6, A_{12} = 4, A_{21} = -3, A_{22} = 2$

$\therefore adjA = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

Now,

$A(adjA) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

$= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Also,

$(adjA) A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

$\begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus, $A(adjA) = (adjA) A = |A| I$

4. Verify $A(adjA) = (adjA) A = |A| I \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Solution:

$$A=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

$$|A|=1(0-0)+1(9+2)+2(0-0)$$

$$=11$$

$$\therefore |A|I=11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Since, cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$

Then,

$$A_{11}=0, A_{12}=-(9+2)=-11, A_{13}=0$$

$$A_{21}=-(-3-0)=3, A_{22}=3-2=1, A_{22}=-(0+1)=-1$$

$$A_{31}=2-0=2, A_{32}=-(-2-6)=8, A_{33}=0+3=3$$

$$\therefore adjA=\begin{bmatrix}0 & 3 & 2\\ -11 & 1 & 8\\ 0 & -1 & 3\end{bmatrix}$$

Now,

$$A(adjA)=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}\begin{bmatrix}0 & 3 & 2\\ -11 & 1 & 8\\ 0 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}0+11+0 & 3-1-2 & 2-8+6\\ 0+0+0 & 9+0+2 & 6+0-6\\ 0+0+0 & 3+0-3 & 2+0+9\end{bmatrix}$$

$$=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Also,

$$\left(adjA\right)A=\begin{bmatrix}0 & 3 & 2\\ -11 & 1 & 8\\ 0 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

$$\begin{bmatrix}0+9+2 & 0+0+0 & 0-6+6\\ -11+3+8 & 11+0+0 & -22-2+24\\ 0-3+3 & 0+0+0 & 0+2+9\end{bmatrix}$$

$$=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Therefore, $A\left(adjA\right)=\left(adjA\right)A=A=|A|I$

5. Find the inverse of each of the matrices (if it exists) $\begin{bmatrix}-1 & 5\\ -3 & 2\end{bmatrix}$

Solution:

Let $A=\begin{bmatrix}-1 & 5\\ -3 & 2\end{bmatrix}$

Here,

$|A|=-2+15=13$

Since, cofactor of a_{ij} is $A_{ij}=\left(-1\right)^{i+j}M_{ij}$

Then,

$A_{11}=2, A_{12}=3, A_{21}=-5, A_{22}=-1$

$\therefore adjA=\begin{bmatrix}2 & -5\\ 3 & -1\end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

6. Find the inverse of each of the matrices (if it exists) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Here,

$$|A| = 1(10-0) - 2(0-0) + 3(0-0) = 10$$

Since, cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$$A_{11} = 10-0, A_{12} = -(0-0) = 0, A_{13} = 0-0 = 0$$

$$A_{21} = -(10-0) = -10, A_{22} = 5-0 = 5, A_{23} = -(0-0) = 0$$

$$A_{31} = 8-6 = 2, A_{32} = -(4-0) = -4, A_{33} = 2-0 = 2$$

$$\therefore adjA = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

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7. Find the inverse of each of the matrices (if it exist). $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Here,

$|A| = 1(-3-0) - 0 + 0 = -3$

Since, cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$A_{11} = -3-0 = -3, A_{12} = -(-3-0) = 3, A_{13} = 6-15 = -9$

$A_{22} = -(0-0) = 0, A_{22} = -1-0 = -1, A_{22} = -(2-0) = -2$

$A_{31} = 0-0 = 0, A_{32} = -(0-0) = 0, A_{33} = 3-0 = 3$

$\therefore adjA = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

8. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Solution:

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Let $A=\begin{bmatrix}2 & 1 & 3\\ 4 & -1 & 0\\ -7 & 2 & 1\end{bmatrix}$

Here,

$|A|=2(-1-0)-1(4-0)+3(8-7)$

$=2(-1)-1(4)+3(1)$

$=-2-4+3$

$=-3$

Since, cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$

Then,

$A_{11}=-1-0=-1, A_{12}=-(4-0)=-4, A_{13}=8-7=1$

$A_{22}=-(1-6)=5, A_{22}=2+21=23, A_{23}=-(4+7)=-11$

$A_{31}=0+3=3, A_{22}=-(0-12)=12, A_{33}=-2-4=-6$

$\therefore adjA=\begin{bmatrix}-1 & 5 & 3\\ -4 & 23 & 12\\ 1 & -11 & -6\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}adjA=-\frac{1}{3}\begin{bmatrix}-1 & 5 & 3\\ -4 & 23 & 12\\ 1 & -11 & -6\end{bmatrix}$

9. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$

Solution:

Let $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$

Expanding along C1

$|A|=1(8-6)-0+3(3-4)=2-3=-1$

Since, cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$

Then,

$A_{11}=8-6, A_{12}=-(0+9)=-9, A_{13}=0-6=-6$

$A_{21}=-(-4+4)=0, A_{22}=4-6=-2, A_{23}=-(-2+3)=-1$

$A_{31}=3-4=-1, A_{32}=-(-3-0)=3, A_{33}=2-0=2$

$\therefore adjA=\begin{bmatrix}2 & 0 & -1\\ -9 & -2 & 3\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}adjA=-1\begin{bmatrix}2 & 0 & -1\\ -9 & -2 & 3\\ -6 & -1 & 2\end{bmatrix}$

$=\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$

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10. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix}1 & 0 & 0\\ 0 & \cos a & \sin a\\ a & \sin a & -\cos a\end{bmatrix}$

Solution:

Let $A=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos a & \sin a\\ a & \sin a & -\cos a\end{bmatrix}a$

Here,

$$|A| = 1\left(-\cos^2 a - \sin^2 a\right) = -\left(\cos^2 a + \sin^2 a\right) = -1$$

Since, cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$$A_{11} = -\cos^2 a - \sin^2 a = -1, A_{12} = 0, A_{13} = 0$$

$$A_{21} = 0, A_{22} = -\cos a, A_{23} = -\sin a$$

$$A_{31} = 0, A_{32} = -\sin a, A_{33} = \cos a$$

$$\therefore adjA = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -1\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & \sin a & -\cos a \end{bmatrix}$$

11. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

Solution:

Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

Here,

$$|A| = 15 - 14 = 1$$

Since, cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$$A_{11} = 5, A_{12} = -2, A_{21} = -7, A_{22} = 3$$

$$\therefore adjA = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

Now, let $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

Here

$$|B| = 54 - 56 = -2$$

$$\therefore adjB = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{|B|} adjB = -\frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}$$

Now,

$$B^{-1}A^{-1} = \begin{bmatrix} -\frac{9}{2} & 4 \\ \frac{7}{2} & -3 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{45}{2} - 8 & \frac{63}{2} + 12 \\ \frac{35}{2} + 6 & -\frac{49}{2} - 9 \end{bmatrix} = \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \text{..............(1)}$$

Then,

$$AB=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$$

$$=\begin{bmatrix}18+49 & 24+63\\ 12+35 & 16+45\end{bmatrix}$$

$$=\begin{bmatrix}67 & 87\\ 47 & 61\end{bmatrix}$$

Therefore, we have $|AB| = 67\times 61-87\times 47=4087-4089=-2$

Also,

$$\therefore adj(AB)=\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}$$

$$\therefore (AB)^{-1}=\frac{1}{|AB|}adj(AB)=-\frac{1}{2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{67}{2} & \frac{87}{2}\\ \frac{47}{2} & -\frac{67}{2}\end{bmatrix}\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2), we have

$$(AB)^{-1}=B^{-1}A^{-1}$$

Hence proved

12. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+71=0$. Hence find A^{-1}

Solution:

$$A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$A^2=A.A$$

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$$=\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2-5A+71$$

$$=\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}-5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}+7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}-\begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}+\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$=\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}+\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $A^2-5A+71=0$

$$\Rightarrow A^2-5A=-71$$

$$\Rightarrow A.A\left(A^{-1}\right)-5AA^{-1}=-71A^{-1}$$ [Multiplying by A^{-1}]

$$\Rightarrow A\left(AA^{-1}\right)-5I=-7A^{-1}$$

$$\Rightarrow AI-5I=-7A^{-1}$$

$$=A^{-1}=-\frac{1}{7}(A-5I)$$

$$=A^{-1}=\frac{1}{7}(5I-A)$$

$$=\frac{1}{7}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}-\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right)$$

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$$=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

13. For the matrix $A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$ find the number a and b such that $A^2+aA+bI=0$

Solution:

$$A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}9+2 & 6+2\\ 3+1 & 2+1\end{bmatrix}$$

$$=\begin{bmatrix}11 & 8\\ 4 & 3\end{bmatrix}$$

Now,

$$A^2+aA+bI=0$$

$$\Rightarrow (AA)A^{-1}+aAA^{-1}+bIA^{-1}=0 \qquad \text{[Multiplying by } A^{-1}\text{]}$$

$$\Rightarrow A(AA^{-1})+aI+b(IA^{-1})=0$$

$$\Rightarrow AI+aI+bA^{-1}=0$$

$$\Rightarrow A+aI=-bA^{-1}$$

$$\Rightarrow A^{-1}=\frac{1}{b}(A+aI)$$

Now,

$$A^{-1}=\frac{1}{|A|}adjA$$

$$=\frac{1}{1}\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$$

Here,

$$\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=-\frac{1}{b}\left(\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}+\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}\right)=-\frac{1}{b}\begin{bmatrix}3+a & 2\\ 1 & 1+a\end{bmatrix}=\begin{bmatrix}\frac{-3-a}{b} & -\frac{2}{b}\\ -\frac{1}{b} & \frac{-1-a}{b}\end{bmatrix}$$

Equating the corresponding elements of the two matrices

$$-\frac{1}{b}=-1\Rightarrow b=1$$

$$\frac{-3-a}{b}=1\Rightarrow -3-a\Rightarrow a=-4$$

Thus, -4 and 1 are the required values of a and b respectively

14. For the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$ show that $A^3-6A^2+5A+11I=0$. Hence, A^{-1}

Solution:

Given, $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$

$$A^2=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+1+2 & 1+2-1 & 1-3+3\\ 1+2-6 & 1+4+3 & 1-6-9\\ 2-1+6 & 2-2-3 & 2+3+9\end{bmatrix}$$

$$=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}$$

$$A^3 = A^2.A=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4+2+2 & 4+4-1 & 4-6+3\\ -3+8-28 & -3+16+14 & -3-24-42\\ 7-3+28 & 7-6-14 & 7+9+42\end{bmatrix}$$

$$\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}$$

$$\therefore A^3-6A^2+5A+11I$$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}-6\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}+5\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}+11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}-\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}+\begin{bmatrix}5 & 5 & 5\\ 5 & 10 & -15\\ 2 & -5 & 15\end{bmatrix}+11\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

$$=\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}-\begin{bmatrix}24 & 12 & 6\\ -18 & 48 & -84\\ 42 & -18 & 84\end{bmatrix}$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0$$

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Thus, $A^2 - 6A^2 + 5A + 11I = 0$

Now,

$$A^3 - 6A^2 + 5A + 11I = 0$$

$$\Rightarrow (AAA)A^{-1} - (AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0 \qquad \text{[Multiplying by } A^{-1}\text{]}$$

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) = -11(IA^{-1})$$

$$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$$

$$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I)\text{.........}(1)$$

Now,

$$A^2 - 6A + 5I$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

From equation (1), we get

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

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$$=\frac{1}{11}\begin{bmatrix}-3 & 4 & 5\\ 9 & -1 & -4\\ 5 & -3 & -1\end{bmatrix}$$

15. If $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ verify that $A^3-6A^2+9A-4I=0$ and hence find A^{-1}

Solution:

Given, $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

$$A^2=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}4+1+1 & -2-2-1 & 2+1+2\\ -2-2-1 & 1+4+1 & -1-2-2\\ 2+1+2 & -1-2-2 & 1+1+4\end{bmatrix}$$

$$\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}12+5+5 & -6-10-5 & 6+5+10\\ -10-6-5 & 5+12+5 & -5-6-10\\ 10+5+6 & -5-10-6 & 5+5+12\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$$

Now,

$A^3-6A^2+9A-4I$

$$=\begin{bmatrix}22&-21&21\\-21&22&-21\\21&-21&22\end{bmatrix}-6\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}+9\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}-4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}22&-21&21\\-21&22&-21\\21&-21&22\end{bmatrix}-\begin{bmatrix}36&-30&30\\-30&36&-30\\30&-30&36\end{bmatrix}+\begin{bmatrix}18&-9&9\\-9&18&-9\\9&-9&18\end{bmatrix}-\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

$$\begin{bmatrix}40&-30&30\\-30&40&-30\\30&-30&40\end{bmatrix}-\begin{bmatrix}40&-30&30\\-30&40&-30\\30&-30&40\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\therefore A^3-6A^2+9A-4I=0$

Now,

$A^3-6A^2+9A-4I=0$

$\Rightarrow (AAA)A^{-1}-6\left(AA^{-1}\right)+9AA^{-1}-4IA^{-1}=0$ [Multiplying by A^{-1}]

$\Rightarrow AA\left(AA^{-1}\right)-6A\left(AA^{-1}\right)+9\left(AA^{-1}\right)=4\left(IA^{-1}\right)$

$\Rightarrow AAI-6AI+9I=4A^{-1}$

$\Rightarrow A^2-6A+9I=4A^{-1}$

$\Rightarrow A^{-1}=\frac{1}{4}\left(A^2-6A+9I\right)$...........(1)

$A^2-6A+9I$

$$=\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}-6\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}+9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-\begin{bmatrix}12 & -6 & 6\\ -6 & 12 & -6\\ 6 & -6 & 12\end{bmatrix}+\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

From equation (1), we have

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

16. Let A be nonsingular square matrix of order 3×3. Then $|adjA|$ is equal to

A) $|A|$ B) $|A|^2$ C) $|A|^3$ D) $3|A|$

Solution:

$$(adjA)=A=|A|I=\begin{bmatrix}|A| & 0 & 0\\ 0 & |A| & 0\\ 0 & 0 & |A|\end{bmatrix}$$

$$\Rightarrow |(adjA)A|=\begin{bmatrix}|A| & 0 & 0\\ 0 & |A| & 0\\ 0 & 0 & |A|\end{bmatrix}$$

$$\Rightarrow |(adjA\backslash =)||A|=|A|^3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=|A|^3(I)$$

$$\therefore |adjA|=|A|^2$$

Hence, (B) is the correct answer

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17. If A is an invertible matrix of order 2, then $\det\left(A^{-1}\right)$ is equal to

A) $\det(A)$ B) $\frac{1}{\det(A)}$ C) 1 D) 0

Solution:

Since A is an invertible matrix, A^{-1} exists and $A^{-1}=\frac{1}{|A|}adjA$

As matrix A is order 2, let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, $|A|=ad-bc$ and $adjA=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Now,

$$A^{-1}=\frac{1}{|A|}adjA$$

$$\begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\therefore \left|A^{-1}\right|=\begin{vmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{vmatrix}$$

$$=\frac{1}{\left|A^2\right|}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

$$=\frac{1}{\left|A^2\right|}(ad-bc)$$

$$=\frac{1}{\left|A^2\right|}|A|$$

$$\frac{1}{|A|}$$

$$\therefore \det\left(A^{-1}\right)=\frac{1}{\det(A)}$$

Hence, (B) is the correct answer

Exercise 4.6

1. Examine the consistency of the system of equations $x+2y=2$, $2x+3y=3$

Solution:

Given,

$x+2y=2$

$2x+3y=3$

Let $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}, X=\begin{bmatrix}2\\ 3\end{bmatrix}$ and $B=\begin{bmatrix}2\\ 3\end{bmatrix}$

Such that, the given system of equations can be written in the form of $AX=B$

Now,

$|A|=1(3)-2(2)=3-4=-1\neq 0$

$\therefore A$ is non-singular

Thus, A^{-1} exists

Therefore, the given system of equations is consistent

2. Examine the consistency of the system of equations $2x-y=5$, $x+y=4$

Solution:

Given,

$2x-y=5$

$x+y=4$

Let $A=\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ z\end{bmatrix}$ and $B=\begin{bmatrix}5\\ 4\end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX = B$

Now,

$|A| = 2(1) - (-1)(1) = 2 + 1 = 3 \neq 0$

$\therefore A$ is non-singular

Thus, A^{-1} exists

Therefore, the given system of equations is consistent

3. Examine the consistency of the system of equations $x + 3y = 5, 2x + 6y = 8$

Solution:

Given,

$x + 3y = 5$
$2x + 6y = 8$

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX = B$

Now,

$|A| = 1(6) - 3(2) = 6 - 6 = 0$

$\therefore A$ is a singular matrix

$(adjA) = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$(adjA)B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$

Thus, the solution of the given system of equations does not exists

Therefore, the given system of equation is inconsistent

4. Examine the consistency of the system of equations

$x+y+z=1$

$2x+3y+2z=2$

$ax+ay+2az=4$

Solution:

Given,

$x+y+z=1$

$2x+3y+2z=2$

$ax+ay+2az=4$

Let $A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Such that, the system of equation can be written in the form of $AX=B$

Now,

$|A|=1(6a-2a)-1(4a-2a)+1(2a-3a)$

$=4a-2a-a=4a-3a=a\neq 0$

$\therefore A$ is a non-singular matrix

Thus, A^{-1} exists

Therefore, the given system of equation is consistent

5. Examine the consistency of the system of equations

$3x-y-2z=2$

$2y-z=-1$

$3x-5y=3$

Solution:

Given,

$$3x-y-2z=2$$

$$2y-z=-1$$

$$3x-5y=3$$

Let $A=\begin{bmatrix}3 & -1 & -2\\ 0 & 2 & -1\\ 3 & -5 & 0\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}$

Such that, this system of equations can be written in the form of $AX=B$

Now,

$$|A|=3(-5)-0+3(1+4)=-15+15=0$$

$\therefore A$ is a singular matrix

Now,

$$(adjA)=\begin{bmatrix}-5 & 10 & 5\\ -3 & 6 & 3\\ -6 & 12 & 6\end{bmatrix}$$

$$\therefore (adjA)B=\begin{bmatrix}-5 & 10 & 5\\ -3 & 6 & 3\\ -6 & 12 & 6\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}=\begin{bmatrix}-10-10+15\\ -6-6+9\end{bmatrix}=\begin{bmatrix}-5\\ -3\end{bmatrix}\neq 0$$

Thus, the solution of the given system of equation does not exist

Therefore, the system of equation is inconsistent

6. Examine the consistency of the system of equations

$$5x-y+4z=5$$

$$2x+3y+5z=2$$

$$5x-2y+6z=-1$$

Solution:

Given,

$$5x - y + 4z = 5$$

$$2x + 3y + 5z = 2$$

$$5x - 2y + 6z = -1$$

Let $A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 3 & -2 & 6 \end{bmatrix}, X \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

Such that, the system of equation can be written in the form of $AX = B$

Now,

$$|A| = 5(18+10) + 1(12-25) + 4(-4-15)$$

$$= 5(28) + 1(-13) + 4(-19)$$

$$= 140 - 13 - 76$$

$$= 51 \neq 0$$

$\therefore A$ is non-singular

Thus, A^{-1} exists

Therefore, the given system of equations is consistent

7. Solve system of linear equations, using matrix method

$$5x + 2y = 4$$

$$7x + 3y = 5$$

Solution:

Given,

$$5x + 2y = 4$$

$$7x + 3y = 5$$

Let $A=\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}4\\ 5\end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX=B$

Now,

$|A|=15-14=1\neq 0$

Thus, A is non-singular

Therefore, its inverse exists

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)$$

$$\therefore A^{-1}=\begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}$$

$$\therefore X=A^{-1}B=\begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}\begin{bmatrix}4\\ 5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}12-10\\ -28+25\end{bmatrix}=\begin{bmatrix}2\\ -3\end{bmatrix}$$

Thus, $x=2$ and $y=-3$

8. Solve system of linear equations, using matrix method

$2x-y=-2$

$3x+4y=3$

Solution:

Given,

$2x-y=-2$

$3x+4y=3$

Let $A=\begin{bmatrix}2 & -1\\3 & 4\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}$ and $B=\begin{bmatrix}-2\\3\end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX=B$

Now,

$|A|=8+3=11\neq 0$

Thus, A is non-singular

Therefore, its inverse exists

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{11}\begin{bmatrix}4 & 1\\-3 & 2\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{11}\begin{bmatrix}4 & 1\\-3 & 2\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-8+3\\6+6\end{bmatrix}$$

$$=\frac{1}{11}\begin{bmatrix}-5\\12\end{bmatrix}$$

$$=\begin{bmatrix}\frac{-5}{11}\\\frac{12}{11}\end{bmatrix}$$

Thus, $x=\frac{-5}{11}$ and $y=\frac{12}{11}$

9. Solve system of linear equations, using matrix method

$4x-3y=3$

$3x-5y=7$

Solution:

Given,

$$4x-3y=3$$

$$3x-5y=7$$

Let $A=\begin{bmatrix}4 & -3\\ 3 & -5\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 7\end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX=B$

Now,

$$|A|=-20+9=-11\neq 0$$

Thus, A is non-singular

Therefore, its inverse exist

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)=-\frac{1}{11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}$$

$$=\frac{1}{11}\begin{bmatrix}15-21\\ 9-28\end{bmatrix}$$

$$=\frac{1}{11}\begin{bmatrix}-6\\ -19\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{6}{11}\\ -\frac{19}{11}\end{bmatrix}$$

Thus, $x=\frac{-6}{11}$ and $y=\frac{-19}{11}$

10. Solve system of linear equations, using matrix method

$5x+2y=3$

$3x+2y=5$

Solution:

Given,

$5x+2y=3$

$3x+2y=5$

Let $A=\begin{bmatrix}5 & 2\\ 3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 5\end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX=B$

Now,

$|A|=10-6=4\neq 0$

Thus A is non-singular

Therefore, its inverse exists

11. Solve system of linear equations, using matrix method

$2x+y+z=1$

$x-2y-z=\frac{3}{2}$

$3y-5z=9$

Solution:

Given,

$2x+y+z=1$

$x-2y-z=\frac{3}{2}$

$3y-5z=9$

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Let $A=\begin{bmatrix}2 & 1 & 1\\ 1 & -2 & -1\\ 0 & 3 & -5\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}1\\ \frac{3}{2}\\ 9\end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX=B$

Now,

$|A|=2(10+3)-1(-5-3)+0=2(13)-1(-8)=26+8=34\neq 0$

Thus A is non-singular

Therefore, its inverse exists

Now,

$A_{11}=13, A_{12}=5, A_{13}=3$

$A_{21}=8, A_{22}=-10, A_{23}=-6$

$A_{31}=1, A_{32}=3, A_{33}=-5$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -16 & -5\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\ 5 & -10 & 3\\ 3 & -16 & -5\end{bmatrix}\begin{bmatrix}1\\ \frac{3}{2}\\ 9\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{34}\begin{bmatrix}13+12+9\\ 5-15+27\\ 3-9-45\end{bmatrix}$$

$$=\frac{1}{34}\begin{bmatrix}34\\ 17\\ -51\end{bmatrix}$$

$$=\begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Thus, $x=1, y=\frac{1}{2}$ and $z=-\frac{3}{2}$

12. Solve system of linear equations, using matrix method

$x-y+z=4$

$2x+y-3z=0$

$x+y+z=2$

Solution:

Given,

$x-y+z=4$

$2x+y-3z=0$

$x+y+z=2$

Let $A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX=B$

Now,

$|A|=1(1+3)+1(2+3)+1(2-1)=4+5+1=10\neq 0$

Thus A is non-singular

Therefore, its inverse exists

Now,

$A_{11}=4, A_{12}=-5, A_{13}=1$

$A_{21}=2, A_{22}=0, A_{23}=-2$

$A_{31}=2, A_{32}=5, A_{33}=3$

$\therefore A^{-1}=\frac{1}{|A|}(adjA)$

$=\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

$\therefore X=A^{-1}B$

$=\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix}$

$=\frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$

$=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Thus, $x=2, y=-1$ and $z=1$

13. Solve system of linear equation, using matrix method

$2x+3y+3z=5$

$x-2y+z=-4$

$3x-y-2z=3$

Solution:

Given,

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$2x+3y+3z=5$

$x-2y+z=-4$

$3x-y-2z=3$

Let $A=\begin{bmatrix}2&3&3\\1&-2&1\\3&-1&-2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}5\\-4\\3\end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX=B$

$|A|=2(4+1)-3(2-3)+3(-1+6)=2(5)-3(-5)=10+15+15=40\neq 0$

Thus, A is non-singular

Therefore, its inverse exists

Now,

$A_{11}=5, A_{12}=5, A_{13}=5$

$A_{21}=3, A_{22}=-13, A_{23}=11$

$A_{31}=9, A_{32}=1, A_{33}=-7$

$\therefore A^{-1}=\frac{1}{|A|}(adjA)$

$=\frac{1}{40}\begin{bmatrix}5&3&9\\5&-13&1\\5&11&-7\end{bmatrix}$

$\therefore X=A^{-1}B$

$=\frac{1}{40}\begin{bmatrix}5&3&9\\5&-13&1\\5&11&-7\end{bmatrix}\begin{bmatrix}5\\-4\\3\end{bmatrix}$

$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{40}\begin{bmatrix}25-12+27\\25+52+3\\25-44-21\end{bmatrix}$

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$$=\frac{1}{40}\begin{bmatrix}40\\80\\-40\end{bmatrix}$$

$$=\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

Thus, $x=1, y=2$ and $z=-1$

14. Solve system of linear equations, using matrix method

$x-y+2z=7$

$3x+4y-5z=-5$

$2x-y+3z=12$

Solution:

Given,

$x-y+2z=7$

$3x+4y-5z=-5$

$2x-y+3z=12$

Let $A=\begin{bmatrix}1 & -1 & 2\\3 & 4 & -5\\2 & -1 & 3\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}7\\-5\\12\end{bmatrix}$

Such that, the given system of equation can be written in the form of $AX=B$

Now,

$|A|=1(12-5)+1(9+10)+2(-3-8)=7+19-22=4\neq 0$

Thus, A is non-singular

Therefore, its inverse exists

Now,

$A_{11}=7, A_{12}=-19, A_{13}=11$

$A_{21}=1, A_{22}=-1, A_{23}=-1$

$A_{31}=-3, A_{32}=11, A_{33}=7$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)$$

$$=\frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore X=A^{-1}B$$

$$=\frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}$$

$$=\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Thus, $x=2, y=1$ and $z=3$

15. The cost of 4kg onion, 3kg wheat and 2kg rice is Rs60. The cost of 2 kg onion, 4kg wheat and 6kg rice is Rs90. The cost of 6kg onion 2kg wheat and 3kg rice is Rs 70. Find cost of each item per kg by matrix method

Solution:

Let the cost of onions, wheat and rice per kg be Rs x, Rs y and Rs z respectively.

Then the given situation can be represented by a system of equations as

$4x+3y+2z=60$

$2x+4y+6z=90$

$6x+2y+3z=70$

Let $A=\begin{bmatrix}4 & 3 & 2\\ 2 & 4 & 6\\ 6 & 2 & 3\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}60\\ 90\\ 70\end{bmatrix}$

Such that, this system of equation can be written in the form of $AX=B$

$|A|=4(12-12)-3(6-36)+2(4-24)=0+90-40=50\neq 0$

Now,

$A_{11}=0, A_{12}=30, A_{13}=-20$

$A_{21}=-5, A_{22}=0, A_{23}=10$

$A_{31}=10, A_{32}=-20, A_{33}=10$

$\therefore adjA=\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}adjA$

$=\frac{1}{50}\begin{bmatrix}0 & -5 & 10\\ 30 & 0 & -20\\ -20 & 10 & 10\end{bmatrix}$

Now,

$X=A^{-1}B$

$$\Rightarrow X = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0-450+700 \\ 1800+0-1400 \\ -1200+900+700 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore x = 5, y = 8$, and $z = 8$

Hence, the cost of onion is Rs 5 per kg, the cost of wheat is Rs 8 per kg and the cost of rice is Rs 8 per kg