Chapter 5: Continuity and differentiability.

Exercise 5. Miscellaneous

1. Differentiate the function w.r.t x

$(3x^2-9x+5)^9$

Solution:

Let $y=(3x^2-9x+5)^9$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}=(3x^2-9x+5)^9$$

$$=9(3x^2-9x+5)^8.\frac{d}{dx}(3x^2-9x+5)$$

$$=9(3x^2-9x+5)^8.(6x-9)$$

$$=9(3x^2-9x+5)^8.3(2x-3)$$

$$=27(3x^2-9x+5)^8.(2x-3)$$

2. Differentiate the function w.r.t x

$\sin^3 x+\cos^6 x$

Solution:

Let $y=\sin^3 x+\cos^6 x$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\sin^3 x)+\frac{d}{dx}(\cos^6 x)$$

$$=3\sin^2 x.\frac{d}{dx}(\sin x)+6\cos^5 x.\frac{d}{dx}(\cos x)$$

$$=3\sin^2 x\cos x+6\cos^5 x.(-\sin x)$$

$$=3\sin x\cos x\left(\sin x-2\cos^4 x\right)$$

3. Differentiate the function w.r.t x

$(5x)^{3\cos 2x}$

Solution:

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both sides, we obtain

$$\log y=3\cos 2x\log 5x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=3\left[\log 5x.\frac{d}{dx}(\cos 2x)+\cos 2x.\frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[\log 5x(-\sin 2x).\frac{d}{dx}(2x)+\cos 2x.\frac{1}{5x}.\frac{d}{dx}(5x)\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[-2\sin x\log 5x+\frac{\cos 2x}{x}\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

$$\therefore\frac{dy}{dx}=(5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

4. Differentiate the function w.r.t x

$\sin^{-1}\left(x\sqrt{x}\right)$, $0\le x\le 1$

Solution:

Let $y=\sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^{3}}}\times\frac{d}{dx}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x^{3}}}\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=\frac{1}{\sqrt{1-x^{3}}}\times\frac{3}{2}.x^{\frac{1}{2}}$$

$$=\frac{3\sqrt{x}}{2\sqrt{1-x^{3}}}$$

$$=\frac{3}{2}\sqrt{\frac{x}{1-x^{3}}}$$

5. Differentiate the function w.r.t x

$$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},-2<x<2$$

Solution:

Let $y=\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right)-\left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^{2}}$$

$$\frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{2x+7}$$

$$=\frac{\sqrt{2x+7}\frac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$=\frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\,(2x+7)}-\frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$=-\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

6. Differentiate the function w.r.t x

$$\cot^{-1}\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right],0<x<\frac{\pi}{2}$$

Solution:

Let $y=\cot^{-1}\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right]$............(1)

Then, $\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right]$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)-(1-\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\frac{dy}{dx}=\frac{1}{2}$$

7. Differentiate the function w.r.t x

$$(\log x)^{\log x}, x>1$$

Solution:

Let $y=(\log x)^{\log x}$

Taking logarithm on both sides, we obtain

$$\log y=\log x.\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[\log x.\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x.\frac{1}{\log x}.\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

8. Differentiate the function w.r.t x

$\cos(a\cos x + b\sin x)$, for some constant a and b

Solution:

Let $y = \cos(a\cos x + b\sin x)$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x).\frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x).\left[a(-\sin x) + b\cos x\right]$$

$$= (a\cos x + b\sin x)\sin(a\cos x + b\sin x)$$

9. Differentiate the function w.r.t x

$(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking logarithm on both sides, we obtain

$$\log y=\log\left[(\sin x-\cos x)^{(\sin x-\cos x)}\right]$$

$$\Rightarrow \log y=(\sin x-\cos x)\log(\sin x-\cos x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[(\sin x-\cos x).\log(\sin x-\cos x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x).\frac{d}{dx}(\sin x-\cos x)+(\sin x-\cos x).\frac{d}{dx}\log(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x).(\cos x+\sin x)+(\sin x-\cos x).\frac{1}{(\text{sinx}-\text{cosx})}\frac{d}{dx}(\sin x-\cos x)$$

$$\Rightarrow \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}\left[(\cos x+\sin x).\log(\sin x-\cos x)+(\cos x+\sin x)\right]$$

$$\therefore \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

10. Differentiate the function w.r.t x

$x^x+x^a+a^x+a^a$, for such fixed $a>0$ and $x>0$

Solution:

Let $y=x^x+x^a+a^x+a^a$

Also, let $x^x=u, x^a=v, a^x=w$ and $a^a=s$

$$\therefore y=u+v+w+s$$

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\frac{ds}{dx}\text{.................}(1)$$

$$u=x^x$$

$$\Rightarrow \log u=\log x^x$$

$$\Rightarrow \log u = x\log x$$

Differentiating both side with respect to x we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x.1 + x.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1+\log x)..........(2)$$

$$v = x^a$$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1}..........(3)$$

$$w = a^x$$

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x\log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w}.\frac{dw}{dx} = \log a.\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w\log a$$

$$\Rightarrow \frac{dw}{dx} = a^x\log a..........(4)$$

$$s = a^a$$

Since a is constant, a^a is also constant

$$\therefore \frac{ds}{dx} = 0..........(5)$$

From 1,2,3,4 and 5, we obtain

$$\frac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x \log a + 0$$

$$= x^x(1+\log x) + ax^{a-1} + a^x \log a$$

11. Differentiate the function w.r.t x

$x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$

Solution:

Let $y = x^{x^2-3} + (x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$\therefore y = u + v$

Differentiating both sides with respect to x we obtain

$$\frac{dv}{dx} = \frac{du}{dx} + \frac{dv}{dx}\text{.............}(1)$$

$u = x^{x^2-3}$

$\therefore \log u = \log\left(x^{x^2-3}\right)$

$\log u = (x^2-3)\text{logx}$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x.\frac{d}{dx}(x^2-3) + (x^2-3).\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x.2x + (x^2-3).\frac{1}{3}$$

$$\Rightarrow \frac{du}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2\times\log x\right]$$

Also,

$$v=(x-3)^{x^2}$$

$$\therefore \log v=\log(x-3)^{x^2}$$

$$\Rightarrow \log v = x^2\log(x-3)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\log(x-3)\frac{d}{dx}(x^2)+x^2\frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\log(x-3)2x+x^2.\frac{1}{x-3}.\frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx}=v\left[2x\log(x-3)+\frac{x^2}{x-3}.1\right]$$

$$\Rightarrow \frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Substituting the expression of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)x^2\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

12. Find $\frac{dy}{dx}$, if $y=12(1-\cos t), x=10(t-\sin t), \frac{\pi}{2}<t<\frac{\pi}{2}$

$-\frac{\pi}{2}<t<\frac{\pi}{2}$

Solution:

It is given that $y=12(1-\cos t), x=10(t-\sin t)$

$$\therefore \frac{dx}{dt}=\frac{d}{dt}[10(t-\sin t)]=10\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dx}=\frac{d}{dx}\left[12\left(1-\cos t\right)\right]=12\frac{d}{dt}\left(1-\cos t\right)=12.\left[0-\left(-\sin t\right)\right]=12\sin t$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10\left(1-\cos t\right)}=\frac{12.2\sin\frac{t}{2}.\cos\frac{t}{2}}{10.2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

13. Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},-1\le x\le 1$

Solution:

It is given that $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1\left(\sqrt{1-x^2}\right)}}.\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}.\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}\left(-2x\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=0$$

14. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for $-1<x<1$, prove that $\frac{dy}{dx}=-\frac{1}{\left(1+x\right)^2}$

Solution:

It is given that, $x\sqrt{1+y}+y\sqrt{1+x}=0$

$$x\sqrt{1+y}=-y\sqrt{1+x}$$

Squaring both sides, we obtain

$x^2(1+y)=y^2(1+x)$

$\Rightarrow x^2+x^2y=y^2+xy^2$

$\Rightarrow x^2-y^2=xy^2-x^2y$

$\Rightarrow x^2-y^2=xy(y-x)$

$\Rightarrow (x+y)(x-y)=xy(y-x)$

$\therefore x+y=-xy$

$\Rightarrow (1+x)y=-x$

$\Rightarrow y=\frac{-x}{(1+x)}$

Differentiating both sides with respect to x, we obtain

$y=\frac{-x}{(1+x)}$

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

Hence, proved

15. If $(x-a)^2+(y-b)^2=c^2$, for some c > 0, prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b

Solution:

It is given that, $(x-a)^2+(y-b)^2=c^2$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\frac{d}{dx}(x-a)+2(y-b)\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a).1+2(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \quad \text{..........(1)}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$=-\frac{\left[(y-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(y-b)\right]}{(y-b)^2}$$

$$=-\left[\frac{(y-b)-(x-a)\frac{dy}{dx}}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a)\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \quad \text{[using (1)]}$$

$$=-\left[\frac{(y-b)^2+(x+a)^2}{(y-b)^2}\right]$$

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[\left(1+\frac{(x-a)^2}{(y-b)^2}\right)\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= -\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{c^2}{(y-b)^3}} = \frac{\frac{c^2}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

= - c, which is constant and is independent of a and b

Hence, proved

16. If $\cos y = x\cos(a+y)$ with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Solution:

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx} = [\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\frac{d}{dx}(x) + x.\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y = \frac{dy}{dx}\cos(a+y) + x.[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \quad \text{...........(1)}$$

Since $\cos y = x\cos(a+y), x = \frac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to $\left[\frac{\cos y}{\cos(a+y)}\sin(a+y)-\sin y\right]\frac{dy}{dx}=\cos(a+y)$

$$\Rightarrow\left[\cos y.\sin(a+y)-\sin y.\cos(a+y)\right]\frac{dy}{dx}=\cos^2(a+y)$$

$$\Rightarrow\sin(a+y-y)\frac{dy}{dx}=\cos^2(a+b)$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos^2(a+b)}{\sin a}$$

Hence, proved

17. If $x=a(\cos t+t\sin t)\, and\ y=a(\sin t-t\cos t)$, find $\frac{d^2y}{dx^2}$

Solution:

It is given that, $x=a(\cos t+t\sin t)\, and\ y=a(\sin t-t\cos t)$

$$\therefore\frac{dx}{dt}=a.\frac{d}{dt}(\text{cot}+t\sin t)$$

$$=a\left[-\sin t+\sin t.\frac{d}{dt}(t)+t.\frac{d}{dt}(\sin t)\right]$$

$$=a[-\sin t+\sin t+t\cos t]=at\cos t$$

$$\frac{dy}{dx}=a.\frac{d}{dt}(\sin t-t\cos t)$$

$$=a\left[\cos t-\left\{\cos t.\frac{d}{dt}(t)+t.\frac{d}{dt}(\cos t)\right\}\right]$$

$$=a\left[\cos t-\{\cos t-t\sin t\}\right]=at\sin t$$

$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{at\sin t}{at\cos t}=\tan t$$

Then, $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(\tan t)=\sec^2 t\frac{dt}{dx}$

$=\sec^2 t\frac{1}{a\,t\cos t}\quad\left[\frac{dx}{dt}=at\cos t\Rightarrow\frac{dt}{dx}=\frac{1}{at\cos t}\right]$

$=\frac{\sec^3 t}{at},0<t<\frac{\pi}{2}$

18. If $f(x)=|x|^3$, show that $f''(x)$ exists for all real x, and find it

Solution:

It is known that, $|x|=\begin{cases}x, & if\ x\geq 0\\ -x, & if\ x<0\end{cases}$

Therefore, when $x\geq 0$, $f(x)=|x|^3=x^3$

In this case, $f'(x)=3x^2$ and hence, $f''(x)=6x$

When $x<0$, $f(x)=|x|^3=(-x^3)=x^3$

In this case, $f'(x)=3x^2$ and hence $f''(x)=6x$

Thus, for $f(x)=|x|^3$, $f''(x)$ exists for all real x and is given by,

$f''(x)=\begin{cases}6x, & if\ x\geq 0\\ -6x, & if\ x<0\end{cases}$

19. Using mathematical induction prove that $\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers n

Solution:

To prove: $P(n):\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers n

For n = 1,

$P(1): \frac{d}{dx}(x) = 1 = 1.x^{1-1}$

$\therefore p(n)$ is true for n = 1

Let p(k) is true for some positive integer k

That is, $p(k): \frac{d}{dx}(x^k) = kx^{k-1}$

It is to be proved that $p(k+1)$ is also true

Consider $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x.x^k)$

$x^k \frac{d}{dx}(x) + x.\frac{d}{dx}(x^k)$

$= x^k.1 + x.k.x^{k-1}$

$= x^k + kx^k$

$= (k+1)x^k$

$= (k+1).x^{(k+1)-1}$

Thus, P (k+1) is true whenever P(k) is true

Therefore, by the principal of mathematical induction, the statement P(n) is true for every

positive integer n

Hence, proved

20. Using the fact that $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and the differentiation, obtain the sum formula for cosines

Solution:

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\frac{d}{dx}(A+B)=\cos B\frac{d}{dx}(\sin A)+\sin A\frac{d}{dx}(\cos B)+\sin B\frac{d}{dx}(\cos A)+\cos A\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\frac{d}{dx}(A+B)=\cos B.\cos A\frac{d}{dx}+\sin A(-\sin B)\frac{dB}{dx}+\sin B(-\sin A)\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$

$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

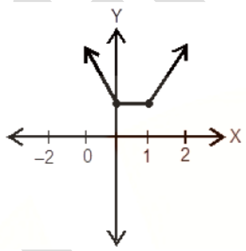
21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer

Solution:

Consider $f(x)=|x|+|x+1|$

Since modulus function is everywhere continuous and sum of two continuous function is also continuous

Differentiability of $f(x)$:Graph of $f(x)$ shows that $f(x)$ is everywhere derivable except possible at x= 0 and x = 1



At x =0, Left hand derivative

$$\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{(|x|+|x-1|)-(1)}{x}=\lim_{x\to0^-}\frac{|(-x)-(x-1)|-1}{x}=\lim_{x\to0^-}\frac{-2x}{x}=-2$$

Right hand derivative =

$$\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{(|x|+|x-1|)-(1)}{x}=\lim_{x\to0^+}\frac{(-x)-(x-1)-1}{x}=\lim_{x\to0^-}\frac{0}{x}=0$$

Since $L.H.D \neq R.H.D$ $f(x)$ is not derivable at x = 0

At x = 1

L.H.D

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{(|x|+|x-1|)}{x-1}=\lim_{x\to 1^-}\frac{(x)-(x-1)-1}{x-1}=\lim_{x\to 1^-}\frac{0}{x-1}=0$$

R.H.D

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{(|x|+|x-1|-1)}{x-1}=\lim_{x\to 1^+}\frac{(x)-(x-1)-1}{x-1}=\lim_{x\to 1^-}\frac{2(x-1)}{x-1}=2$$

Since $L.H.D \neq R.H.D$ $f(x)$ is not derivable at x = 1

$\therefore f(x)$ is continuous everywhere but not derivable at exactly two points

22. If $y=\begin{bmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{bmatrix}$, prove that $\frac{dy}{dx}=\begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$

Solution:

$$y=\begin{bmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{bmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

Then, $\frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$$

Thus, $\frac{dy}{dx}=\begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$

23. If $y=e^{a\cos^{-1}x}, -1\le x\le 1$, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Solution:

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both sides, we obtain

$\log y = a\cos^{-1}x\log e$

$\log y = a\cos^{-1}x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{y}\frac{dy}{dx}=ax\frac{1}{\sqrt{1-x^2}}$

$\frac{dy}{dx}=\frac{-ay}{\sqrt{1-x^2}}$

By squaring both the sides, we obtain

$\left(\frac{dy}{dx}\right)^2=\frac{a^2y^2}{1-x^2}$

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$

$(1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$

Again, differentiating both sides with respect to x, we obtain

$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x)+(1-x^2)\times 2\frac{dy}{dx}\frac{d^2y}{dx^2}=a^2.2y\frac{dy}{dx}$

$$\Rightarrow x\frac{dy}{dx}+\left(1-x^2\right)\frac{d^2y}{dx^2}=a^2.y \qquad \left[\frac{dy}{dx}\neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

Hence, proved

Exercise 5.1

1. Prove that the function $f(x)=5x-3$ is continuous at $x=0, x=-3 \, and \, at \, x=5$

Solution:

The given function is $f(x)=5x-3$

At $x=0, f(0)=5\times0-3=3$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}(5x-3)=5\times0-3=3$$

$$\therefore \lim_{x\to\infty} f(x)=f(0)$$

Therefore, f is continuous at $x=0$

At $x=-3, f(-3)=5x(-3)-3=18$

$$\lim_{x\to3} f(x)=\lim_{x\to3} f(5x-3)=5x(-3)-3=-18$$

$$\therefore \lim_{x\to3} f(x)=f(-3)$$

Therefore, f is continuous at $x=-3$

At $x=5, f(x)=f(5)=5\times5-3=25-3=22$

$$\lim_{x\to5} f(x)=\lim_{x\to5}(5x-3)=5\times5-3=22$$

$$\therefore \lim_{x\to5} f(x)=f(5)$$

Therefore, is f continuous at $x=5$

2. Examine the continuity of the function at $f(x)=2x^2-1 \; x=3$

Solution:

The given function is $f(x)=2x^2-1$

At $x=3,\ f(x)=f(3)=2\times3^2-1=17$

$$\lim_{x\to3} f(x)=\lim_{x\to3}\left(2x^2-1\right)=2\times3^2-1=17$$

$$\therefore \lim_{x\to3} f(x)=f(3)$$

Thus, f is continuous at $x=3$.

3. Examine the following functions for continuity

a) $f(x)=x-5$

b) $f(x)=\dfrac{1}{x-5},\ x\neq5$

c) $f(x)=\dfrac{x^2-25}{x+5},\ x\neq5$

d) $f(x)=|x-5|$

Solution:

a) The given function is $f(x)=x-5$

It is evident that f is defined at every real number k and its value at k is $k-5$

It is also observed that $\lim_{x\to k} f(x)=\lim_{x\to k} f(x-5)=k=k-5=f(k)$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f is continuous at every real number and therefore, it is a continuous function.

b) The given function is $f(x)=\dfrac{1}{x-5},\ x\neq5$ for any real number $k\neq5$, we obtain

$$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{1}{x-5}=\frac{1}{k-5}$$

Also, $f(k)=\dfrac{1}{k-5}$ $\quad (As\ k\neq5)$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

c) The given function is $f(x)=\frac{x^2-25}{x+5},\ x\neq 5$

For any real number $c\neq -5$, we obtain

$$\lim_{x\to c} f(x)=\lim_{x\to c}\frac{x^2-25}{x+5}=\lim_{x\to c}\frac{(x+5)(x-5)}{x+5}=\lim_{x\to c}(x-5)=(c-5)$$

Also, $f(c)=\frac{(c+5)(c-5)}{c+5}=c(c-5)(as\,c\neq 5)$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Hence f is continuous at every point in the domain of f and therefore. It is continuous function.

d) The given function is $f(x)=|x-5|=\begin{cases}5-x, \text{ if } x<5\\ x-5, \text{if } x\geq 5\end{cases}$

This function f is defined at all points of the real line.

Let c be a point on a real time. Then, $c<5$ or $c=5\ or\, c>5$

Case I: $c<5$

Then, $f(c)=5-c$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(5-x)=5-c$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all real numbers less than 5.

Case II : $c=5$

Then, $f(c)=f(5)=(5-5)=0$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5}(5-x)=(5-5)=0$$

$$\lim_{x\to 5^+} f(x)=\lim_{x\to 5}(x-5)=0$$

$\therefore \lim_{x \to c^+} f(x) = \lim_{x \to c^+}(x) = f(c)$

Therefore, f is continuous at $x = 5$

Case III : $c > 5$

Then, $f(c) = f(5) = c - 5$

$\lim_{x \to c} f(x) = \lim_{x \to c} f(x-5) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

4. Prove that the function $f(x) = x^n$ is continuous at is a $x = n$ positive integer.

Solution:

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

Then, $\lim_{x \to n} f(n) = \lim_{x \to n} f(x^n) = n^n$

$\therefore \lim_{x \to n} f(x) = f(n)$

Therefore, f is continuous at n, where n is a positive integer.

5. Is the function f defined by $f(x) = \begin{cases} x, if\ x \le 1 \\ 5, if\ x > 1 \end{cases}$

Continuous at $x = 0$? At $x = 1$?. At $x = 2$?

Solution:

The given function f is $f(x) = \begin{cases} x,\ if\ x \le 1 \\ 5, if\ x > 1 \end{cases}$

At $x=0$

It is evident that f is defined at 0 and its value of 0 is 0

Then, $\lim_{x\to 0} f(x)=\lim_{x\to 0} x=0$

$\therefore \lim_{x\to 0} f(x)=f(0)$

Therefore, f is continuous at $x=0$

At $x=1$

f is defined at 1 and its value at is 1.

The left hand limit of at f i $x=1$ s,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x=1$

The right hand limit of at f is $x=1$,

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} f(5)$

$\therefore \lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)$

Therefore, is f is not continuous at $x=1$

At $x=2$

f is defined at 2 and its value at 2 is 5.

Then, $\lim_{x\to 2} f(x)=\lim_{x\to 2} f(5)=5$

$\therefore \lim_{x\to 2} f(x)=f(2)$

Therefore f is continuous at $x=2$

6. Find all points of discontinuous of f, where f is defined by

$$f(x)=\begin{cases} 2x+3, & if\ \ x\le 2 \\ 2x-3 & if\ x>2 \end{cases}$$

Solution:

The given function f if $f(x)=\begin{cases}2x+3, & if\ x\le 2\\ 2x-3 & if\ x>2\end{cases}$

It is evident that the given function f is defined at all the points of the real time.

Let c be a point on the real line. Then, three cases arise.

I. $c<2$

II. $c>2$

III. $c=2$

Case (i) $c<2$

Then, $f(x)=2x+3$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x+3)=2c+3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x<2$

Case (ii) $c>2$

Then, $f(c)=2c-3$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x-3)=2c-3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x>2$

Case (iii) $c=2$

Then, the left hand limit of f at $x=2$ is

$$\lim_{x\to 2} f(x)=\lim_{x\to 2^-}(2x+3)=2\times 2+3=7$$

The right hand limit of f at is $x=2$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x+3)=2\times 2-3=1$$

It is observed that the left and right hand limit of f at $x=2$ do not coincide.

Hence, $x=2$ is the only point of discontinuity of f.

7. Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases} |x|+3, & if\ x\le -3 \\ -2x, & if\ -3<x<3 \\ 6x+2 & if\ x\ge 3 \end{cases}$$

Solution:

The given function f is defined at all the points at the real line.

Let c be a point on the real line.

Case I:

If $c<-3$, then $f(c)=-c+3$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-x+3)=-c+3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x<-3$

Case II :

If , $c=-3$ then $f(-3)=-(-3)+3=6$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(-x+3)=-(-3)+3=6$$

$$\therefore \lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} f(-2x)=2x(-3)=6$$

$$\therefore \lim_{x\to 3} f(x)=f(-3)$$

Therefore, f is continuous at $x=-3$

Case III :

If , $-3 < c < 3$ then $f(c) = -2c$ and $\lim_{x \to c} f(x) = \lim_{x \to 3c}(-2x) = -2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in (-3, 3)

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} f(-2x) = -2 \times 3 = 6$

The right hand limit of f at $x = 3$ is

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(6x+2) = 6 \times 3 + 2 = 20$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Case V:

If $c > 3$, then $f(c) = 6c + 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(6x+2) = 6c + 2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore f is continuous at all points x, such that $x > 3$.

Hence, $x = 3$ is the only point of discontinuity of f .

8. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \frac{|x|}{x}, & if\ x \neq 0 \\ 0 & if\ \ x = 0 \end{cases}$

Solution:

The given function f is $f(x) = \begin{cases} \frac{|x|}{x}, & if\ x \neq 0 \\ 0 & if\ \ x = 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x)\begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1 & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1 & \text{if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 1$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore f is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$$

The right hand limit of f at $x = 0$ is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1) = 1$$

It is observed that the left and right hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$.

Case III:

If $c > 0$, $f(c) = 1$

$\lim_{x\to c} f(x) = \lim_{x\to c}(1) = 1$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of f.

9. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \frac{x}{|x|}, & if\ x < 0 \\ -1 & if\ x \geq 0 \end{cases}$

Solution:

The given function f is $f(x) = \begin{cases} \frac{x}{|x|}, & if\ x < 0 \\ -1 & if\ x \geq 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{x}{|x|}, & if\ x < 0 \\ -1 & if\ x \geq 0 \end{cases} \Rightarrow f(x) = -1\ for\ all\ x \in R$$

Let c be any real number. Then $\lim_{x\to c} f(x) = \lim_{x\to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x\to c} f(x)$

Therefore, the given function is continuous function.

Hence, the given function has no point of discontinuity.

10. Find all the points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x+1, & if\ x \geq 1 \\ x^2+1, & if\ x < 1 \end{cases}$$

Solution:

The given function f is $f(x)=\begin{cases} x+1, & if \ x\geq 1 \\ x^2+1, & if \ x<1 \end{cases}$

The given function f is defined at all the points of the real lime.

Let c be a point on the real time.

Case I :

If $c<1$, then $f(c)=c^2+1$ and $\lim_{x\to c} f(x)=\lim_{x\to c} f(x^2+1)=c^2+1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<1$

Case II :

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of f at $x=1$ is

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2+1)=1^2+1=2$

The right hand limit of f at $x=1$ is

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x^2+1)=1^2+1=2$

$\therefore \lim_{x\to 1} f(x)=f(c)$

Therefore, f is continuous at $x=1$

Case III:

If $c>1$, then $f(c)=c+1$

$\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>1$.

Hence, the given function f has no points of discontinuity.

11. Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases} x^3-3, & if\ \ x\le 2 \\ x^2+1, & if\ \ x>2 \end{cases}$$

Solution:

The given function f is $f(x)=\begin{cases} x^3-3, & if\ \ x\le 2 \\ x^2+1, & if\ \ x>2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c<2$, then $f(c)=c^3-3$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(x^3-3)=c^3-3$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such at that $x<2$

Case II :

If $c=2$, then $f(c)=f(2)=2^3-3=5$

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^-}(x^3-3)=2^3-3=5$

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x^2+1)=2^2+1=5$

$\therefore \lim_{x\to 2} f(x)=f(2)$

Therefore, f is continuous at $x=2$

Case III :

If $c>2$, then $f(c)=c^2+1$

$\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^2+1\right) = c^2+1$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 2$

Thus, the given function f is continuous at every point on the real time.

Hence, f has no point of discontinuity.

12. Find all points of discontinuity of f, where f is define by $f(x) = \begin{cases} x^{10}-1, & if \ x \le 1 \\ x^2, & if \ x > 1 \end{cases}$

Solution:

The given function f is $f(x) = \begin{cases} x^{10}-1, & if \ x \le 1 \\ x^2, & if \ x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c < 1$, then $f(c) = c^{10} - 1$ *and* $\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^{10}-1\right) = c^{10}-1$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II :

If $c = 1$, then the left hand limit of f at $x = 1$ is

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(x^{10}-1\right) = 10^{10} - 1 = 1 - 1 = 0$

The right hand limit of f at $x = 1$ is ,

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(x^2\right) = 1^2 = 1$

It is observed that the left and right hand limit of f at $x=1$ do not coincide.

Therefore, f is not continuous at $x=1$

Case III :

If $c>1$, then $f(c)=c^2$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\right)=c^2$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore f is continuous at al points x, such that $x>1$

Thus, from the above observation, it can be concluded that $x=1$ is the only point of discontinuity of f.

13. Is the function define by $f(x)=\begin{cases} x+5, & if\ \ x\le 1 \\ x-5, & if\ x>1 \end{cases}$ a continuous function?

Solution:

The given function is $f(x)=\begin{cases} x+5, & if\ \ x\le 1 \\ x-5, & if\ x>1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I ;

If $c<1$, then $f(c)=c+5$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(x+5)=c+5$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x<1$

Case II :

If $c=1$, then $f(1)=1+5=6$

The left hand limit of f at $x=1$ is

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+5) = 1+5 = 6$$

The right hand limit of f at $x=1$ is $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x-5) = 1-5 = -4$

It is observed that the left and right hand limit of f at $x=1$ do not coincide.

Therefore f is not continuous at $x=1$

Case III :

If $c>1$, then $f(c)=c-5$ *and* $\lim_{x\to c} f(x) = \lim_{x\to c}(x-5) = c-5$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore f is continuous at all points x, such that $x>1$

Thus, from the above observations, it can be concluded that $x=1$ is the only point of discontinuity of f.

14. Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 3, & if\ 0\le x\le 1 \\ 4, & if\ 1<x<3 \\ 5, & if\ 3\le x\le 10 \end{cases}$$

Solution:

The given function is $f(x) = \begin{cases} 3, & if\ 0\le x\le 1 \\ 4, & if\ 1<x<3 \\ 5, & if\ 3\le x\le 10 \end{cases}$

The given function is defined at all the points of the interval $[0, 10]$.

Let c be a point in the interval $[0, 10]$

Case I ;

If , $0\le c<1$ then $f(c)=c+5$ $f(c)=3$ *and* $\lim_{x\to c} f(x) = \lim_{x\to c}(3) = 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in the interval $[0,1)$

Case II :

If $c = 1$, then $f(3) = 3$

The left hand limit of f at $x = 1$ is

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3) = 3$

The left hand limit of f at $x = 1$ is $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3) = 3$

The right hand limit of f at $x = 1$ is $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4) = 4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore f is not continuous at $x = 1$

Case III :

If $c > 1$, then $f(c) = 4$ *and* $\lim_{x \to c} f(x) = \lim_{x \to c} (4) = 4$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore f is continuous at all points of the interval (1, 3)

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of f at $x = 3$ is $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$

The right hand limit of f at $x = 3$ is $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V :

If $3 < c \le 10$, then $f(c) = 5$ *and* $\lim_{x \to c} f(x) = \lim_{x \to c}(5) = 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at $x = 1$ *and* $x = 3$

15. Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & if\ x < 0 \\ 0, & if\ 0 \le x \le 1 \\ 4x. & if\ x > 1 \end{cases}$$

Solution:

The given function is $f(x) = \begin{cases} 2x, & if\ x < 0 \\ 0, & if\ 0 \le x \le 1 \\ 4x. & if\ x > 1 \end{cases}$

The given function is defined at all the points of the real line.

Let c be a point on the real line

Case I ;

If , $c < 0$ then $f(c) = 2c$, $\lim_{x \to c} f(x) = \lim_{x \to c}(2x) = 2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points , x such that x < 0

Case II :

If $c = 0$, then $f(c) = f(0) = 3$

The left hand limit of f at $x = 0$ is $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(2x) = 2 \times 0 = 0$

The right hand limit of f at $x = 0$ is $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(0) = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

Case III :

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(0) = 0$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore f is continuous at all points of the interval (0, 1)

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of f at $x = 1$ is $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(0) = 0$

The right hand limit of f at $x = 1$.is $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(4x) = 4 \times 1 = 4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case V :

If $c < 1$, then $f(c) = 4c$ *and* $\lim_{x \to c} f(x) = \lim_{x \to c}(4x) = 4c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Hence, f is not continuous only at $x = 1$

16. Discuss the continuity of the function f , where f is defined by

$$f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$$

Solution:

The given function f is $f(x)=\begin{cases}-2, & if\ x\le -1\\ 2x, & if\ -1<x\le 1\\ 2, & if\ x>1\end{cases}$

The given function is defined at all point of the real time. Let c be a point on the real time.

Case I :

If $c<-1$, then $f(c=-2)$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(-2)=-2$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x , such that $x<-1$

Case –II :

If $c=-1, then\ f(c)=f(-1)=-2$

The left hand limit of $f\ at\ x=-1$ is,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(-2)=-2$

The right hand limit of f at $x=-1$ is ,

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 2x(-1)=-2$

$\lim_{x\to 1} f(x)=f(-1)$

Therefore , f is continuous at $x=-1$

Case III :

If $-1<c<1, then\ f(c)=2c$

$\lim_{x\to c} f(x)=\lim_{x\to c}(2x)=2c$

$\lim_{x\to c} f(x)=f(c)$

Therefore , f is continuous at all points of the interval $(-1,1)$.

Case - IV :

If $c=1$, then $f(c)=f(1)=2\times 1=2$

The left hand limit of f at $x=1$ is ,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(2x)=2\times 1=2$

The right hand limit of f at $x=1$ is ,

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 2=2$

$\lim_{x\to 1} f(x) = f(c)$

Therefore, f is continuous at $x = 2$

Case –V :

If $c > 1, f(c) = 2\, and \lim_{x\to 2} f(x) = \lim_{x\to 2}(2) = 2$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore , f is continuous at all points , x , such that $x > 1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real time.

17. Find the relationship be *a* and *b* so that the function *f* defined by *f (x)* = $\begin{cases} ax+1, if\ x \le 3 \\ bx+3, if\ x > 3 \end{cases}$ is continuous at *x=3*.

Solution: The given function *f* is *f (x)* = $\begin{cases} ax+1, if\ x \le 3 \\ bx+3, if\ x > 3 \end{cases}$ If *f* is continuous at *x=3*, then

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} f(x) = f(3)$

Also,

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} f(ax+1) = 3a+1$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} f(ax+1) = 3b+3$

$f(3) = 3a+1$

therefore, from (1), we obtain

$3a+1 = 3b+3 = 3a+1$

$\Rightarrow 3a+1 = 3b+3$

$\Rightarrow 3a=3b+2$

$\Rightarrow a=b+\frac{2}{3}$

Therefore, the required relationship is given by, $a=b+\frac{2}{3}$

18. For what value of λ is the function defined by *f (x)* = $\begin{cases} \lambda(x^2-2x), if\ x \le 0 \\ 4x+1, \quad if \quad x>0 \end{cases}$

continuous at $x = 0$? What about continuity at $x = 1$?

Solution: The given function *f* is $f(x) = \begin{cases} \lambda(x^2-2x), if\ x \le 0 \\ 4x+1, \quad if \quad x>0 \end{cases}$

If *f* is continuous at *x* = *0*, then

$$\lim_{x\to 0^-} f(x) \lim_{x\to 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} \lambda(x^2-2x) = \lim_{x\to 0^+}(4x+1) = \lambda(0^2-2\times 0)$$

$$\Rightarrow \lambda(02-2\times 0) = 4\times 0+1 = 0$$

$$\Rightarrow 0=1=0, \textit{which is not possible}$$

Therefore, there is no value of λ for which *f* is continuous at *x* = *0*

At *x* = *1*,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$$\lim_{x\to 1}(4x+1) = 4\times 1+1 = 5$$

19. Show that the function defined by $g(x) = x-[x]$ is discontinuous at all integral point. Here $[x]$ denoted the greatest integer less than or equal to *x*.

Solution:

The given function is $g(x) = x-[x]$

It is evident that *g* is defined at all integral points.

Let *n* be a integer

Then,

$$g(n) = n-[n] = n-n = 0$$

The left hand limit of *f* at $x = n$ is

$$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}[x-[x]] = \lim_{x\to n^-} x - \lim_{x\to n^-}[x] = n-(n-1) = 1$$

The right hand limit of *f* at $x = n$ is

$$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}[x-[x]] = \lim_{x\to n^-} x - \lim_{x\to n^-}[x] = n-n = 0$$

It is observed that the left and right hand limits of *f* at $x = n$ do not coincide.

Therefore, *f* is not continuous at $x = n$

Hence, *g* is discontinuous at all integral points

20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Solution:

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at $x = \pi$

At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} f\left(x^2 - \sin x + 5\right)$

Put $x = \pi + h$

If $x \to \pi$, then it is evident that $h \to 0$

$$\therefore \quad \lim_{x\to\pi} f(x) = \lim_{x\to\pi}\left(x^2 - \sin x\right) + 5$$

$$= \lim_{h\to 0}\left[(\pi + h)^2 - \sin(\pi + h) + 5\right]$$

$$= (\pi + 0)^2 - \lim_{h\to 0}[\sin\pi \cosh + \cos\pi + \sinh] + 5$$

$$= \pi^2 - \lim_{h\to 0}\sin\pi \cosh - \lim_{h\to 0}\cos\pi + \sinh + 5$$

$$= \pi^2 - \sin\pi\cos 0 - \cos\pi\sin 0 + 5$$

$$= \pi^2 - 0\times 1 - (-1)\times 0 + 5$$

$$= \pi^2 + 5$$

$$\therefore \lim_{x\to x} f(x) = f(\pi)$$

Therefore, the given function f is continuous at $x = \pi$

21. Discuss the continuity of the following functions.

A) $f(x) = \sin x + \cos x$

B) $f(x) = \sin x - \cos x$

C) $f(x) = \sin x \times \cos x$

Solution:

It is known that if g and h are two continuous functions, then $g + h, g - h$ and $g.h$ are also continuous.

It has to proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x=c+h$

If $x \to c$, then $h \to 0$

$g(c)=\sin c$

$\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} g\sin x$

$=\lim\limits_{h\to 0}\sin(c+h)$

$=\lim\limits_{h\to 0}[\sin c\cosh+\cos c\sinh]$

$=\lim\limits_{h\to 0}(\sin c\cosh)+\lim\limits_{h\to 0}(\cos c\sinh)$

$=\sin c\cos 0+\cos c\sin 0$

$=\sin c+0$

$=\sin c$

$\therefore\ \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is a continuous function.

Let $h(x)=\cos x$

It is evident that $h(x)=\cos x$ is defined for every real number.

Let c be a real number. $x=c+h$

If $x \to c$, then $h \to 0$

$h(c)=\cos c$

$\lim\limits_{x\to c}(x)=\lim\limits_{x\to c}\cos x$

$=\lim\limits_{h\to 0}\cos(c+h)$

$=\lim\limits_{h\to 0}[\cos c\cosh-\sin c\sinh]$

$=\lim\limits_{h\to 0}\cos c\cosh-\lim\limits_{h\to 0}\sin c\sinh$

$=\cos c\cos 0-\sin c\sin 0$

$=\cos c\times 1-\sin c\times 0$

$=\cos c$

$\therefore \lim\limits_{h\to 0}(x)=h(c)$

Therefore, h is a continuous function.

Therefore, it can be concluded that

a) $f(x)=g(x)+h(x)=\sin x+\cos x$ is a continuous function

b) $f(x)=g(x)-h(x)=\sin x-\cos x$ is a continuous function

c) $f(x)=g(x)\times h(x)=\sin x\times\cos x$ is a continuous function

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Solution:

It is known that if p and h are two continuous functions, then

i. $\dfrac{h(x)}{g(x)}.g(x) \neq 0$ is continuous

ii. $\dfrac{1}{g(x)}.g(x) \neq 0$ is continuous

iii. $\dfrac{1}{h(x)}.h(x) \neq 0$ is continuous

It has to be proved first that $g(x) - \sin x$ and $h(x) - \cos x$ are continuous functions.

Let $g(x) - \sin x$

It has evident that $g(x) - \sin x$ is defined foc every real number.

Let c be a real number. Put $x - c + h$

If $x \to C,$ then $h \to 0$

$g(c) - \sin x$

$\lim\limits_{x+c} g(c) - \lim\limits_{x+c} \sin x$

$- \lim\limits_{x+c} \sin(c+h)$

$- \lim\limits_{x+c} [\sin c \cosh + \cos c \sinh]$

$- \lim\limits_{x+c} (\sin c \cosh) + \lim\limits_{x+c} (\cos c \sinh)$

$- \sin c \cos 0 + \cos c \sin 0$

$- \sin c + 0$

$- \sin c$

$\therefore \lim\limits_{0+c} g(x) - g(c)$

Therefore, g is a continuous function.

Let $h(x) - \cos x$

It is evident that $h(x) - \cos x$ is defined for every real number.

Let c be a real number. Put $x - c + h$

If $x \to c,$ then $h \to 0x$

$h(c) - \cos c$

$- \lim\limits_{x \to 0} \cos(c+h)$

$- \lim\limits_{x \to 0} [\cos c \cosh - \sin c \sinh]$

$- \lim\limits_{x \to 0} \cos c \cosh - \lim\limits_{x \to 0} \sin c \sinh$

$- \cos c \cos 0 - \sin c \sin 0$

$- \cos c\, x 1 - \sin c\, x 0$

$= \cos c$

$\therefore \lim\limits_{x+c} h(x) - h(c)$

Therefore, $h(x) - \cos x$ is continuous function.

It can be conclned that,

$\cos ex - \dfrac{1}{\sin x}, \sin x \neq 0$ is continuous

$\Rightarrow \cos ec\,x, x \neq nx(n \in Z)$ is continuous

Therefore, secant is continuous except at $X - np, nIZ$

$\sec x = \dfrac{1}{\cos x}, \cos x \neq 0$ is continuous

$\Rightarrow \sec x, x \neq (2n+1)\dfrac{\pi}{2}(n \in Z)$ is continuous

Therefore, secant is continuous except at $x - (2n+1)\dfrac{\pi}{2}(n \in Z)$

$\cot x = \dfrac{\cos x}{\sin x}, \sin x \neq 0$ is continuous

$\Rightarrow \cot x, x \neq n\pi(n \in Z)$ is continuous

Therefore, cotangent is continuous except at $x - np, nIZ$

23. Find the points of discontinuity of f, where $f(x) = \begin{cases} \dfrac{\sin x}{x}, & if \quad x < 0 \\ x+1, & if \quad x \geq 0 \end{cases}$

Solution:

The given function f is $f(x) = \begin{cases} \dfrac{\sin x}{x}, & if \quad x < 0 \\ x+1, & if \quad x \geq 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number

Case I:

If $c < 0$, then $f(c) = \dfrac{\sin c}{c}$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}\left(\dfrac{\sin x}{x}\right) = \dfrac{\sin c}{c}$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, then $f(c) = c+1$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x+1) = c+1$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $f(c) = f(0) = 0+1 = 1$

The left hand limit of f at $x = 0$ is,

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{\sin x}{x} = 1$

The right hand limit of f at $x = 0$ is,

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+}(x+1) = 1$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be conducted that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

24. Determine if f defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & if \neq 0 \\ 0, & if \neq 0 \end{cases}$ is a continuous function?

Solution:

The given function f is $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & if \neq 0 \\ 0, & if \neq 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin\frac{1}{c}$

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^2 \sin\frac{1}{x}\right) = \left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c} \sin\frac{1}{x}\right) = c^2 \sin\frac{1}{c}$$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all p[oints $x \neq 0$

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{2}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1, \ x \neq 0$

$$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$\therefore \lim_{x\to 0} f(x) = 0$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x) = 0$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus f is a continuous function.

25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & if \quad x \neq 0 \\ 1 \quad , & if \quad x = 0 \end{cases}$$

Solution:

The given function f is $f(x) = \begin{cases} \sin x - \cos x, & if \quad x \neq 0 \\ 1 \quad , & if \quad x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$\lim_{x\to c} f(x) = \lim_{x\to c}(\sin x - \cos x) = \sin c - \cos c$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x \neq 0$

Case II:

If $c = 0$, then $f(0) = -1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

26. Find the value of k so that the function f is continuous at the indicated point.

$$f(x)\begin{cases} \dfrac{k\cos x}{\pi-2x}, & if \quad x\neq\dfrac{\pi}{2} \\ 3, & if \quad x=\dfrac{\pi}{2} \quad at x=\dfrac{\pi}{2} \end{cases}$$

Solution:

The given function f is $f(x)\begin{cases} \dfrac{k\cos x}{\pi-2x}, & if \quad x\neq\dfrac{\pi}{2} \\ 3, & if \quad x=\dfrac{\pi}{2} \end{cases}$

The given function f is continuous at $x=\frac{\pi}{2}$, it is defined at $x=\frac{\pi}{2}$ and if the value of the f at $x=\frac{\pi}{2}$ equals the limit of f at $x=\frac{\pi}{2}$.

It is evident that f is defined at $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=3$

$$\lim_{x\to}\frac{\pi}{2}f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

Put $x=\frac{\pi}{2}+h$

Then, $x\to\frac{\pi}{2}\Rightarrow h\to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}}f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{x\to 0}\frac{-\sinh}{-2h}=\frac{k}{2}\lim_{x\to 0}\frac{\sinh}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}}f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow\frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of k is 6.

27. Find the value of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx^2, & if\ x\le 2 \\ 3, & if\ x>2 \end{cases} \text{ at } x=2$$

Solution:

The given function is $f(x)=\begin{cases} kx^2, & if\ x\le 2 \\ 3, & if\ x>2 \end{cases}$

The given function f is continuous at $x=2$, if f is defined at $x=2$ and if the value of f at $x=2$ equals the limit of f at $x=2$

It is evident that f is defined at $x=2$ and $f(2)=k(2)^2=4k$

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\Rightarrow \lim_{x\to 2^-}(kx)^2=\lim_{x\to 2^+}(3)=4k$

$\Rightarrow k\times 2^2=3=4k$

$\Rightarrow 4k=3=4k$

$\Rightarrow 4k=3$

$\Rightarrow k=\frac{3}{4}$

Therefore, the required value of k is $\frac{3}{4}$.

28. Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & if\ x\le \pi \\ \cos x, & if\ x>\pi \end{cases} at\ x=\pi$$

Solution:

The given function is $f(x)=\begin{cases} kx+1, & if\ x\le \pi \\ \cos x, & if\ x>\pi \end{cases}$

The given function f is continuous at $x=\pi$ and, if f is defined at $x=\pi$ and if the value of f at $x=\pi$ equals the limit of f at $x=\pi$

It is evident that f is defined at $x=\pi$ and $f(\pi)=k\pi+1$

$\lim_{x\to \pi^-} f(x)=\lim_{x\to \pi^+} f(x)=f(\pi)$

$\Rightarrow \lim_{x\to \pi^-}(kx+1)=\lim_{x\to \pi^+}\cos x=k\pi+1$

$\Rightarrow k\pi+1=\cos\pi=k\pi+1$

$\Rightarrow k\pi+1=-1=k\pi+1$

$\Rightarrow k\pi+1=-1=k\pi+1$

$\Rightarrow k=-\frac{2}{\pi}$

Therefore, the required value of k is $-\frac{2}{\pi}$.

29. Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & if \quad x\le 5 \\ 3x-5, & if \quad x>5 \end{cases}$$

Solution:

The given function of *f* is $f(x)=\begin{cases} kx+1, & if \quad x\le 5 \\ 3x-5, & if \quad x>5 \end{cases}$

The given function *f* is continuous at $x=5$, if *f* is defined at $x=5$ and if the value of *f* at $x=5$ equals the limit of *f* at $x=5$

It is evident that *f* is defined at $x=5$ and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$

$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$

$$\Rightarrow 5k+1=15-5=5k+1$$

$$\Rightarrow 5k+1=10$$

$$\Rightarrow 5k=9$$

$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of *k* is $\frac{9}{5}$

30. Find the vales of *a* and *b* such that the function defined

$$f(x)=\begin{cases} 5, & if \quad x\le 2 \\ ax+b, & if \quad 2<x<10 \\ 21 & if \quad x\ge 10 \end{cases}$$ is continuous function.

Solution:

The given function *f* is $f(x)=\begin{cases} 5, & if \quad x\le 2 \\ ax+b, & if \quad 2<x<10 \\ 21 & if \quad x\ge 10 \end{cases}$

It is evident that the given function *f* is defined at all points of the real line.

If *f* is a continuous function, then *f* is a continuous at all real numbers.

In a particular, *f* is continuous at $x=2$ and $x=10$

Since *f* is continuous at $x=2$, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$

$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$

$$\Rightarrow 5=2a+b=5$$

$$\Rightarrow 2a+b=5 \qquad \text{.......... (1)}$$

Since *f* is continuous at $x=10$, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$

$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 2^+}(21)=21$$

$$\Rightarrow 10a+b-21=21$$

$$\Rightarrow 10a+b=21 \qquad \text{.......... (2)}$$

On subtracting equation (1) from equation (2), we obtain

$$8a=16$$

$$\Rightarrow a=2$$

By putting $a=2$ in equation (1), we obtain

$$2\times 2+b=5$$

$$\Rightarrow 4+b=5$$

$$\Rightarrow b=1$$

Therefore, the values of *a* and *b* for which *f* is a continuous function are 2 and 1 respectively.

31. Show that the function defined by $f(x)=\cos(x^2)$ is a continuous function.

Solution:

The given function is $f(x)=\cos(x^2)$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f = goh$, where $g(x) = \cos x$ and $h(x) = x^2$

$\left[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)\right]$

It has to be first proves that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that *g* is defined foe every real number.

Let *c* be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \to c$, then $h \to 0$

$\lim_{x \to c} g(x) = \lim_{x \to c} \cos x$

$= \lim_{h \to 0} \cos(c + h)$

$= \lim_{h \to 0}[\cos c \cosh - \sin c \sinh]$

$= \lim_{h \to 0} \cos c \cosh - \lim_{h \to 0} \sin c \sinh$

$= \cos c \cos 0 - \sin c \sin 0$

$= \cos c \times 1 - \sin c \times 0$

$= \cos c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, $g(x) = \cos x$ is a continuous function

$h(x) = x^2$

Clearly, *h* is defined for every real number.

Let *k* be a real number, then $h(k) = k^2$

$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$

$\therefore \lim_{x \to k} h(x) = h(k)$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*, such that (goh) is defined at *c*, it *g* is continuous at *c* and it *f* is continuous at *c*.

Therefore, $f(x) = (goh)(x) = \cos(x^3)$ is a continuous function.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function

Solution:

The given function is $f(x) = |\cos x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f = goh$, where $g(x) = |x|$ and $h(x) = \cos x$

$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)\right]$

It has to be first proves that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$, can be written as

$$g(x) = \begin{cases} -x & if \quad x < 0 \\ x & if \quad x \geq 0 \end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, *g* is continuous at all points *x*, such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x>0$

Case III:

If $c=0$, then $g(c)=g(0)=0$ and $\lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$

$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$

$$\therefore \lim_{x\to c^-} g(x)=\lim_{x\to c^+} g(x)=g(0)$$

Therefore, g is continuous at $x=0$

From the above three observations, it can be concluded that g is continuous at all points.

$$h(x)=\cos x$$

It is evident that $h(x)=\cos x$ is defined for every real number.

Let c be a real number. Put $x=c+h$

If $x\to c$, then $h\to 0$

$$h(c)=\cos c$$

$$\lim_{x\to c} h(x)=\lim_{x\to c}\cos x$$

$$=\lim_{h\to 0}\cos(c+h)$$

$$=\lim_{h\to 0}[\cos c\cosh-\sin c\sinh]$$

$$=\lim_{h\to 0}\cos c\cosh-\lim_{h\to 0}\sin c\sinh$$

$$=\cos c\cos 0-\sin c\sin 0$$

$$=\cos c\times 1-\sin c\times 0$$

$$=\cos c$$

$$=\lim_{h\to c} h(x)=h(c)$$

Therefore, $h(x)=\cos x$ is continuous function/

It is known that for real values functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\cos x)=|\cos x|$ is a continuous function.

33. Examine that $\sin|x|$ is a continuous function

Solution:

Let $f(x)=\sin|x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f=goh$, where $g(x)=|x|$ and $h(x)=\sin x$

$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$

It has to be prove first that $g(x)=|x|$ and $h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x & if \quad x<0\\ x & if \quad x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, that $x<0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x>0$

Case III:

If $c=0$, then $g(c)=g(0)=0$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$

$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$

$$\therefore \lim_{x\to c^-} g(x)=\lim_{x\to c^+} g(x)=g(0)$$

Therefore, g is continuous at $x=0$

From the above three observations, it can be concluded that g is continuous at all points.

$$h(x)=\sin x$$

It is evident that $h(x)=\sin x$ is defined for every real number.

Let c be a real number. Put $x=c+k$

If $x\to c$, then $k\to 0$

$$h(c)=\sin c$$

$$\lim_{x\to c} h(x)=\lim_{x\to c}\sin x$$

$$=\lim_{k\to 0}\sin(c+k)$$

$$=\lim_{k\to 0}[\sin c\cos k+\cos c\sin k]$$

$$=\lim_{k\to 0}(\sin c\cos k)-\lim_{k\to 0}(\cos c\sin k)$$

$$=\sin c\cos 0+\cos c\sin 0$$

$$=\sin c+0$$

$$=\sin c$$

$$=\lim_{x\to c} h(x)=g(c)$$

Therefore, h is continuous function.

It is known that for real values functions *g* and *h*, such that (goh) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at $g(c)$, then (foh) is continuous at *c*.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\sin x)=|\sin x|$ is a continuous function.

34. Find all the points of discontinuity of *f* defined by $f(x)=|x|-|x+1|$.

Solution:

The given function is $f(x)=|x|-|x+1|$

The two functions, *g* and *h*, are defined as

$g(x)=|x|$ and $h(x)=|x+1|$

Then $f=g-h$

The continuous of *g* and *h* is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x & if \quad x<0\\ x & if \quad x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=g(0)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, *g* is continuous at all points *x*, that $x<0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, *g* is continuous at all points *x*, such that $x>0$

Case III:

If $c=0$, then $g(c)=g(0)=0$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$

$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$

$$\therefore \lim_{x\to c^-} g(x)=\lim_{x\to c^+} g(x)=g(0)$$

Therefore, g is continuous at $x=0$

From the above three observations, it can be concluded that g is continuous at all points.

$$h(x)=|x+1|$$

$$h(x)=\begin{cases}-(x+1), & if, \quad x<-1\\ x+1, & if, \quad x\geq -1\end{cases}$$

Clearly, h is defined for every real number.

Let c be a real number

Case I:

If $c<-1$, then $h(c)=-(c+1)$ and $\lim_{x\to c} h(x)=\lim_{x\to c}[-(x+1)]=-(c+1)$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, h is continuous at all points x, such that $x<-1$

Case II:

If $c>-1$, then $h(c)=c+1$ and $\lim_{x\to c} h(x)=\lim_{x\to c} x+1=(c+1)$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, g is continuous at all points x, such that $x>-1$

Case III:

If $c=-1$, then $h(c)=h(-1)=-1+1=0$

$$\lim_{x\to 1^-} h(x)=\lim_{x\to 1^-}[-(x+1)]=-(-1+1)0$$

$$\lim_{x\to 1^+} h(x)=\lim_{x\to 1^+}(x+1)=(-1+1)=0$$

$$\therefore \lim_{x\to 1^-} h = \lim_{x\to 1^+} h(x) = h(-1)$$

Therefore, *h* is continuous at $x = -1$

From the above three observations, it can be concluded that *h* is continuous at all points of the real line.

g and *h* are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, *f* has no point of discontinuity.

Exercise 5.2

1. Differentiate the function with respect to x. $\sin\left(x^2+5\right)$

Solution:

Let $f(x)=\sin\left(x^2+5\right), u(x)=x^2+5, \text{ and } v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, f is a composite of two functions

Put $t=u(x)=x^2+5$

Then, we obtain

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$$

Therefore, by chain rule, $\frac{df}{dx}=\frac{dv}{dt}\frac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$

Alternate method

$$\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]=\cos\left(x^2+5\right)\frac{d}{dx}\left(x^2+5\right)$$

$$=\cos\left(x^2+5\right)\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]$$

$$=\cos\left(x^2+5\right)[2x+0]$$

$$=2x\cos\left(x^2+5\right)$$

2. Differentiate the functions with respect of x. $\cos(\sin x)$

Solution:

Let $f(x)=\cos(\sin x), u(x)=\sin x, \text{ and } v(t)=\cos t$

Then, $(vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$

Thus, f is a composite function of two functions

Put $t=u(x)=\sin x$

$$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$

$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

By chain rule, $\frac{df}{dx},\frac{dv}{dt}.\frac{dt}{dx}=-\sin(\sin x)\cos x=-\cos x\sin(\sin x)$

Alternate method

$$\frac{d}{dx}[\cos(\sin x)]=-\sin(\sin x)\frac{d}{dx}(\sin x)=-\sin(\sin x)-\cos x=-\cos x\sin(\sin x)$$

3. Differentiate the functions with respect of x. $\sin(ax+b)$

Solution:

Let $f(x)=\sin(ax+b), u(x)=ax+b, \text{ and } v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$

Thus, f is a composite function of two functions u and v

Put $t=u(x)=ax+b$

Therefore, $\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx}=\frac{dv}{dt}\frac{dt}{dx}=\cos(ax+b)a=a\cos(ax+b)$$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b)\frac{d}{dx}(ax+b)$$

$$=\cos(ax+b)\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$

$$=\cos(ax+b)(a+0)$$

$$=a\cos(ax+b)$$

4. Differentiate the functions with respect of x. $\sec\left(\tan\left(\sqrt{x}\right)\right)$

Solution:

Let $f(x)=\sec\left(\tan\left(\sqrt{x}\right)\right), u(x)=\sqrt{x}, v(t)=\tan t, \textit{ and } w(s)=\sec s$

Then, $(wovou)(x)=w\left[v\left(u(x)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, f is a composite function of three functions, u, v and w

Put $s=v(t)=\tan t \textit{ and } t=u(x)=\sqrt{x}$

Then, $\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t)\tan(\tan t) \quad [s=\tan t]$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right) \quad \left[t=\sqrt{x}\right]$$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}.x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\frac{ds}{dt}\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right).\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

5. Differentiate the functions with respect of X. $\frac{\sin(ax+b)}{\cos(cx+d)}$

Solution:

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where $g(x)=\sin(ax+b)$ and

$h(x)=\cos(cx+d)$

$\therefore h(x)=\cos(cx+d)$

$\therefore f=\frac{g'h-gh'}{h^2}$

Consider $g(x)=\sin(ax+b)$

Let $u(x)=ax+b, v(t)=\sin t$

Then $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

$\therefore$ g is a composite function of two functions, u and v

Put $t=u(x)=ax+b$

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Therefore, by chain rule, we obtain

$$g'=\frac{dg}{dx}=\frac{dv}{dt}\frac{dt}{dx}=\cos(ax+b).a=a\cos(ax+b)$$

Consider $h(x)=\cos(cx+d)$

Let $p(x)=cx+d, q(y)=\cos y$

Then, $(qop)(x)=q(p(x))=q(cx+d)=\cos(cx+d)=h(x)$

$\therefore$ h is a composite function of two functions, p and q

Put $y=p(x)=cx+d$

$$\frac{dq}{dy}=\frac{d}{dy}(\cos y)=-\sin y=-\sin(cx+d)$$

$$\frac{dy}{dx}=\frac{d}{dx}(cx+d)=\frac{d}{dx}(cx)+\frac{d}{dx}(d)=c$$

Therefore, by chain rule, we obtain

$$h'=\frac{dh}{dx}=\frac{dq}{dy}\frac{dy}{dx}=-\sin(cx+d)\times c=-c\sin(cx+d)$$

$$\therefore f' = \frac{a\cos(ax+b)\cos(cx+d) - \sin(ax+b)\{-c\sin cx+d\}}{\left[\cos(cx+d)\right]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\frac{\sin(cx+d)}{\cos(cx+d)} \times \frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+b)\sec(cx+d)$$

6. Differentiate the function with respect to x. $\cos x^3.\sin^2\left(x^5\right)$

Solution:

$$\cos x^3.\sin^2\left(x^5\right)$$

$$\frac{d}{dx}\left[\cos x^3.\sin^2\left(x^5\right)\right] = \sin^2\left(x^5\right) \times \frac{d}{dx}\left(\cos x^3\right) + \cos^3 \times \frac{d}{dx}\left[\sin^2\left(x^5\right)\right]$$

$$= \sin^2\left(x^5\right) \times \left(-\sin x^3\right) \times \frac{d}{dx}\left(x^3\right) + \cos x^3 + 2\sin\left(x^5\right)\frac{d}{dx}\left[\sin x^5\right]$$

The given function is $= \sin x^3 \sin^2\left(x^5\right) \times 3x^2 + 2\sin x^5 \cos x^3.\cos x^5 \times \frac{d}{dx}\left(x^5\right)$

$$= 3x^2 \sin x^3 \sin^3\left(x^5\right) + 2\sin x^5 \cos x^5 \cos x^3 \times 5x^4$$

$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2\left(x^5\right)$$

7. Differentiate the functions with respect to x.

$$\sqrt[2]{\cot\left(x^2\right)}$$

Solution:

$$\frac{d}{dx}\left[\sqrt[2]{\cot\left(x^2\right)}\right]$$

$$= 2.\frac{1}{\sqrt[2]{\cot\left(x^2\right)}} \times \frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \times -\cos ec^2\left(x^2\right) \times \frac{d}{dx}\left(x^2\right)$$

$$= \sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \times \frac{1}{\sin^2\left(x^2\right)} \times (2x)$$

$$= \frac{-2x}{\sqrt{\cos x^2 \sqrt{\sin x^2 \sin x^2}}}$$

$$= \frac{-2\sqrt{2}x}{\sqrt{2\sin x^2 \cos x^2 \sin x^2}}$$

$$= \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{\sin 2x^2}}$$

8. Differentiate the functions with respect to x

$\cos\left(\sqrt{x}\right)$

Solution:

Let $f(x) = \cos\left(\sqrt{x}\right)$

Also, let $u(x) = \sqrt{x}$

And, $v(t) = \cos t$

Then, $(vou)(x) = v\left(u(x)\right)$

$= v\left(\sqrt{x}\right)$

$= \cos\sqrt{x}$

$= f(x)$

Clearly, f is a composite function of two functions, u and v, such that $t = u(x) = \sqrt{x}$

Then, $\frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

And, $\frac{dv}{dt}=\frac{d}{dt}(\cos t)=-\sin t=-\sin\sqrt{x}$

By using chain rule, we obtain

$$\frac{dt}{dx}=\frac{dv}{dt}\frac{dt}{dx}$$

$$=-\sin\left(\sqrt{x}\right)\frac{1}{2\sqrt{x}}$$

$$=-\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$

$$=-\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right]=-\sin\left(\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=-\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=-\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

9. Prove that the function f given by $f(x)=|x-1|, x\in R$ is not differentiable at x = 1

Solution:

The given function is $f(x)=|x-1|, x\in R$

It is known that a function f is differentiable at a point x = c in its domain if both

$$\lim_{k\to 0^-}\frac{f(c+h)-f(c)}{k}\ and\ \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$$ are finite and equal

To check the differentiability of the given function at $x=1$,

Consider the left hand limit of f at $x=1$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{f|I+h-1||1-1|}{h}$$

$$\lim_{h\to 0^-}\frac{|h|-0}{h}=\lim_{h\to 0^-}\frac{-h}{h} \qquad (h<0\Rightarrow |h|=-h)$$

= - 1

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{f|I+h-1|-|1-1|}{h}$$

$$=\lim_{h\to 0^+}\frac{|h|-0}{h}=\lim_{h\to 0^+}\frac{h}{h} \qquad (h>0\Rightarrow |h|=h)$$

= 1

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at $x=1$

10. Prove that the greatest integer function defined by $f(x)=[x], 0<x<3$ is not differentiable at $x=1 \, and \, x=2$

Solution:

The given function f is $f(x)=[x], 0<x<3$

It is known that a function f is differentiable at a point $x=c$ in its domain if both $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ *and* $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal

To check the differentiable of the given function at x = 1, consider the left hand limit of f at x = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at x = 1

$$= \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[1+h][1]}{h}$$

$$= \lim_{h \to 0^+} \frac{1-1}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right limits of f at x = 1 are not equal, f is not differentiable at x = 1

To check the differentiable of the given function at x=2, consider the left hand limit of f at x = 2

$$= \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^-} \frac{1-2}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

Consider the right hand limit of f at x= 1

$$= \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^+} \frac{1-2}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at x = 2 are not equal, f is not differentiable at $x = 2$

Exercise 5.3

11. Find $\frac{dy}{dx}: 2x+3y=\sin x$

Solution:

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dy}(2x+3y)=\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dy}(2x)+\frac{d}{dx}(3y)=\cos x$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$

$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$

$$\therefore \frac{dx}{dy}=\frac{\cos x-2}{3}$$

12. Find $\frac{dy}{dx}: 2x+3y=\sin y$

Solution:

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{d}{dx}=\cos y\frac{dy}{dx}$$ (by using chain rule)

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

13. Find $\frac{dy}{dx}: ax + by^2 = \cos y$

Solution:

The given relationship is $ax + by^2 = \cos y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y)$$

$$\Rightarrow a + b\frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2) = 2y\frac{d}{dx}(\cos y)$........(1)

And $\frac{d}{dx}(\cos y) = \sin y\frac{d}{dx}$........(2)

From (1) and (2), we obtain

$$a + bx\,2y\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by + \sin y)\frac{dy}{dx} = -a$$

$$\therefore \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

14. Find $\frac{dy}{dx}: xy + y^2 = \tan x + y$

Solution:

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(xy+y^2\right)=\frac{d}{dx}\left(\tan x+y\right)$$

$$\Rightarrow \frac{d}{dx}(xy)+\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(\tan x)+\frac{dy}{dx}$$

$$\Rightarrow \left[y.\frac{d}{dx}(x)+x.\frac{d}{dx}\right]+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$ [using product rule and chain rule]

$$y.1+x\frac{dy}{dx}+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

$$\Rightarrow (x+2y-1)\frac{dy}{dx}=\sec^2 x-y$$

$$\therefore \frac{dy}{dx}=\frac{\sec^2 x-y}{(x+2y-1)}$$

15. Find $\frac{dy}{dx}: x^2+xy+y^2=100$

Solution:

The given relationship is $x^2+xy+y^2=100$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(x^2+xy+y^2\right)=\frac{d}{dx}(100)$$ [Derivative of constant function is 0]

$$\Rightarrow \frac{d}{dx}x^2+\frac{d}{dx}(xy)+\frac{d}{dx}\left(y^2\right)=0$$

$$\Rightarrow 2x+\left[y.\frac{d}{dx}(x)+x.\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0$$ [Using product rule and chain rule]

$$\Rightarrow 2x+y.1+x.\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y+(x+2y).\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

16. Find $\frac{dy}{dx}: x^2 + x^2y + xy^2 + y^3 = 81$

Solution:

The given relationship is $x^2 + x^2y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(x^3 + x^2y + xy^2 + y^3\right) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^2y\right) + \frac{d}{dx}(xy)^2 + \frac{d}{dx}\left(y^3\right) = 0$$

$$\Rightarrow 3x^2 + \left[y\frac{d}{dx}\left(x^2\right) + x^2\frac{dy}{dx}\right] + \left[y^2\frac{d}{dx}(x) + x\frac{d}{dx}\left(y^2\right)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left[y.2x + x^2\frac{dx}{dy}\right] + \left[y^2.1 + x.2y\frac{dy}{dx}\right] + 3y^2\frac{dx}{dy} = 0$$

$$\Rightarrow \left(x^2 + 2xy + 3y^2\right)\frac{dy}{dx} + \left(3x^2 + 2xy + y^2\right) = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\left(3x^2 + 2xy + y^2\right)}{\left(x^2 + 2xy + 3y^2\right)}$$

17. Find $\frac{dx}{dy}: \sin^2 y + \cos xy = \pi$

Solution:

The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi) \ldots\ldots\ldots\ldots (1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right)+\frac{d}{dx}\left(\cos xy\right)=0$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right)+2\sin y\frac{d}{dx}\left(\sin y\right)=2\sin y\cos y\frac{dy}{dx}$$

$$\frac{d}{dx}\left(\cos xy\right)+-\text{sinx}\, y\left(xy\right)=-\text{sinx}\, y\left[y\frac{d}{dx}\left(x\right)+x\frac{dy}{dx}\right]\text{.........}\left(2\right)$$

$$=-\sin xy\left[y.1+x\frac{dy}{dx}\right]=-y\sin xy-x\sin xy\frac{dy}{dx}\text{ .... (3)}$$

From (1), (2) and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx}-y\sin xy-x\sin xy\frac{dy}{dx}=0$$

$$\Rightarrow\left(2\sin y\cos y-x\sin xy\right)\frac{dy}{dx}=y\sin xy$$

$$\Rightarrow\left(\sin 2y-x\sin xy\right)\frac{dx}{dy}=y\sin xy$$

$$\therefore \frac{dx}{dy}=\frac{y\sin xy}{\sin 2y-x\sin xy}$$

18. Find $\frac{dy}{dx}:\sin^2 x+\cos^2 y=1$

Solution:

The given relationship is $\sin^2 x+\cos^2 y=1$

Differentiating this relationship with respect to x, we obtains

$$\frac{dy}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\cos^2 y=0$$

$$\Rightarrow 2\sin x.\frac{d}{dx}(\sin x)+2\cos y.\frac{d}{dx}(\cos y)=0$$

$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y).\frac{dy}{dx}=0$$

$$\Rightarrow \sin x-\sin 2y\frac{dy}{dx}=0$$

$$\therefore \frac{dx}{dy}=\frac{\sin 2x}{\sin 2y}$$

19. Find $\frac{dy}{dx}: y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution:

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)\text{...........}(1)$$

The function $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{(1+x^2)\frac{d}{dx}(2x)-2x.\frac{d}{dx}(1+x^2)}{(1+x^2)}$$

$$=\frac{(1+x^2).2-2x[0+2x]}{(1+x^2)^2}=\frac{2+2x^2-4x^3}{(1+x^2)^2}=\frac{2(1+x^2)}{(1+x^2)^2}.........(2)$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{\left((1+x^2)^2-4x^2\right)}{(1+x^2)^2}}........(3)$$

$$=\sqrt{\frac{(1-x^2)^2}{(1-x^2)^2}}=\frac{1-x^2}{1+x^2}$$

From (1) (2) and (3) we obtain

$$\frac{1-x^2}{1+x^2}x\frac{dy}{dx}=\frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

20. Find $\frac{dx}{dy}: y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

Solution:

The given relationship is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan y=\frac{3x-x^3}{1-3x^2}............(1)$$

It is known that, $\tan y=\frac{3\tan\frac{y}{3}-\tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}.............(2)$

Comparing equations (1) and (2), we obtain

$$x = \tan\frac{y}{3}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3}.\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3}.\frac{1}{3}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2\frac{y}{3}} = \frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dx}{dy} = \frac{3}{1+x^2}$$

21. Find $\frac{dy}{dx} : y\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Solution:

The given relationship is,

$$y\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}} = \frac{1-x^2}{1+x^2}$$

On comparing LHS and RHS of the above relationship, we obtain

$$\tan\frac{y}{2} = x$$

Differentiating this relationship with respect to x, we obtain

$$\sec^2\frac{y}{2}.\frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{d}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

22. Find $\frac{dy}{dx}: \mathrm{y} = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Solution:

The given relationship is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow = \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)............(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y)=\cos y.\frac{dy}{dx}$$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$=\sqrt{\frac{\left(1+x^2\right)^2-\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\sqrt{\frac{4x^2}{\left(1+x^2\right)^2}}=\frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y)=\frac{2x}{1+x^2}\frac{dy}{dx}..........(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{\left(1+x^2\right)\left(1-x^2\right)'-\left(1-x^2\right)\left(1+x^2\right)'}{\left(1+x^2\right)^2}$$ [using quotient rule]

$$=\frac{\left(1+x^2\right)(-2x)-\left(1-x^2\right)(2x)}{\left(1+x^2\right)^2}$$

$$=\frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$=\frac{-4x}{\left(1+x^2\right)^2}.............(3)$$

From (1), (2) and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Alternate method

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow (1+\sin y)x^2 = 1-\sin y$$

$$\Rightarrow x^2=\frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2=\frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{x}\right)^2}$$

$$\Rightarrow x=\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x=\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{\pi}{2}\right)$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}.\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1=\sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right).\frac{d}{dx}\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

$$\Rightarrow 1=\left[1+\tan^2\left(\frac{\pi}{4}-\frac{y}{2}\right).\left(-\frac{1}{2}.\frac{dy}{dx}\right)\right]$$

$$\Rightarrow 1=(1+x^2)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dx}{dy}=\frac{-2}{1+x^2}$$

23. Find $\frac{dy}{dx}: y=\cos^{-1}\left(\frac{2x}{1+x^2}\right), -1<x<1$

Solution:

The given relationship is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y.\frac{dy}{dx}=\frac{(1+x^2)\frac{d}{dx}(2x)-2x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx}=\frac{(1+x^2)\times 2-2x.2x}{(1+x^2)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx}=-\left[\frac{2(1-x)^2}{(1+x^2)^2}\right]$$

$$\Rightarrow \sqrt{\frac{(1-x^2)^2-4x^2}{(1+x^2)^2}}\,\frac{dy}{dx}=\frac{-2(1-x^2)}{(1+x^2)}$$

$$\Rightarrow \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}\,\frac{dy}{dx}=\frac{-2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\frac{dy}{dx}=\frac{-2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

24. Find $\frac{dy}{dx}: y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Solution:

Relationship is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y = 2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to x, we obtain

$$\cos y = \frac{dy}{dx} = 2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{d}{dx}\right]$$

$$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx} = 2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$$

$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)}\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{\left(1-2x\right)^2}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \left(1-2x\right)\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

25. Find $\frac{dy}{dx}: y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Solution:

The given relationship is $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y=\frac{1}{2x^2-1}$$

$$\Rightarrow \cos y=2x^2-1$$

$$\Rightarrow 2x^2=1+\cos y$$

$$\Rightarrow 2x^2=2\cos^2\frac{y}{2}$$

$$\Rightarrow x=\cos\frac{y}{2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\cos\frac{y}{2}\right)$$

$$\Rightarrow 1=\sin\frac{y}{2}.\frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin\frac{y}{2}}=\frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{\sin\frac{y}{2}}=\frac{-2}{\sqrt{1-\cos^2\frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$$

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Exercise 5.4

1. Differentiating the following w.r.t $x : \frac{e^x}{\sin x}$

Solution:

Let $y = \frac{e^x}{\sin x}$

Differentiating wrt x, we obtain

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}\left(e^x\right) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x.\left(e^x\right) - e^x.(\cos x)}{\sin^2 x}$$

$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in Z$$

2. Differentiating the following $e^{\sin^{-1} x}$

Solution:

Let $y = e^{\sin^{-1} x}$

Differential wrt x, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1} x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} x}.\frac{d}{dx}\left(\sin^{-1} x\right)$$

$$\Rightarrow e^{\sin^{-1} x}.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1,1)$$

3. Differentiating the following wrt $x : e^{x^3}$

Solution:

Let $y = e^{x^3}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx} = \left(e^{x^3}\right) = e^{x^3} . \frac{d}{dx}\left(x^3\right) = e^{x^3} .3x^2 = 3x^2 e^{x^3}$$

4. Differentiating the following w.r.t $x : \sin\left(\tan^{-1} e^{-x}\right)$

Solution:

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx} : \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$

$$= \cos\left(\tan^{-1} e^{-x}\right). \frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$

$$= \cos\left(\tan^{-1} e^{-x}\right). \frac{1}{1+\left(e^{-x}\right)^2} . \frac{d}{dx}\left(e^{-x}\right)$$

$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} .\text{e}^{-x} . \frac{d}{dx}(-x)$$

$$= \frac{e^{-x} \cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}(-1)$$

$$= \frac{-e^{-x} \cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

5. Differentiating the following w.r.t $x: \log\left(\cos e^x\right)$

Solution:

Let $y=\log\left(\cos e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\cos e^x\right)\right]$$

$$=\frac{1}{\cos e^x}\cdot\frac{d}{dx}\left(\cos e^x\right)$$

$$=\frac{1}{\cos e^x}\cdot\left(-\sin e^x\right)\frac{d}{dx}\left(e^x\right)$$

$$=\frac{-\sin e^x}{\cos e^x}.e^x$$

$$=-e^x\tan e^x, e^x\neq(2n+1)\frac{\pi}{2}, n\in N$$

6. Differentiating the following w.r.t $x: e^x+e^{x^2}+..........+e^{x^5}$

Solution:

$$\frac{d}{dx}\left(e^x+e^{x^2}+..........+e^{x^5}\right)$$

$$=\frac{d}{dx}\left(e^x\right)+\frac{d}{dx}\left(e^{x^2}\right)+\frac{d}{dx}\left(e^{x^3}\right)+\frac{d}{dx}\left(e^{x^4}\right)+\frac{d}{dx}\left(e^{x^5}\right)$$

$$=e^x+\left[e^{x^2}\frac{d}{dx}\left(x^2\right)\right]+\left[e^{x^3}\frac{d}{dx}\left(x^3\right)\right]+\left[e^{x^4}\frac{d}{dx}\left(x^4\right)\right]+\left[e^{x^5}\frac{d}{dx}\left(x^5\right)\right]$$

$$=e^x+\left(e^{x^2}2x\right)+\left(e^{x^3}3x^2\right)+\left(e^{x^4}4x^3\right)+\left(e^{x^5}5x^4\right)$$

$$=e^x+2xe^{x^2}+3x^2e^{x^3}+5x^4e^{x^5}$$

7. Differentiating the following w.r.t $x : \sqrt{e^{\sqrt{x}}}, x > 0$

Solution:

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to x, we obtain

$y^2 = e^{\sqrt{x}}$ [by applying the chain rule]

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x > 0$$

8. Differentiating the following w.r.t $x : \log(\log x), x > 1$

Solution:

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right]$$

$$= \frac{1}{\log x}.\frac{d}{dx}(\log x)$$

$$=\frac{1}{\log x}.\frac{1}{x}$$

$$\frac{1}{x\log x}, x>1$$

9. Differentiating the following w.r.t $x:\frac{\cos x}{\log x}, x>0$

Solution:

Let $y=\frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\log x-\cos x\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$\frac{-\sin x\log x-\cos x\frac{1}{x}}{(\log x)^2}$$

$$=\frac{-[x\log x.\sin x+\cos x]}{x(\log x)^2}, x>0$$

10. Differentiating the following w.r.t $x:\cos\left(\log x+e^x\right), x>0$

Solution:

Let $y=\cos\left(\log x+e^x\right)$

By using the chain rule, we obtain

$$y=\cos\left(\log x+e^x\right)$$

$$\frac{dy}{dx}=-\sin\left[\log x+e^x\right]\frac{d}{dx}\left(\log x+e^x\right)$$

$$=\sin\left(\log x+e^x\right)\left[\frac{d}{dx}(\log x)+\frac{d}{dx}\left(e^x\right)\right]$$

$$= -\sin\left(\log x + e^x\right)\left(\frac{1}{x} + e^x\right)$$

$$= \left(\frac{1}{x} + e^x\right)\sin\left(\log x + e^x\right), x > 0$$

Exercise 5.5

1. Differentiate the following with respect to x.

$\cos x.\cos 2x.\cos 3x$

Solution:

Let $y=\cos x.\cos 2x.\cos 3x$

Taking logarithm or both the side, we obtain

$$\log y=\log(\cos x.\cos 2x.\cos 3x)$$

$$\Rightarrow \log y=\log(\cos x)+\log(\cos 2x)+\log(\cos 3x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\cos x}\frac{d}{dx}(\cos x)+\frac{1}{\cos 2x}\frac{d}{dx}(\cos 2x)+\frac{1}{\cos 3x}\frac{d}{dx}(\cos 3x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{\sin x}{\cos x}-\frac{\sin 2x}{\cos 2x}\frac{d}{dx}(2x)-\frac{\sin 3x}{\cos 3x}\frac{d}{dx}(3x)\right]$$

$$\therefore \frac{dy}{dx}=-\cos x.\cos 2x.\cos 3x[\tan x+2\tan 2x+3\tan 3x]$$

2. Differentiate the functions with respect to x.

$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution:

Let $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm or both the side, we obtain

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y=\frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-1}\frac{d}{dx}(x-1)+\frac{1}{x-2}\frac{d}{dx}(x-2)-\frac{1}{x-3}\frac{d}{dx}(x-3)-\frac{1}{x-4}\frac{d}{dx}(x-4)-\frac{1}{x-5}\frac{d}{dx}(x-5)\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

3. Differentiate the functions with respect to x.

$(\log x)^{\cos x}$

Solution:

Let $y=(\log x)^{\cos x}$

Taking logarithm or both the side, we obtain

$$\log y=\cos x.\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin \times \log(\log x)\right]$$

4. Differentiate the functions with respect to x.

$x^x - 2^{\sin x}$

Solution:

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u \; and \; 2^{\sin x} = v$

$\therefore y = u - v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$u = x^x$

Taking logarithm on both sides, we obtain

$\log u = x\log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x)\times\log x + x\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1\times\log x + x\times\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to x, we obtain

$\log v = \sin x.\log 2$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \log 2 \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dv}{dx} = x^2(1+\log x) - 2^{\sin x} \cos x \log 2$$

5. Differentiate the functions with respect to x.

$$(x+3)^2.(x+4)^3.(x+5)^4$$

Solution:

Let $y = (x+3)^2(x+4)^3(x+5)^4$

Taking logarithm on both sides, we obtain

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$

$$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 2.\frac{1}{x+3}\frac{d}{dx}(x+3) + 3.\frac{1}{x+4}\frac{d}{dx}(x+4) + 4.\frac{1}{x+5}\frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3.\left[2(x^2+9x+20)+3(x^2+9x+15)+4(x^2+7x+12)\right]$$

$$\therefore \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3\left(9x^2+70x+133\right)$$

6. Differentiate the function with respect to x.

$$\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$$

Solution:

Let $y=\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$

Also, let $u=\left(x+\frac{1}{x}\right)^x$ and $v=x^{\left(1+\frac{1}{x}\right)}$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ....(1)$$

Then, $u=\left(x+\frac{1}{x}\right)^x$

Taking log on both sides

$$\Rightarrow \log u=\log\left(x+\frac{1}{x}\right)^x$$

$$\Rightarrow \log u=x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log\left(x+\frac{1}{x}\right) + \frac{x}{\left(x+\frac{1}{x}\right)} x\left(x+\frac{1}{x^2}\right)\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x \left[\log\left(x+\frac{1}{x}\right) + \frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x \left[\log\left(x+\frac{1}{x}\right) + \frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] \quad \text{.............}(2)$$

$$v = x^{\left(x+\frac{1}{x}\right)}$$

Taking log on both sides, we obtain

$$v = x^{\left(x+\frac{1}{x}\right)}$$

$$\Rightarrow \log v = \left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right] \times \log x + \left(1+\frac{1}{x}\right)\frac{d}{dx}\log x$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left(-\frac{1}{x^2}\right)\log x + \left(1+\frac{1}{x}\right)\frac{1}{x}$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{-\log x + x + 1}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = x^{\left(x+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right) \quad ....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(x+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

7. Differentiate the functions with respect to x.

$(\log x)^x + x^{\log x}$

Solution:

Let $y = (\log x)^x + x^{\log x}$

Also, let $u = (\log x)^x$ *and* $v = x^{\log x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ....(1)$$

$u = (\log x)^x$

$\Rightarrow \log u = \log\left[(\log x)^x\right]$

$\Rightarrow \log u = x\log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\,x\log(\log x) + x.\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1x\log(\log x) + x.\frac{1}{\log x}\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{x}{\log x}\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\frac{\log(\log x).\log x + 1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x-1}\left[1 + \log x.\log(\log x)\right] \quad ....(2)$$

$$v = x^{\log x}$$

$$\Rightarrow \log v = \log\left(x^{\log x}\right)$$

$$\Rightarrow \log v = \log x \log x = (\log x)^2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left[(\log x)^2\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = 2(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x - 1}\log x \quad ....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx} = (\log x)^{x-1}\left[1 + \log x.\log(\log x)\right] + 2x^{\log x - 1}\log x$$

8. Differentiate the functions with respect to x

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution:

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ *and* $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \quad ....(1)$$

$$u = (\sin x)^x$$

$$\Rightarrow \log u = \log(\sin x)^x$$

$$\Rightarrow \log u = x \log(\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\sin x) + x \times \frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{du}{dx} = u\left[1.\log(\sin x) + x.\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[\log(\sin x) + \frac{x}{\sin x}.\cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x) \quad ....(2)$$

$$v = \sin^{-1}\sqrt{x}$$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \quad ....(3)$$

Therefore, from (1), (2) and (3) we obtain

$$\frac{dy}{dx} = (\sin x)^2 (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x-x^2}}$$

9. Differentiate the function with respect to x.

$$x^{\sin x} + (\sin x)^{\cos x}$$

Solution:

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also $u = x^{\sin x}$ *and* $v = (\sin x)^{\cos x}$

$\therefore y = u + v$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ....(1)$

$u = x^{\sin x}$

$\Rightarrow \log u = \log\left(x^{\sin x}\right)$

$\Rightarrow \log u = \sin x \log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(\sin x).\log x + \sin x \frac{d}{dx}(\log x)$

$\Rightarrow \frac{du}{dx} = u = \left[\cos x \log x + \sin x.\frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right] \quad ....(2)$

$v = (\sin x)^{\cos x}$

$\Rightarrow \log v = \log(\sin x)^{\cos x}$

$\Rightarrow \log v = \cos x \log(\sin x)$

Differentiating both sides with respect to x, we obtain

$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cos x)\times\log(\sin x) + \cos x \times \frac{d}{dx}[\log(\sin x)]$

$\Rightarrow \frac{dv}{dx} = v\left[-\sin x.\log(\sin x) + \cos x.\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$

$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x.\log\sin x + \frac{\cos x}{\sin x}\cos x\right]$

$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}[-\sin x \log\sin x + \cot x \cos x]$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}[\cot x \cos x - \sin x \log \sin x] \quad ......(3)$$

Therefore, from (1), (2) and (3) we obtain

$$\frac{dy}{dx} = x^{\sin x}\left(\cos x \log x + \frac{\sin x}{x}\right) + (\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x]$$

10. Differentiate the function with respect to x.

$$x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

Solution:

Let $y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Also, let $u = x^{x\cos x}$ and $v = \frac{x^2+1}{x^2-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\therefore y = u + v$$

$$u = x^{x\cos x}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\cos x \log x + x\frac{d}{dx}(\cos x)\log x + x\cos x\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[1.\cos x.\log x + x.(-\sin x)\log x + x\cos x.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}(\cos x \log x - x\sin x \log x + \cos x)$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}[\cos x(1+\log x) - x\sin x \log x] \quad ....(2)$$

$$v = \frac{x^2+1}{x^2-1}$$

$$\Rightarrow \log v = \log\left(x^2+1\right) - \log\left(x^2-1\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} = \frac{dv}{dx} = \frac{2x}{x^2+1} - \frac{2x}{x^2-1}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{2x\left(x^2-1\right) - 2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{x^2+1}{x^2-1} \times \left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{\left(x^2-1\right)^2} \qquad \text{..........}(3)$$

Therefore, from (1), (2) and (3) we obtain

$$\frac{dy}{dx} = x^{x\cos x}\left[\cos x\left(1+\log x\right) - x\sin x \log x\right] - \frac{4x}{\left(x^2-1\right)^2}$$

11. Differentiate the function with respect to x.

$$\left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$$

Solution:

Let $y = \left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$

Also, let $u = \left(x\cos x\right)^x$ *and* $v = \left(x\sin x\right)^{\frac{1}{x}}$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{.......}(1)$$

$$u = \left(x\cos x\right)^2$$

$$\Rightarrow \log u = \log\left(x\cos x\right)^x$$

$$\Rightarrow \log u = x\log(x\cos x)$$

$$\Rightarrow \log u = x[\log x + \log\cos x]$$

$$\Rightarrow \log u = x\log x + x\log\cos x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x+\log x) + \frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\left\{\log x\frac{d}{dx}(x) + x\frac{d}{dx}(\log x)\right\} + \left\{\log\cos x\frac{d}{dx}(x) + x\frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[\left(\log x.1 + x.\frac{1}{x}\right) + \left\{\log\cos x.1 + x.\frac{1}{\cos x}.\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[\left(\log x.1 + x.\frac{1}{x}\right) + \left\{\log\cos x.1 + x.\frac{1}{\cos x}.\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[(\log x + 1) + \left\{\log\cos x + \frac{x}{\cos x}(-\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[(1+\log x) + (\log\cos x - x\tan x)\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[1 - x\tan x + (\log x + \log\cos x)\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right] \quad ....(2)$$

$$v = (x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}(\log x + \log\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log \sin x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}(\log x)\right] + \left[\log(\sin x).\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{x}\right] + \left[\log(\sin x).\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1-\log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}.\cos x\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(x\sin x)^{\frac{1}{x}} + \left[\frac{1-\log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x - \log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x) + x\cot x}{x^2}\right] \quad .....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx} = (x\cos x)^2\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

12. Find $\frac{dy}{dx}$ of function.

$x^y + y^x = 1$

Solution:

The given function is $x^y + y^x = 1$

Let $x^y = u \text{ and } y^x = v$

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Then, the function becomes $u+v=1$

$$\therefore \frac{du}{dx}+\frac{dv}{dx}=0 \qquad ....(1)$$

$$u=x^{y}$$

$$\Rightarrow \log u=\log\left(x^{y}\right)$$

$$\Rightarrow \log u=y\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\log x\frac{dy}{dx}+y.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\log x\frac{dy}{dx}+y.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{y}\left(\log x\frac{dy}{dx}+\frac{y}{x}\right) \qquad ....(2)$$

$$v=y^{x}$$

$$\Rightarrow \log v=\log\left(y^{x}\right)$$

$$\Rightarrow \log v=x\log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\log y\frac{d}{dx}(x)+x.\frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx}=v\left(\log y.1+x.\frac{1}{y}.\frac{d}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx}=y^{x}\left(\log y+\frac{x}{y}\frac{dy}{dx}\right) \qquad ....(3)$$

Therefore, from (1), (2) and (3) we obtain

$$x^y\left(\log x\frac{dy}{dx}++\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow\left(x^2+\log x+xy^{y-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

13. Find $\frac{dy}{dx}$ of function $y^x=x^y$

Solution:

The given function is $y^x=x^y$

Taking logarithm on both sides, we obtain

$$x\log y=y\log x$$

Differentiating both sides with respect to x, we obtain

$$\log y.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log y)=\log x.\frac{d}{dx}(y)+y.\frac{d}{dx}(\log x)$$

$$\Rightarrow\log y.1+x.\frac{1}{y}.\frac{dy}{dx}=\log x.\frac{dy}{dx}+y.\frac{1}{x}$$

$$\Rightarrow\log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

14. Find $\frac{dy}{dx}$ of friction $(\cos x)^y = (\cos y)^x$

Solution:

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both sides, we obtain

$$y = \log\cos x = x\log\cos y$$

Differentiating both sides with respect to x, we obtain

$$\log\cos x.\frac{dy}{dx} + y.\frac{d}{dx}(\log\cos x) = \log\cos y.\frac{d}{dx}(x) + x.\frac{d}{dx}(\log\cos y)$$

$$\Rightarrow \log\cos x.\frac{dy}{dx} + y.\frac{1}{\cos x}.\frac{d}{dx}(\cos x) = \log\cos y.1 + x.\frac{1}{\cos y}\frac{d}{dx}(\cos y)$$

$$\Rightarrow \log\cos x\frac{dy}{dx} + \frac{y}{\cos x}(-\sin x) = \log\cos y + \frac{x}{\cos y}(-\sin y)\frac{dy}{dx}$$

$$\Rightarrow \log\cos x\frac{dy}{dx} - y\tan x = \log\cos y - x\tan y\frac{dy}{dx}$$

$$\Rightarrow (\log\cos x + x\tan y)\frac{dy}{dx} = y\tan x + \log\cos y$$

$$\therefore \frac{dy}{dx} = \frac{y\tan x + \log\cos y}{x\tan y + \log\cos x}$$

15. Find $\frac{dy}{dx}$ of function $xy = e^{(x-y)}$

Solution:

The given function is $xy = e^{(x-y)}$

Taking logarithm on both sides, we obtain

$$\log(xy) = \log\left(e^{x-y}\right)$$

$$\Rightarrow \log x + \log y = (x-y)\log e$$

$$\Rightarrow \log x + \log y = (x-y)\times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

16. Find the derivative of the function given by $f(x) = (1-x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$

Solution:

The given relationship is $f(x) = (1-x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both sides, we obtain

$$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)] = \frac{d}{dx}\log(1+x) + \frac{d}{dx}\log(1+x^2) + \frac{d}{dx}\log(1+x^4) + \frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)}.f'(x) = \frac{1}{1+x}.\frac{d}{dx}(1+x) + \frac{1}{1+x^2}.\frac{d}{dx}(1+x^2) + \frac{1}{1+x^4}.\frac{d}{dx}(1+x^4) + \frac{1}{1+x^8}.\frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{1+x} + \frac{1}{1+x^2}.2x + \frac{1}{1+x^4}.4x^3 + \frac{1}{1+x^8}.8x^7\right]$$

$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Hence, $f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+1^4}+\frac{8\times1^7}{1+1^8}\right]$

$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

17. Differentiate $(x^2-5x+8)(x^3+7x+9)$ in three easy mentioned below

i. By using product rule

ii. By expanding the product to obtain a single polynomial

iii. By logarithm differentiate

Do they all given the same answer?

Solution:

Let $y=(x^2-5x+8)(x^3+7x+9)$

(i) Let $x=x^2-5x+8\ and\ u=x^3+7x+9$

$\therefore y=uv$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dv}v+u\frac{dv}{dx}$ (By using product rule)

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(x^2-5x+8).(x^3+7x+9)+(x^2+5x+8).\frac{d}{dx}(x^3+7x+9)$$

$$\Rightarrow \frac{dy}{dx}=(2x-5)(x^3+7x+9)+(x^2-5x+8)(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx}=2x(x^3+7x+9)-5(x^3+7x+9)+x^2(3x^2+7)-5x(3x^2+7)-8(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx} = \left(2x^4 + 14x^2 + 18x\right) - 5x^3 - 35x - 45 + \left(3x^2 + 7x^2\right) - 15x^3 - 35x + 24x^2 + 56$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(ii) $y = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)$

$$= x^2\left(x^3 + 7x + 9\right) - 5x\left(x^3 + 7x + 9\right) + 8\left(x^3 + 7x + 9\right)$$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

$$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72\right)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^5\right) - 5\frac{d}{dx}\left(x^4\right) + 15\frac{d}{dx}\left(x^3\right) - 26\frac{d}{dx}\left(x^2\right) + 11\frac{d}{dx}(x) + \frac{d}{dx}(72)$$

$$= 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(iii) Taking logarithm on both sides, we obtain

$$\log y = \log\left(x^2 - 5x + 8\right) + \log\left(x^3 + 7x + 9\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\log\left(x^2 - 5x + 8\right) + \frac{d}{dx}\log\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}\cdot\frac{d}{dx}\left(x^2 - 5x + 8\right) + \frac{1}{x^3 + 7x + 9}\cdot\frac{d}{dx}\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x^2 - 5x + 8} \times (2x - 5) + \frac{1}{x^3 + 7x + 9} \times \left(3x^2 + 7\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{2x - 5}{x^3 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^3-5x+8\right)+\left(x^3+7x+9\right)}\right]$$

$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x^5+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$

$$\Rightarrow \frac{dy}{dx}=5x^2-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the result of $\frac{dy}{dx}$ are same

18. If u, v and w are function of x, then show that $\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u\frac{dv}{dx}.\mathrm{w}+\mathrm{u}.\mathrm{v}\frac{dw}{dx}$.

In two ways – first by repeated application of product rule, second by logarithmic differentiation.

Solution:

Let $y=u.v.w=u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}.(v.w)+u.\frac{d}{dx}(v.w)$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v.w+u\left[\frac{dv}{dx}.\mathrm{w}+\mathrm{v}.\frac{dv}{dx}\right]$$ (Again applying product rule)

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y=u.v.w,$ we obtain

$$\log y=\log u+\log v+\log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=\frac{d}{dx}(\log u)+\frac{d}{dx}(\log v)+\frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v\frac{dw}{dx}$$

Exercise 5.6

1. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$, $x = 2at^2, y = at^4$

Solution:

The given equation are $x = 2at^2$ and $y = at^4$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a.\frac{d}{dt}\left(t^2\right) = 2a.2t = 4at$$

$$\frac{dy}{dx} = \frac{d}{dt}\left(at^4\right)a.\frac{d}{dt}\left(t^4\right) = a.4.t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

2. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = a\cos\theta, y = b\cos\theta$

Solution:

The given equations are $x = a\cos\theta$ and $y = b\cos\theta$

Then, $\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right) = a\left(-\sin\theta\right) = -a\sin\theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos\theta\right) = b\left(-\sin\theta\right) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

3. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = \sin t, y = \cos 2t$

Solution:

The given equations are $x = \sin t$ and $y = \cos 2t$

Then, $\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t.\frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2.2\sin t\cos t}{\cos t} = -4\sin t$$

4. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x = 4t, y = \frac{4}{t}$

Solution:

The equation are $x = 4t$ and $y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4.\frac{d}{dt}\left(\frac{1}{t}\right) = 4.\left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

5. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=\cos\theta-\cos 2\theta, y=\sin\theta-\sin 2\theta$

Solution:

The given equations are $x=\cos\theta-\cos 2\theta$ and $y=\sin\theta-\sin 2\theta$

Then, $\frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos 2\theta)=\frac{d}{d\theta}(\cos\theta)-\frac{d}{d\theta}(\cos 2\theta)$

$=-\sin\theta(-2\sin 2\theta)=2\sin 2\theta-\sin\theta$

$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin 2\theta)=\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\sin 2\theta)$

$=\cos\theta-2\cos\theta$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\cos\theta-2\cos\theta}{2\sin 2\theta-\sin\theta}$$

6. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=a(\theta-\sin\theta), y=a(1+\cos\theta)$

Solution:

The given equations are $x=a(\theta-\sin\theta)$ and $y=a(1+\cos\theta)$

Then, $\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}(\theta)-\frac{d}{d\theta}(\sin\theta)\right]=a(1-\cos\theta)$

$\frac{dy}{d\theta}=a\left[\frac{d}{d\theta}(1)+\frac{d}{d\theta}(\cos\theta)\right]=a[0+(-\sin\theta)]=-a\sin\theta$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=\frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=-\cot\frac{\theta}{2}$$

7. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x=\frac{\sin^3 t}{\sqrt{\cos x2t}},\ y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Solution:

The given equations are $x=\frac{\sin^3 t}{\sqrt{\cos x2t}}$ and $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$=\frac{\sqrt{\cos 2t}-\frac{d}{dt}(\sin^3 t)-\sin^3 t-\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\sin^2 t.\frac{d}{dt}(\sin t)-\sin^3 tx\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}.\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t\sqrt{\cos 2t}}$$

$$=\frac{3\cos 2t\sin^2 t\cot t+\sin^2 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}.\frac{d}{dt}(\cos^3 t)-\cos^3 t.\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t.\frac{d}{dt}(\cos t)-\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}\cos^2 t.(-\sin t)-\cos^3 t.\frac{1}{\sqrt{\cos 2t}}(-2\sin 2t)}{\cos 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{\cos 2t.\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx}\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-3\cos 2t.\cos^2 t+\cos^3 t\sin 2t}{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}$$

$$=\frac{3\cos 2t.\cos^2 t\sin t+\cos^3 t(2\sin t\cos t)}{3\cos 2\text{t}\sin^2 t.\cos t+\sin^3 t(2\sin t\cos t)}$$

$$=\frac{\sin t\cos t\left[-3\cos 2t.\cos t+2\cos^3 t\right]}{\sin t\cos t\left[3\cos 2t\sin t+2\sin^3 t\right]}$$

$$=\frac{\left[-3\left(2\cos^2 t-1\right)\cos t+2\cos^3 t\right]}{\left[3\left(1-2\sin^2 t\right)\sin t+2\sin^3 t\right]}\quad\begin{bmatrix}\cos 2t=\left(2\cos^2 t-1\right)\\ \cos 2t=\left(1-2\sin^2 t\right)\end{bmatrix}$$

$$=\frac{-4\cos^3 t+3\cos t}{3\sin t-4\sin^3 t}\quad\begin{bmatrix}\cos 3t=4\cos^3 t-3\cos t\\ \sin 3t=3\sin t-4\sin^2 t\end{bmatrix}$$

$$=\frac{-\cos 3t}{\sin 3t}$$

$$=-\cot 3t$$

8. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x=a\left(\cos t+\log\frac{t}{2}\right), y=a\sin t$$

Solution:

The given equations are $x = a\left(\cos t + \log \frac{t}{2}\right)$ and $y = a\sin t$

Then, $\frac{d}{dx} = a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}.\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \cot\frac{t}{2}.\sec^2\frac{t}{2}.\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cot\frac{t}{2}}{\sin\frac{t}{2}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

9. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=a\sec, y=b\tan\theta$

Solution:

The given equations are $x=a\sec$ and $y=b\tan\theta$

Then, $\frac{dx}{d\theta}=a.\frac{d}{d\theta}(\sec\theta)=a\sec\theta\tan\theta$

$\frac{dy}{d\theta}=b.\frac{d}{d\theta}(\tan\theta)=b\sec^2\theta$

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b}{a}\sec\theta\cot\theta=\frac{b\cos\theta}{a\cos\theta\sin\theta}=\frac{b}{a}\times\frac{1}{\sin\theta}=\frac{b}{a}\cos ec\theta$$

10. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=a(\cos\theta+\theta\sin\theta), y=a(\sin\theta-\theta\cos\theta)$

Solution:

The given equations are $x=a(\cos\theta+\theta\sin\theta)$ and $y=a(\sin\theta-\theta\cos\theta)$

Then, $\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}\cos\theta+\frac{d}{d\theta}(\theta\sin\theta)\right]=a\left[-\sin\theta+\theta\frac{d}{d\theta}(\sin\theta)+\sin\theta\frac{d}{d\theta}(\theta)\right]$

$=a[-\sin\theta+\theta\cos\theta+\sin\theta]=a\theta\cos\theta$

$$\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\theta\cos\theta)\right]=a\left[\cos\theta-\left\{\theta\frac{d}{d\theta}(\cos\theta)+\cos\theta.\frac{d}{d\theta}(\theta)\right\}\right]$$

$=a[\cos\theta+\theta\sin\theta-\cos\theta]$

$=a\theta\sin\theta$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta \sin\theta}{a\theta \sin\theta} = \tan\theta$$

11. If $x = \sqrt{a^{\sin -1t}}, y = \sqrt{a^{\cos -1t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Solution:

The given equation are $x = \sqrt{a^{\sin -1t}}$ *and* $y = \sqrt{a^{\cos -1t}}$

$$x = \sqrt{a^{\sin -1t}} \text{ and } y = \sqrt{a^{\cos -1t}}$$

$$\Rightarrow x = \left(a^{\sin -1t}\right) \text{ and } y = \left(a^{\cos -1t}\right)^{\frac{1}{2}}$$

$$\Rightarrow x = a^{\frac{1}{2}\sin -1t} \text{ and } y = a^{\frac{1}{2}\cos -1t}$$

Consider $x = a^{\frac{1}{2}\sin -1t}$

Taking logarithm on both sides, we obtain

$$\log x = \frac{1}{2}\sin^{-1} t \log a$$

$$\therefore \frac{1}{x}\frac{dx}{dt} = \frac{1}{2}\log a.\frac{d}{dt}\left(\sin^{-1} t\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x}{2}\log a.\frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{x\log a}{2\sqrt{1-t^2}}$$

Then, consider

$$y = a^{\frac{1}{2}\cos^{-1} t}$$

Taking logarithm on both sides, we obtain

$$\log y=\frac{1}{2}\cos^{-1}t\log a$$

$$\therefore \frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\log a.\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\frac{x\log a}{2\sqrt{1-t^2}}}=-\frac{y}{x}$$

Hence proved

Exercise 5.7

1. Find the second order derivatives of the function x^2+3x+2

Solution:

Let $y=x^2+3x+2$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(3x)+\frac{d}{dx}(2)=2x+3+0=2x+3$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)=\frac{d}{dx}(2x)+\frac{d}{dx}(3)=2+0=2$$

2. Find the second order derivative of the function x^{20}

Solution:

Let $y=x^{20}$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{20}\right)=20x^{19}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(20x^{19}\right)=20\frac{d}{dx}\left(x^{19}\right)=20.19.x^{18}=380x^{18}$$

3. Find the second order derivatives of the function $x.\cos x$

Solution:

Let $y=x.\cos x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x.\cos x)=\cos x.\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x.1+x(-\sin x)=\cos x-x\sin x$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}[\cos x-\sin x]=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)$$

$$= -\sin x - \left[\sin x \frac{d}{dx}(x) + x\frac{d}{dx}(\sin x)\right]$$

$$= -\sin x - (\sin x + \cos x)$$

$$= -(x\cos x + 2\sin x)$$

4. Find the second order derivatives of the function $\log x$

Solution:

Let $y = \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

5. Find the second order derivatives of the function $x^3 \log x$

Solution:

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right] = \log x.\frac{d}{dx}\left(x^3\right) + x^3 \frac{d}{dx}(\log x)$$

$$= \log x.3x^2 + x^3.\frac{1}{x} = \log x.3x^2 + x^2$$

$$= x^2(1 + 3\log x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[x^2(1 + 3\log x)\right]$$

$$= (1 + 3\log x).\frac{d}{dx}\left(x^2\right) + x^2 \frac{d}{dx}(1 + 3\log x)$$

$$=(1+3\log x).2x+x^3.\frac{3}{x}$$

$$=2x+6\log x+3x$$

$$=5x+6x\log x$$

$$=x(5+6\log x)$$

6. Find the second order derivative of the function $e^x \sin 5x$

Solution:

Let $y=e^x \sin 5x$

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x \sin 5x\right)=\sin 5x\frac{d}{dx}\left(e^x\right)+e^x\frac{d}{dx}(\sin 5x)$$

$$=\sin 5x.e^x+e^x.\cos 5x.\frac{d}{dx}(5x)=e^x\sin 5x+e^x\cos 5x.5$$

$$=e^x(\sin 5x+5\cos 5x)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x(\sin 5x+5\cos 5x)\right]$$

$$(\sin 5x+5\cos 5x)\frac{d}{dx}\left(e^x\right)+e^x.\frac{d}{dx}(\sin 5x+5\cos 5x)$$

$$(\sin 5x+5\cos 5x)e^x+e^x\left[\cos 5x.\frac{d}{dx}(5x)+5(-\sin 5x).\frac{d}{dx}(5x)\right]$$

$$=e^x(\sin 5x+5\cos 5x)+e^x(5\cos 5x-25\sin 5x)$$

Thus, $e^x(10\cos 5x-24\sin 5x)=2e^x(5\cos 5x-12\sin 5x)$

7. Find the second order derivatives of the function $e^{6x}\cos 3x$

Solution:

Let $y=e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{6x}\cos 3x\right)=\cos 3x.\frac{d}{dx}\left(e^{6x}\right)+e^{6x}\frac{d}{dx}\left(\cos 3x\right)$$

$$=\cos 3x.\mathrm{e}^{6x}\frac{d}{dx}(6x)+e^{6x}\left(-\sin 3x\right)\frac{d}{dx}(3x)$$

$$=6e^{6x}\cos 3x-3e^{6x}\sin 3x.........(1)$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left(6e^{6x}\cos 3x-3e^{6x}\sin 3x\right)=6.\frac{d}{dx}\left(e^{6x}\cos 3x\right)-3.\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$=6.\left[6e^{6x}\cos 3x-3e^{6x}\sin 3x\right]-3\left[\sin 3x.\frac{d}{dx}\left(e^{6x}\right)+e^{6x}.\frac{d}{dx}\left(\sin 3x\right)\right]\qquad[\text{using}\,(1)]$$

$$=36e^{6x}\cos 3x-18e^{6x}\sin 3x-3\left[\sin 3x.e^{6x}.6+e^{6x}.\cos 3x-3\right]$$

$$=36e^{6x}\cos 3x-18e^{6x}\sin 3x-18e^{6x}\sin 3x-9e^{6x}\cos 3x$$

$$=27e^{6x}\cos 3x-36e^{6x}\sin 3x$$

$$=9e^{6x}\left(3\cos 3x-4\sin 3x\right)$$

8. Find the second order derivatives of the function $\tan^{-1}x$

Solution:

Let $y=\tan^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}\left(1+x^2\right)^{-1}=(-1)\left(1+x^2\right)^{-2}\frac{d}{dx}\left(1+x^2\right)-\frac{1}{\left(1+x^2\right)^2}\times 2x=-\frac{2x}{\left(1+x^2\right)^2}$$

9. Find the second order derivative of the function $\log(\log x)$

Solution:

Let $y=\log(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log(\log x)\right]=\frac{1}{\log x}.\frac{d}{dx}(\log x)=\frac{1}{\log x}=(x\log x)^{-1}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]=(-1)(x\log x)^{-2}\frac{d}{dx}(x\log x)$$

$$=\frac{-1}{(x\log x)^2}.\left[\log x.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log x)\right]$$

$$=\frac{-1}{(x\log x)^2}.\left[\log x.1x+\frac{1}{x}\right]=\frac{-1(1+\log x)}{(x\log x)^2}$$

10. Find the second order derivatives of the function $\sin(\log x)$

Solution:

Let $y=\sin(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin(\log x)\right]=\cos(\log x).\frac{d}{dx}(\log x)=\frac{\cos(\log x)}{x}$$

$$\therefore\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$=\frac{x.\frac{d}{dx}\left[\cos(\log x)\right]-\cos(\log x).\frac{d}{dx}(x)}{x^2}$$

$$=\frac{x\left[-\sin(\log x).\frac{d}{dx}(\log x)\right]-\cos(\log x).1}{x^2}$$

$$= \frac{-x\sin(\log x).\frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-\left[\sin(\log x) + \cos(\log x)\right]}{x^2}$$

11. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Solution:

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$

$$= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(5\sin x + 3\cos x)\right]$$

$$= -\left[5.\frac{d}{dx}(\sin x) + 3.\frac{d}{dx}(\cos x)\right]$$

$$= \left[5\cos x + 3(-\sin x)\right]$$

$$= -[5\cos x - 3\sin x]$$

$$= -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence, proved

12. If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone

Solution:

It is given that, $y = \cos^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} = -\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$= \left(-\frac{1}{2}\right).\left(1-x^2\right)^{\frac{-3}{2}}.\frac{d}{dx}\left(1-x^2\right)$$

$$= \frac{1}{\sqrt{\left(1-x^2\right)^3}} \times \left(-2x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{\sqrt{\left(1-x^2\right)^3}} ....(i)$$

$y = \cos^{-1} x \Rightarrow x = \cos y$

Putting $x = \cos y$ in equation (1), we obtain

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$\frac{-\cos y}{\sin^3 y}$$

$$= \frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \cot y . \operatorname{cosec}^2 y$$

13. If $y=3\cos(\log x)+4\sin(\log x)$, show that $x^2y_2+xy_1+y=0$

Solution:

It is given that $y=3\cos(\log x)+4\sin(\log x)$ and $x^2y_2+xy_1+y=0$

Then,

$$y_1=3.\frac{d}{dx}\left[\cos(\log x)\right]+4.\frac{d}{dx}\left[\sin(\log x)\right]$$

$$=3.\left[-\sin(\log x).\frac{d}{dx}(\log x)\right]+4.\left[\cos(\log x).\frac{d}{dx}(\log x)\right]$$

$$\therefore y_1=\frac{-3\sin(\log x)}{x}+\frac{4\cos(\log x)}{x}=\frac{4\cos(\log x)-3\sin(\log x)}{x}$$

$$\therefore y_2=\frac{d}{dx}\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)$$

$$=x\frac{\left[4\{\cos(\log x)\}-\{-3\sin(\log x)\}'-\{4\cos(\log x)\}-3\sin(\log x)\right].1}{x^2}$$

$$=x\frac{\left[-4\sin(\log x).(\log x)'-3\cos(\log x)(\log x)'\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=x\frac{\left[-4\sin(\log x)\frac{1}{x}-3\cos(\log x)\frac{1}{x}\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{4\sin(\log x)-3\cos(\log x)-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{-\sin(\log x)-7\cos(\log x)}{x^2}$$

$$\therefore x^2y_2+xy_1+y$$

$$=x^2\left(\frac{-\sin(\log x)-7\cos(\log x)}{x^2}\right)+x\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)+3\cos(\log x)+4\sin(\log x)$$

$$=\sin(\log x)-7\cos(\log x)+4\cos(\log x)-3\sin(\log x)+3\cos(\log x)+4\sin(\log x)$$

=0

Hence, proved

14. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Solution:

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A.\frac{d}{dx}(e^{mx}) + B.\frac{d}{dx}(e^{nx}) = A.e^{mx}.\frac{d}{dx}(mx) + B.e^{nx}\frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(Ame^{mx} + Bne^{nx}) = Am.\frac{d}{dx}(e^{mx}) + Bn.\frac{d}{dx}(e^{nx})$$

$$= Am.e^{mx}.\frac{d}{dx}(mx) + bn.e^{nx}.\frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n).(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{mx})$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2ex^{mx} - Bmne^{mx} - Bn^2e^{mx} + Amne^{mx} + Bmne^{nx}$$

=0

Hence, proved

15. If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Solution:

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx}=500.\frac{d}{dx}\left(e^{7x}\right)+600.\frac{d}{dx}\left(e^{-7x}\right)$$

$$=500.e^{7x}.\frac{d}{dx}(7x)+600.e^{-7x}.\frac{d}{dx}(-7x)$$

$$=3500e^{7x}-4200e^{-7x}$$

$$\therefore\frac{d^2y}{dx^2}=3500.\frac{d}{dx}\left(e^{7x}\right)-4200\frac{d}{dx}\left(e^{-7x}\right)$$

$$3500.e^{7x}\frac{d}{dx}(7x)-4200.e^{-7x}.\frac{d}{dx}(-7x)$$

$$=7\times3500.e^{7x}+7\times4200e^{-7x}$$

$$=49\times500e^{7x}+49\times600e^{-7x}$$

$$=49\left(500e^{7x}+600e^{-7x}\right)$$

$$=49y$$

Hence, proved

16. If $e^y(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

Solution:

The given relationship is $e^y(x+1)=1$

$$e^y(x+1)=1$$

$$\Rightarrow e^y=\frac{1}{x+1}$$

Taking logarithm on both sides, we obtain

$$y=\log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to x we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{(x+1)}\right)=(x+1)\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved

17. If $y=(\tan^{-1}x)^2$, show that $(x^2+1)^2 y_2+2x(x^2+1)y_1=2$

Solution:

The given relationship is $y=(\tan^{-1}x)^2$

Then,

$$y_1=2\tan^{-1}x\frac{d}{dx}(\tan^{-1}x)$$

$$\Rightarrow (1+x^2)y_1=2\tan^{-1}x$$

Again differentiating with respect to x on both sides, we obtain

$$(1+x^2)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow (1+x^2)y_2+2x(1+x^2)=2$$

Hence proved

Exercise 5.8

1. Differentiate the following with respect to *x*.
$\cos x.\cos 2x.\cos 3x$

Solution-1

Let $y=\cos x.\cos 2x.\cos 3x$
Taking logarithm or both the side, we obtain
$$\log y=\log(\cos x.\cos 2x.\cos 3x)$$
$$\Rightarrow \log y=\log(\cos x)+\log(\cos 2x)+\log(\cos 3x)$$
Differentiating both sides with respect to *x*, we obtain
$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\cos x}.\frac{d}{dx}(\cos x)+\frac{1}{\cos 2x}.\frac{d}{dx}(\cos 2x)+\frac{1}{\cos 3x}.\frac{d}{dx}(\cos 3x)$$
$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{\sin x}{\cos x}-\frac{\sin 2x}{\cos 2x}.\frac{d}{dx}(2x)-\frac{\sin 3x}{\cos 3x}.\frac{d}{dx}(3x)\right]$$
$$\therefore \frac{dy}{dx}=-\cos x.\cos 2x.\cos 3x[\tan x+2\tan 2x+3\tan 3x]$$

2. Differentiate the function with respect to *x*. $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution-2

Let $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$
Taking logarithm or both the side, we obtain
$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$
$$\Rightarrow \log y=\frac{1}{2}\log\left[\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\right]$$
$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\begin{array}{c}\frac{1}{x-1}.\frac{d}{dx}(x-1)+\frac{1}{x-2}.\frac{d}{dx}(x-2)-\frac{1}{x-3}.\frac{d}{dx}(x-3)\\ -\frac{1}{1-4}.\frac{d}{dx}(x-4)-\frac{1}{x-5}.\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{1-4}-\frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{1-4}-\frac{1}{x-5}\right]$$

3. Differentiate the function with respect to x. $(\log x)^{\cos x}$

Solution-3

Let $y=(\log x)^{\cos x}$

Taking logarithm or both the side, we obtain

$$\log y=\cos x.\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{\log x}-\sin x\log(\log x)\right]$$

.

4. Differentiate the function with respect to x. $x^x-2^{\sin x}$

Solution-4

Let $y=x^x-2^{\sin x}$

Also, let $x^x=u$ and $2^{\sin x}=v$

$\therefore y=u-v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$$

$$u=x^{x}$$

Taking logarithm on both sides, we obtain

$$\frac{1}{u}\frac{du}{dx}=\left[\frac{d}{dx}(x)\times\log x+x\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx}=u\left[1\times\log x+x\times\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{x}(\log x+1)$$

$$\Rightarrow \frac{du}{dx}=x^{x}(1+\log x)$$

$v=2^{\sin x}$

Taking logarithm on both the sides with respect to *x*, we obtain

$\log v=\sin x.\log 2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}.\frac{dv}{dx}=\log 2.\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx}=v\log 2\cos x$$

$$\Rightarrow \frac{dv}{dx}=2^{\sin x}\cos x\log 2$$

$$\therefore \frac{dy}{dx}=x^{2}(1+\log x)-2^{\sin x}\cos x\log 2$$

5. Differentiate the function with respect to *x*. $(x+3)^{2}.(x+4)^{3}.(x+5)^{4}$

Solution-5

Let $y=(x+3)^{2}.(x+4)^{3}.(x+5)^{4}$

Taking logarithm on both sides, we obtain.

$$\log y=\log(x+3)^{2}+\log(x+4)^{3}+\log(x+5)^{4}$$

$$\Rightarrow \log y=2\log(x+3)+3\log(x+4)+4\log(x+5)$$

Differentiating both sides with respect to *x*, we obtain

$$\Rightarrow \frac{dy}{dx}=(x+3)(x+4)^{2}(x+5)^{3}.\left[2\left(x^{2}+9x+20\right)+3\left(x^{2}+9x+15\right)+4\left(x^{2}+7x+12\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4.\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3.\left[2\left(x^2+9x+20\right)+3\left(x^2+9x+15\right)+4\left(x^2+7x+12\right)\right]$$

$$\therefore \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3\left(9x^2+70x+133\right)$$

6. Differentiate the function with respect to *x*. $\left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Solutoin-6

Let $y=\left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Also, let $u=\left(x+\frac{1}{x}\right)^x$ and $v=x^{\left(1+\frac{1}{x}\right)}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \text{................ (1)}$$

Then, $u=\left(x+\frac{1}{x}\right)^x$

Taking log on both sides

$$\Rightarrow \log u=\log\left(x+\frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times(\log)\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=1\times(\log)\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}.\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow\frac{du}{dx}=u\left[(\log)\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}x\left(x+\frac{1}{x^2}\right)\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[(\log)\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)\left[\frac{x^2-1}{x^2+1}+(\log)\left(x+\frac{1}{x}\right)\right] \qquad ....(2)$$

$$v=x^{\left(1+\frac{1}{x}\right)}$$

Taking log on both sides, we obtain

$$\log v=\log x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow\log v=\left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right).\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(x+\frac{1}{x}\right).\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right] \qquad .... (3)$$

Therefore from (1) , (2) and (3), we obtain

$$\frac{dy}{dx}=\left(1+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2}+\log x\left(x+\frac{1}{x}\right)\right]+x^{\left(x+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

7. Differentiating both sides with respect to x. $(\log x)^{x}+x^{\log x}$

Solution-7

Let $y=(\log x)^{x}+x^{\log x}$

Also, let $u=(\log x)^{x}$ and $v=x^{\log x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ....(1)$$

$u=(\log x)^{x}$

$\Rightarrow \log u=\log\left[(\log x)^{x}\right]$

$\Rightarrow \log u=x\log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)x\log(\log x)+x.\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{du}{dx}=u\left[1x\log(\log x)+x.\frac{1}{\log x}.\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\frac{\log(\log x).\log x+1}{\log x}\right]$$

$$\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right] \qquad ....(2)$$

$v=x^{\log x}$

$\Rightarrow \log v=\log\left(x^{\log x}\right)$

$\Rightarrow \log v=\log x\log x=(\log x)^{2}$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}.\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^{2}\right]$$

$$\Rightarrow \frac{1}{v}.\frac{dv}{dx}=2(\log x).\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x-1}.\log x \qquad ....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dv}{dx} = (\log x)^{x-1}\left[1+\log x.\log(\log x)\right] + 2x^{\log x-1}.\log x$$

8. Differentiating both sides with respect to x. $(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution-8

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad .... (1)$$

$$u = (\sin x)^x$$

$$\Rightarrow \log u = \log(\sin x)^x$$

$$\Rightarrow \log u = x\log(\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)x\log(\sin x) + x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1.\log(\sin x) + x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x}.\cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x(x\cot x + \log\sin x) \qquad ....(2)$$

$$v = \sin^{-1}\sqrt{x}$$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}.\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \qquad ....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx}=(\sin x)^2(x\cot x+\log\sin x)+\frac{1}{2\sqrt{x-x^2}}$$

9. Differentiate the function with respect to *x*. $x^{\sin x}+(\sin x)^{\cos x}$

Solution-9

Let $y=x^{\sin x}+(\sin x)^{\cos x}$

Also $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \text{.............(1)}$$

$u=x^{\sin x}$

$\Rightarrow \log u=\log\left(x^{\sin x}\right)$

$\Rightarrow \log u=\sin x\log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x).\log x+\sin x.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u=\left[\cos x\log x+\sin x.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{\sin x}=\left[\cos x\log x+\frac{\sin x}{x}\right]$$

$v=(\sin x)^{\cos x}$

$\Rightarrow \log v=\log(\sin x)^{\cos x}$

$\Rightarrow \log v=\cos x\log(\sin x)$

Differentiating both sided with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}(\cos x)\times\log(\sin x)+\cos x\times\frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{dv}{dx}=v\left[-\sin x.\log(\sin x)+\cos x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}[-\sin x\log\sin x+\cot x\cos x]$$

$$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}[\cot x\cos x-\sin x\log\sin x]$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dv}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}\left[\cos x\cot x-\sin x\log\sin x\right]$$

10. Differentiate the function with respect to x. $x^{\cos x}+\frac{x^2+1}{x^2-1}$

Solution-10

Let $y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Also, let $u=x^{x\cos x}$ and $v=\frac{x^2+1}{x^2-1}$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dy}{dx}$$

$$\therefore y=u+v$$

$$u=x^{x\cos x}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x).\cos x\log x+x.\frac{d}{dx}(\cos x).\log x+x\cos x.\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1.\cos x\log x+x.(-\sin x).\log x+x\cos x.\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}(\cos x\log x-x\sin x.\log x+\cos x)$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]\quad ....(2)$$

$$v=\frac{x^2+1}{x^2-1}$$

$$\Rightarrow\log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}=\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{2x\left(x^2-2\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx}=\frac{-4x}{\left(x^2-1\right)^2} \qquad ....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x\left(1+\log x\right)-x\sin x\log x\right]-\frac{4x}{\left(x^2-1\right)^2}$$

11. Differentiate the function with respect to x $\left(x\cos x\right)^x+\left(x\sin x\right)^{\frac{1}{x}}$

Solution-11

Let $y=\left(x\cos x\right)^x+\left(x\sin x\right)^{\frac{1}{x}}$

Also, let $u=\left(x\cos x\right)^x$ and $v=\left(x\sin x\right)^{\frac{1}{x}}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dy}{dx} \qquad ....(1)$$

$u=\left(x\cos x\right)^2$

$\Rightarrow \log u=\log\left(x\cos x\right)^x$

$\Rightarrow \log u=x\log\left(x\cos x\right)^x$

$\Rightarrow \log u=x[\log x+\log\cos x]$

$\Rightarrow \log u=x\log x+x\log\cos x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}\left(x+\log x\right)+\frac{d}{dx}\left(x\log\cos x\right)$$

$$\Rightarrow \frac{du}{dx}=u\left[\left\{\log x.\frac{d}{dx}\left(x\right)+x.\frac{d}{dx}\left(\log x\right)\right\}+\left\{\log\cos x.\frac{d}{dx}\left(x\right)+x.\frac{d}{dx}\left(\log\cos x\right)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=\left(x\cos x\right)^x\left[\left(\log x.1+x.\frac{1}{x}\right)+\left\{\log\cos x.1+x.\frac{1}{\cos x}.\frac{d}{dx}\left(\cos x\right)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=\left(x\cos x\right)^x\left[\left(\log x+1\right)+\left\{\log\cos x+\frac{x}{\cos x}.\left(-\sin x\right)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=\left(x\cos x\right)^x\left[\left(1+\log x\right)+\left(1+\log\cos x-x\tan x\right)\right]$$

$$\Rightarrow \frac{du}{dx}=\left(x\cos x\right)^x\left[1-x\tan x+\left(\log x+\log\cos x\right)\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x \left[1 - x\tan x + +\log(x\cos x)\right] \qquad ....(2)$$

$$v = (x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}(\log x + \log\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log\sin x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left(\frac{1}{x}\log(\sin x)\right)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}(\log x)\right] + \left[\log(\sin x).\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{x}\right] + \left[\log(\sin x).\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1 - \log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}.\cos x\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(x\sin x)^{\frac{1}{x}} + \left[\frac{1 - \log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1 - \log x - \log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1 - \log(x\sin x) + x\cot x}{x^2}\right] \qquad ....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dv}{dx} = (x\cos x)^2\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

12. Find $\frac{dy}{dx}$ of function. $x^y + y^x = 1$

Solution-12

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \text{.........(1)}$$

$$u = x^y$$

$$\Rightarrow \log u = \log\left(x^y\right)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad \text{...........(2)}$$

$$v = y^x$$

$$\Rightarrow \log v = \log\left(y^x\right)$$

$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \log y \frac{d}{dx}(x) + x.\frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v\left[\log y.1 + x.\frac{1}{y}.\frac{dy}{dx}\right]$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad \text{...........(3)}$$

Therefore, from (1), (2) and (3), we obtain

$$x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) + y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \left(x^2 + \log x + xy^{y-1}\right)\frac{dy}{dx} = -\left(yx^{y-1} + y^x \log y\right)$$

$$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

13. Find $\frac{dy}{dx}$ of function $y^x = x^y$

Solution:13

The given function $y^x = x^y$

Taking logarithm on both sides, we obtain

$x\log y = y\log x$

Differentiating both sides with respect to *x*, we obtain

$$\log y.\frac{d}{dx}(x) + x.\frac{d}{dx}(\log y) = \log x.\frac{d}{dx}(y).\frac{d}{dx}(\log x)$$

$$\Rightarrow \log y.1 + x.\frac{1}{y}.\frac{dy}{dx} = \log x.\frac{dy}{dx} + y.\frac{1}{x}$$

$$\Rightarrow \log y + \frac{x}{y}.\frac{dy}{dx} = \log x.\frac{dy}{dx} + \frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \left(\frac{x - y\log x}{y}\right)\frac{dy}{dx} = \frac{y - \log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right).$$

14. Find $\frac{dy}{dx}$ of function $(\cos x)^y = (\cos y)^x$

Solution:14

The given function $(\cos x)^y = (\cos y)^x$

Taking logarithm on both sides, we obtain

$y = \log\cos x = x\log\cos y$

Differentiating both sides with respect to *x*, we obtain

$$\log\cos x.\frac{d}{dx} + y.\frac{d}{dx}(\log\cos x) = \log\cos y.\frac{d}{dx}(x). + x\frac{d}{dx}(\log\cos y)$$

$$\Rightarrow \log\cos x.\frac{dy}{dx} + y\frac{1}{\cos x}.\frac{d}{dx}(\cos x) = \log\cos y.1 + x.\frac{1}{\cos y}.\frac{d}{dx}(\cos y)$$

$$\Rightarrow \log\cos x.\frac{dy}{dx} + \frac{y}{\cos x}.(-\sin x) = \log\cos y + \frac{x}{\cos y}(-\sin y)\frac{dy}{dx}$$

$$\Rightarrow \log\cos x.\frac{dy}{dx} - y\tan x = \log\cos y - x\tan y\frac{dy}{dx}$$

$$\Rightarrow (\log\cos x + x\tan y)\frac{dy}{dx} = y\tan x + \log\cos y$$

$$\therefore \frac{dy}{dx} = \frac{y\tan x + \log\cos y}{x\tan y + \log\cos x}$$

15. Find $\frac{dy}{dx}$ of function $xy = e^{(x-y)}$

Solution-15

The given function is $xy = e^{(x-y)}$

Taking logarithm on both sides, we obtain.

$$\log(xy) = \log\left(e^{x-y}\right)$$

$$\Rightarrow \log x + \log y = (x-y)\log e$$

$$\Rightarrow \log x + \log y = (x-y)\times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y-1)}$$

16. Find the derivative of the function given by

$f(x) = (1-x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$ and hence find $f'(1)$

Solution-16

The given relationship is $f(x) = (1-x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$

Taking logarithm on both sides, we obtain

$\log f(x) = \log(1-x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$

Differentiating both sides with respect to x, we obtain.

$$\frac{1}{f(x)}\cdot\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\log(1-x) + \frac{d}{dx}\log(1+x^2) + \frac{d}{dx}\log(1+x^4) + \frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)}\cdot f'(x) = \frac{1}{1+x}\cdot\frac{d}{dx}(1-x) + \frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2) + \frac{1}{1+x^4}\cdot\frac{d}{dx}\log(1+x^4) + \frac{1}{1+x^8}\cdot\frac{d}{dx}\log(1+x^8$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{1+x} + \frac{1}{1+x^2}.2x + \frac{1}{1+x^4} + \frac{1}{1+x^8}.8x^7\right]$$

$$\therefore f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Hence, $f'(1) = (1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1} + \frac{2\times 1}{1+1^2} + \frac{4\times 1^3}{1+x^4} + \frac{8\times 1^7}{1+x^8}\right]$

$$= 2\times 2\times 2\times 2\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$$

$$= 16\left[\frac{1+2+4+8}{2}\right]$$

$$= 16\times\frac{15}{2} = 100$$

17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below

i. By using product rule

ii. By expanding the product to obtain a single polynomial

iii. By

iii. By logarithm Differentiate

Do they all given the same answer?

Solution-17

Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

(i) Let $x = x^2 - 5x + 8$ and $u = x^3 + 7x + 9$

$\therefore$ $y = uv$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dv}v + u\frac{dv}{dx} \qquad \text{(By using product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 - 5x + 8).(x^3 + 7x + 9) + (x^2 - 5x + 8).\frac{d}{dx}(x^3 + 7x + 9)$$

$$\Rightarrow \frac{dy}{dx} = (2x-5)(x^3+7x+9)+(x^2-5x+8)(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3+7x+9)-5(x^3+7x+9)+x^2(3x^2+7)-5x(3x^2+7)-8(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx} = (2x^4+14x^2+18x)-5x^3-35x-45+(3x^4+7x^2)-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx} = 5x^4-20x^3+45x^2-52x+11$$

(ii) $y=(x^2-5x+8)(x^3+7x+9)$

$$= x^2(x^3+7x+9)-5x(x^3+7x+9)+8(x^3+7x+9)$$

$$= x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$

$$= x^5-5x^4+15x^3-26x^2+11x+72$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} = (x^5-5x^4+15x^3-26x^2+11x+72)$$

$$= \frac{d}{dx} = (x^5)-5\frac{d}{dx}(x^4)+15\frac{d}{dx}(x^3)-26\frac{d}{dx}(x^2)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)$$

$$= 5x^4-5\times 4x^3+15\times 3x^2-26\times 2x+11\times 1+0$$

$$= 5x^4-20x^3+45x^2-52x+11$$

(iii) Taking logarithm on both sides, we obtain.

$$\log y = \log(x^2+5x+8)+\log(x^3+7x+9)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\log(x^2-5x+8)+\frac{d}{dx}\log(x^3+7x+9)$$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\log(x^2-5x+8).\frac{d}{dx}\log(x^3+7x+9)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2-5x+8}.\frac{d}{dx}(x^2-5x+8)+\frac{1}{x^3+7x+9}.\frac{d}{dx}\log(x^3+7x+9)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x^2-5x+8}\times(2x-5)+\frac{1}{x^3+7x+9}\times(3x^2+7)\right]$$

$$\Rightarrow \frac{dy}{dx} = (x^2-5x+8)(x^3+7x+9)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x^2-5x+8)(x^3+7x+9)\left[\frac{(2x-5)(x^3+7x+9)+(3x^2+7)(x^2-5x+8)}{(x^2-5x+8)+(x^3+7x+9)}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3+7x+9)-5(x^3+7x+9)+3x^2(x^2-5x+8)+7(x^2-5x+8)$$

$$\Rightarrow \frac{dy}{dx} = (2x^4+14x^2+18x)-5x^3-35x-45+(3x^4-15x^3+24x^2)+(7x^2-35x+56)$$

$$\Rightarrow \frac{dy}{dx} = 5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the result of $\frac{dy}{dx}$ are same

18. If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u\frac{dv}{dx}.w + u.v\frac{dw}{dx}$$

In two ways-first by repeated application of product rule, second by logarithmic differentiation.

Solution – 18

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}.(v.w)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v.w + u\left[\frac{dv}{dx}.w + v.\frac{dv}{dx}\right] \quad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y = u.v.w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}\frac{d}{dx} + \frac{1}{v}\frac{d}{dx} + \frac{1}{w}\frac{d}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx}u.w + u\frac{dv}{dx}w + u.v\frac{dw}{dx}$$