Chapter 6: Applications of Derivatives.

Exercise 6. Miscellaneous

1. Using differentials, find the approximate value of each of the following

$$(a)\left(\frac{17}{81}\right)^{\frac{1}{4}} \qquad (b)(33)^{\frac{1}{5}}$$

Solution:

$$(a)\ y = x^{\frac{1}{4}}$$

$$x = \frac{16}{81}$$

$$\Delta x = \frac{1}{81}$$

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x)$$

$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) = \frac{27}{4\times 8}\times\frac{1}{81} = \frac{1}{32\times 3} = \frac{1}{96} = 0.010$$

Approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3}+0.010=0.667+0.010$

$=0.677$

$(b)\ y=x^{\frac{1}{5}}$

$x=32$

$\Delta x=1$

$$\Delta y=(x+\Delta x)^{\frac{1}{5}}-x^{\frac{1}{5}}=(33)^{\frac{1}{5}}-(32)^{\frac{1}{5}}=(33)^{-\frac{1}{5}}-\frac{1}{2}$$

$$\therefore (33)^{\frac{1}{5}}=\frac{1}{2}+\Delta y$$

$$dy=\left(\frac{dy}{dx}\right)=(\Delta x)=\frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x)$$

$$=\frac{1}{5(2)^{6}}(1)=\frac{1}{320}=-0.003$$

Approximate value of $(33)^{\frac{1}{5}}$ is $\frac{1}{2}+(-0.003)=0.5-0.003=0.497$

2. Show that the function given by $f(x)=\frac{\log x}{x}$ has maximum at $x=e$

Solution:

$$f(x)=\frac{\log x}{x}$$

$$f'(x)=\frac{x\left(\frac{1}{x}\right)-\log x}{x^{2}}=\frac{1-\log x}{x^{2}}$$

$$f'(x)=0$$

$$\Rightarrow 1-\log x=0$$

$$\Rightarrow \log x = 1$$

$$\Rightarrow \log x = \log e$$

$$\Rightarrow x = e$$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1-\log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

$$f''(e) = \frac{-3+2\log e}{e^3} = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$$

f is the maximum at x = e

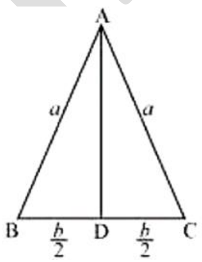
3. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3cm per second. How fast is the area decreasing when the two equal sides area equal to the base?

Solution:

Let ΔABC be isosceles where BC is the base of fixed length b

Let the length of the two equal sides of ΔABC be a

Draw $AD \perp BC$



$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

Area of triangle $=\frac{1}{2}b\sqrt{a^2-\frac{b^2}{4}}$

$$\frac{dA}{dt}=\frac{1}{2}b.\frac{2a}{2\sqrt{a^2-\frac{b^2}{4}}}\frac{da}{dt}=\frac{ab}{\sqrt{4a^2-b^2}}\frac{da}{dt}$$

$$\frac{da}{dt}=-3cm/s$$

$$\therefore\frac{da}{dt}=\frac{-3ab}{\sqrt{4a^2-b^2}}$$

When $a=b,$

$$\frac{dA}{dt}=\frac{-3b^2}{\sqrt{4b^2-b^2}}=\frac{-3b^2}{\sqrt{3b^2}}=-\sqrt{3}b$$

4. Find the equation of the normal to $y^2=4x$ curve at the point (1, 2)

Solution:

$$2y\frac{dy}{dx}=4$$

$$\Rightarrow\frac{dy}{dx}=\frac{4}{2y}=\frac{2}{y}$$

$$\therefore\left.\frac{dy}{dx}\right]_{(1,2)}=\frac{2}{2}=1$$

Slope of the normal at (1, 2) is $\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}}=\frac{-1}{1}=-1$

Equation of normal at (1, 2) is $y-2=-1(x-1)$

$$y-2=-x+1$$

$$x+y-3=0$$

5. Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta, y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin

Solution:

$$x = a\cos\theta + a\theta\sin\theta$$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$y = a\sin\theta - a\theta\cos\theta$$

$$\therefore \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

$$\therefore \frac{dy}{d\theta} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Slope of normal at any point θ is $\frac{1}{\tan\theta}$

Equation of normal at a given point (x, y) is given by,

$$y - a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta - a\left(\sin^2\theta + \cos^2\theta\right) = 0$$

$$\Rightarrow x\cos\theta + y\sin\theta - a = 0$$

Perpendicular distance of normal from origin is

$$\frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = |-a|$$, which is independent of θ

Perpendicular distance of normal from origin is constant

6. Find the intervals in which the function f given by $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is

(i) Increasing (ii) Decreasing

Solution:

$$f\left(x\right)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

$$\therefore f'\left(x\right)=\frac{\left(2+\cos x\right)\left(4\cos x-2-\cos x+x\sin x\right)-\left(4\sin x-2x-x\cos x\right)\left(-\sin x\right)}{\left(2+\cos x\right)^{2}}$$

$$=\frac{\left(2+\cos x\right)\left(3\cos x-2+x\sin x\right)+\sin x\left(4\sin x-2x-x\cos x\right)}{\left(2+\cos x\right)^{2}}$$

$$=\frac{6\cos x-4+2x\sin x+3\cos^{2}x-2\cos x+x\sin x\cos x+4\sin^{2}x-2\sin^{2}x-2x\sin x-x\sin x\cos x}{\left(2+\cos x\right)^{2}}$$

$$=\frac{4\cos x-4+3\cos^{2}x+4\sin^{2}x}{\left(2+\cos x\right)^{2}}$$

$$=\frac{4\cos x-\cos^{2}x}{\left(2+\cos x\right)^{2}}=\frac{\cos x\left(4-\cos x\right)}{\left(2+\cos x\right)^{2}}$$

$$f'\left(x\right)=0$$

$$\Rightarrow \cos x=0, \cos x=4$$

$$\cos x\neq 4$$

$$\cos x=0$$

$$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$$

In $\left(0,\frac{\pi}{2}\right)$ *and* $\left(\frac{3\pi}{2},2\pi\right)$, $f'\left(x\right)>0$

$f\left(x\right)$ is increasing for $0<x<\frac{x}{2}$ *and* $\frac{3\pi}{2}<x<2\pi$

In $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $f'\left(x\right)<0$

$f(x)$ is decreasing for $\frac{\pi}{2} < x < \frac{3\pi}{2}$

7. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is

(i) Increasing (ii) Decreasing

Solution:

$$f(x) = x^3 + \frac{1}{x^3}$$

$$\therefore f'(x) = 3x^2 + \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

$$f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = x \pm 1$$

In $(-\infty, 1)$ and $(1, \infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x) > 0$

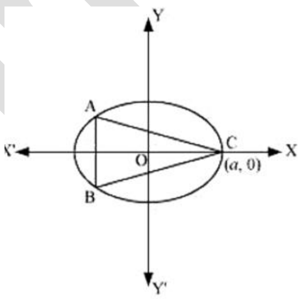
when $x < -1$ and $x > 1$, f is increasing

In $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) = 0$

Thus, when $-1 < x < 1$, f is decreasing

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis

Solution:



ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let ABC, be the triangle inscribed in the ellipse where vertex C is at (a, 0)

Since the ellipse is symmetrical with x – axis and y – axis

$$y_1 = \pm \frac{b}{a}\sqrt{a^2 - x_1^2}$$

Coordinates of A are $\left(-x_1, \frac{b}{a}\sqrt{a^2 - x_1^2}\right)$ and coordinates of B are $\left(x_1, -\frac{b}{a}\sqrt{a^2 - x_1^2}\right)$

As the point $(-x_1, y_1)$ lies on the ellipse, the area of triangle ABC is

$$A = \frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2 - x_1^2}\right) + (-x_1)\left(-\frac{b}{a}\sqrt{a^2 - x_1^2}\right) + (-x_1)\left(-\frac{b}{a}\sqrt{a^2 - x_1^2}\right)\right|$$

$$\Rightarrow A = ba\sqrt{a^2 - x_1^2} + x_1\frac{b}{a}\sqrt{a^2 - x_1^2}$$

$$\therefore \frac{dA}{dx_1} = \frac{-2xb}{2\sqrt{a^2 - x_1^2}} + \frac{b}{a}\sqrt{a^2 - x_1^2} - \frac{2bx_1^2}{a^2\sqrt{a^2 - x_1^2}}$$

$$= \frac{b}{2\sqrt{a^2 - x_1^2}}\left[-x_1 a + \left(a^2 - x_1^2\right) - x_1^2\right]$$

$$= \frac{b\left(-2x_1^2 - x_1^2 + a^2\right)}{a\sqrt{a^2 - x_1^2}}$$

$$\frac{dA}{dx_1} = 0$$

$$\Rightarrow -2x_1^2 - x_1 a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)\left(a^2\right)}}{2(-2)}$$

$$= \frac{a \pm \sqrt{9a^2}}{-4}$$

$$= \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

x_1 cannot be equal to a

$$\therefore x_1=\frac{a}{2}\Rightarrow y_1=\frac{b}{a}\sqrt{a^2-\frac{a^2}{4}}=\frac{ba}{2a}\sqrt{3}=\frac{\sqrt{3}b}{2}$$

Now, $$\frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{\sqrt{a^2-x_1^2}\,(-4x_1-a)-\left(-2x_1^2-x_1a+a^2\right)\frac{(-2x_1)}{2\sqrt{a^2-x_1^2}}}{a^2-x_1^2}\right\}$$

$$=\frac{b}{a}\left\{\frac{\left(a^2-x_1^2\right)\left(-4x_1-a\right)+x_1\left(-2x_1^2-x_1a+a^2\right)}{\left(a^2-x_1^2\right)^{\frac{2}{3}}}\right\}$$

$$=\frac{b}{a}\left\{\frac{2x^3-3a^2x-a^3}{\left(a^2-x_1^2\right)^{\frac{2}{3}}}\right\}$$

When $x_1=\frac{a}{2}$,

$$\frac{d^2A}{dx_1^2}=\frac{b}{a}\left[\frac{2\frac{a^3}{8}-3\frac{a^3}{2}-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right]=\frac{b}{a}\left\{\frac{\frac{a^3}{4}-\frac{3}{2}a^3-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$=\frac{b}{a}\left[\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right]<0$$

Area is the maximum when $x_1=\frac{a}{2}$

Maximum area of the triangle is

$$A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$

$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

9. A tank with rectangular base and rectangular sides, open at the top is to constructed so that its depth is 2m and volume is $8m^3$. If building of tank costs Rs 70 per sq meters for the base and Rs 45 per sq meters for sides. What is the cost of least expensive tank?

Solution:

Let l, b and h represent the length, breadth and height of the tank respectively

height $(h) = 2m$

Volume of the tank = $8m^3$

Volume of the tank = $l \times b \times h$

$8 = l \times b \times 2$

$\Rightarrow lb = 4 \Rightarrow \frac{4}{l}$

Area of the base $= lb = 4$

Area of 4 walls $(A) = 2h(l+b)$

$\therefore A = 4\left(l + \frac{4}{l}\right)$

$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$

Now, $\frac{dA}{dl} = 0$

$\Rightarrow l - \frac{4}{l^2} = 0$

$\Rightarrow l^2 = 4$

$\Rightarrow l = \pm 2$

Therefore, we have $l = 4$

$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$

$\frac{d^2A}{dl^2} = \frac{32}{l^3}$

$l = 2, \frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$

Area is the minimum when $l = 2$

We have $l = b = h = 2$

Cost of building base = $Rs70 \times (lb) = Rs70(4) = Rs280$

Cost of building walls =

$Rs\ 2h(l+h) \times 45 = Rs\ 90(2)(2+21) = Rs\ 8(90) = Rs\ 720$

Required total cost $= Rs(280+720) = Rs\ 1000$

10. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their area is least when the side of square is double the radius of the circle

Solution:

$2\pi r + 4a = k$ (where k is constant)

$\Rightarrow a = \frac{k - 2\pi r}{4}$

Sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k = 2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(2\pi)}{16} = 2\pi r = -\frac{\pi(k - 2\pi r)}{4}$$

Now, $\frac{dA}{dr} = 0$

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

$8r = k - 2\pi r$

$\Rightarrow (8 + 2\pi)r = k$

$\Rightarrow r = \frac{k}{8+2\pi} = \frac{k}{2(4+\pi)}$

Now, $\frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$

$\therefore$ where $r = \frac{k}{2(4+\pi)}, \frac{d^2A}{dr^2} > 0$

Area is least when $r = \frac{k}{2(4+\pi)}$

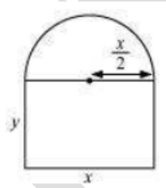
Where $r = \frac{k}{2(4+\pi)}, a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4} = \frac{8k + 2\pi k - 2\pi k}{2(4+\pi)\times 4} = \frac{k}{4+\pi} = 2r$

11. A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening

Solution:

x and y be length and breadth of rectangular window

Radius of semicircular opening $= \frac{x}{2}$



$\therefore x + 2y + \frac{\pi x}{2} = 10$

$\Rightarrow x\left(1 + \frac{\pi}{2}\right) + 2y = 10$

$\Rightarrow 2y = 10 - x\left(1 + \frac{\pi}{2}\right)$

$\Rightarrow y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi}{8} \times x^2$$

$$= 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{8} \times x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$\frac{d^2A}{dx^2} = -\left(1 - \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

$$\frac{dA}{dx} = 0$$

$$\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow x\left(1 + \frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

$$x = \frac{20}{\pi + 4}, \frac{d^2A}{dx^2} < 0$$

Area is maximum when length $x = \frac{20}{\pi + 4}$ m.

Now, $y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4}m$

The required dimensions

Length $= \dfrac{20}{\pi+4} m$ and breadth $= \dfrac{10}{\pi+4} m$

12. A point of the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

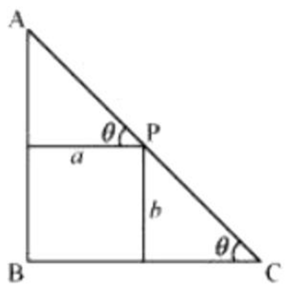
Solution:

ΔABC right – angled at B

AB = x and BC = y

P be a point on hypotenuse such that P is at a distance of a and b from the sides AB and BC respectively

$\angle c=\theta$



$AC=\sqrt{x^2+y^2}$

$PC=b\cos ec\theta$

$AP=a\sec\theta$

$AC=AP+PC$

$AC=b\cos ec\theta+\sec\theta$ ....(1)

$$\therefore \frac{d(AC)}{d\theta}=-b\cos ec\theta\cot\theta+a\sec\theta\tan\theta$$

$$\therefore \frac{d(AC)}{d\theta}=0$$

$\Rightarrow a\sec\theta\tan\theta=b\cos ec\theta\cot\theta$

$$\Rightarrow \frac{a}{\cos\theta}\frac{\sin\theta}{\cos\theta}=\frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta=b\cos^3\theta$$

$$\Rightarrow (a)^{\frac{1}{3}}\sin\theta=(b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta=\left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta=\frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}\ and\ \cos\theta=\frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}\ ......(2)$$

Clearly $\frac{d^2(AC)}{d\theta^2}<0\ when\ \tan\theta=\left(\frac{b}{a}\right)^{\frac{1}{3}}$

The length of the hypotenuse is the maximum when $\tan\theta=\left(\frac{b}{a}\right)^{\frac{1}{3}}$

Now, when $\tan\theta=\left(\frac{b}{a}\right)^{\frac{1}{3}}$

$$\tan\theta=\left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$AC=\frac{b\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{b^{\frac{1}{3}}}+\frac{a\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{a^{\frac{1}{3}}}$$

$$=a\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right)$$

$$=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Maximum length of the hypotenuse is $=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

13. Find the points at which the function f given by $f(x)=(x-2)^4(x+1)^3$ has

(i) local maxima (ii) local minima (iii) point of inflexion

Solution:

$$f(x)=(x-2)^4(x+1)^3$$

$$\therefore f'(x)=4(x-2)^3(x+1)^3+3(x+1)^2(x-2)^4$$

$$=(x-2)^3(x+1)^2\left[4(x+1)+3(x-2)\right]$$

$$=(x-2)^3(x+1)^2(7x-2)$$

$$f'(x)=0 \Rightarrow x=-1 \; and \; x=\frac{2}{7} \; or \; x=2$$

For x close to $\frac{2}{7}$ and to left of $\frac{2}{7}$, $f'(x)>0$

For x close to $\frac{2}{7}$ and to right of $\frac{2}{7}$, $f'(x)>0$

$x=\frac{2}{7}$ is point of local minima

As the value of x varies $f'(x)$ does not changes its sing

$x=-1$ is point of inflexion

14. Find the absolute maximum and minimum values of the function f given by $f(x)=\cos^2 x+\sin x, x\in[0,\pi]$

Solution:

$$f(x)=\cos^2 x+\sin x$$

$$f'(x)=2\cos x(-\sin x)+\cos x$$

$$= -2\sin x\cos x + \cos x$$

$$f'(x) = 0$$

$$\Rightarrow 2\sin x\cos x = \cos x \Rightarrow \cos x(2\sin x - 1) = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \ or \ \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{6}, or \ \frac{\pi}{2} \ as \ x \in [0, \pi]$$

$$f\left(\frac{\pi}{6}\right) = \cos^2\frac{\pi}{6} + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$$

$$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$$

$$f(\pi) = \cos^2\pi + \sin\pi = (-1)^2 + 0 = 1$$

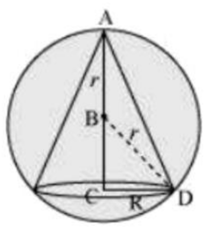
$$f\left(\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$$

Absolute maximum value of f is $\frac{5}{4}$ at $x = \frac{\pi}{6}$

Absolute minimum value of f is 1 at $x = 0, x = \frac{\pi}{2}, and \ \pi$

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$

Solution:



$$V = \frac{1}{3}\pi R^2 h$$

$$BC=\sqrt{r^2-R^2}$$

$$h=r+\sqrt{r^2-R^2}$$

$$\therefore V=\frac{1}{3}\pi R^2\left(r+\sqrt{r^2-R^2}\right)=\frac{1}{3}\pi R^2 r+\frac{1}{3}\pi R^2\sqrt{r^2-R^2}$$

$$\therefore \frac{dV}{dR}=\frac{2}{3}\pi Rr+\frac{2\pi}{3}\pi R\sqrt{r^2-R^2}+\frac{R^2}{3}.\frac{(-2R)}{2\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi}{3}\pi R\sqrt{r^2-R^2}-\frac{R^2}{3\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi Rr\left(r^2-R^2\right)-\pi R^3}{3\sqrt{r^2-R^2}}$$

$$=\frac{2}{3}\pi Rr+\frac{2\pi Rr^2-3\pi Rr^3}{3\sqrt{r^2-R^2}}$$

$$=\frac{dV}{dR^2}=0$$

$$\Rightarrow \frac{2\pi rR}{3}=\frac{3\pi R^2-2\pi Rr^2}{3\sqrt{r^2-R^2}}$$

$$\Rightarrow 2r\sqrt{r^2-R^2}=\left(3R^2-2r^2\right)^2$$

$$\Rightarrow 4r^2\left(r^2-R^2\right)=\left(3R^2-2r^2\right)^2$$

$$\Rightarrow 14r^2-4r^2R^2=9R^4+4r^4-12R^2r^2$$

$$\Rightarrow 9R^4-8r^2R^2=0$$

$$\Rightarrow 9R^2=8r^2$$

$$\Rightarrow R^2=\frac{8r^2}{9}$$

$$\frac{d^2V}{dR^2}=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)-\left(2\pi R^3-3\pi R^3\right)(-6)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

$$=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)-\left(2\pi R^3-3\pi R^3\right)(3R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

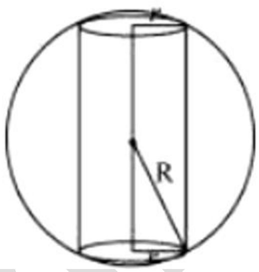
When $R^2=\frac{8r^2}{9},\frac{d^2V}{dR^2}<0$

Volume is the maximum when $R^2=\frac{8r^2}{9}$

$R^2=\frac{8r^2}{9}$, height of the cone $= r+\sqrt{r^2-\frac{8R^2}{9}}=r+\sqrt{\frac{r^2}{9}}=r+\frac{r}{3}=\frac{4r}{3}$

16. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$, also find the maximum volume

Solution:



$h=2\sqrt{R^2-r^2}$

$V=\pi r^2h=2\pi r^2\sqrt{R^2-r^2}$

$$\therefore\frac{dV}{dr}=4\pi r\sqrt{R^2-r^2}+\frac{2\pi r^2(-2r)}{2\sqrt{R^2-r^2}}$$

$$=4\pi r\sqrt{R^2-r^2}+\frac{2\pi r^3}{\sqrt{R^2-r^2}}$$

$$= \frac{4\pi r\left(R^2 - r^2\right) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi r R^2 - 6\pi r^3\right)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{\left(R^2 - r^2\right)}$$

$$= \frac{\left(R^2 - r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi r R^2 - 6\pi r^3\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0$$

Volume is maximum when $r^2 = \frac{2R^2}{3}$

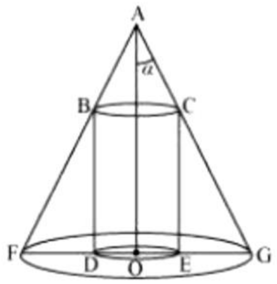
$$r^2 = \frac{2R^2}{3}$$

Height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$

Volume of the cylinder is maximum when height of cylinder is $\frac{2R}{\sqrt{3}}$

17. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi veritical angle a is one – third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^2 \tan^2 a$

Solution:



$r = h\tan a$

Since ΔAOG is similar to ΔCEG

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r-R}$$

$$\Rightarrow H = \frac{h}{r}(r-R) = \frac{h}{h\tan a}(h\tan a - R) = \frac{1}{\tan a}(h\tan a - R)$$

Volume of the cylinder is

$$V = \pi R^2 H = \frac{\pi R^2}{\tan a}(h\tan a - R) = \pi R^2 h - \frac{\pi R^3}{\tan a}$$

$$\therefore \frac{dV}{dR} = 2\pi Rh \frac{3\pi R^2}{\tan a}$$

$$\frac{dV}{dR} = 0$$

$$\Rightarrow 2\pi Rh = \frac{3\pi R^2}{\tan a}$$

$$\Rightarrow 2h\tan a = 3R$$

$$\Rightarrow R=\frac{2h}{3}\tan a$$

$$\frac{d^2V}{dR^2}=2\pi Rh-\frac{6\pi R}{\tan a}$$

And, for $R=\frac{2h}{3}\tan a$, we have

$$\frac{d^2V}{dR^2}=2\pi h-\frac{6\pi}{\tan a}\left(\frac{2h}{3}\tan a\right)=2\pi h-4\pi h=-2\pi h<0$$

Volume of the cylinder is greatest when $R=\frac{2h}{3}\tan a$

$$R=\frac{2h}{3}\tan a, H=\frac{1}{\tan a}\left(h\tan a-\frac{2h}{3}\tan a\right)=\frac{1}{\tan a}\left(\frac{h\tan a}{3}\right)=\frac{h}{3}$$

The maximum volume of cylinder can be obtained as

$$\pi\left(\frac{2h}{3}\tan a\right)^2\left(\frac{h}{3}\right)=\pi\left(\frac{4h^2}{9}\tan^2 a\right)\left(\frac{h}{3}\right)=\frac{4}{27}\pi h^3\tan^2 a$$

18. A cylindrical tank of radius 10m is being filled with wheat at the rate of 314 cubic mere per hour. Then the depth of the wheat is increasing at the rate of

$(A)1\,m/h$ $(B)0.1\,m/h$ $(C)1.1\,m/h$ $(D)0.5\,m/h$

Solution:

$$V=\pi(radius)^2\times height$$

$$=\pi(10)^2 h \qquad (radius=10m)$$

$$=100\pi h$$

$$\frac{dV}{dt}=100\pi\frac{dh}{dt}$$

Tank is being filled with wheat at rate of 314 cubic meters per hour

$$\frac{dV}{dt}=314\,m^3/h$$

$$314 = 100\pi \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

The depth of wheat is increasing at 1 m/h

The correct answer is A

19. The slope of the tangent to the curve $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is

$(A)\frac{22}{7}$ $(B)\frac{6}{7}$ $(C)\frac{7}{6}$ $(D)\frac{-6}{7}$

Solution:

$$\therefore \frac{dx}{dt} = 2t + 3 \; and \; \frac{dy}{dt} = 4t - 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{4t-2}{2t+3}$$

Given point is $(2, -1)$

At x = 2

$$t^2 + 3t - 8 = 2$$

$$\Rightarrow t^2 + 3t - 10 = 0$$

$$\Rightarrow (t-2)(t+5) = 0$$

$$\Rightarrow t = 2 \; or \; t = -5$$

At $y = -1$, we have

$$2t^2 - 2t - 5 = -1$$

$$\Rightarrow 2t^2 - 2t - 4 = 0$$

$$\Rightarrow 2(t^2 - t - 2) = 0$$

$\Rightarrow (t-2)(t+1)=0$

$\Rightarrow t=2 \text{ or } t=-1$

Common value of t is 2

Slope of tangent to given curve at point $(2,-1)$ is

$$\left.\frac{dy}{dx}\right]_{t=2}=\frac{4(2)-2}{2(2)+3}=\frac{8-2}{4+3}=\frac{6}{7}$$

The correct answer is B

20. The line $y=mx+1$ is tangent to the given curve $y^2=4x$ if the value on m is

$(A)\,1\ \ (B)\,2\ \ (C)\,3\ \ (D)\,\frac{1}{2}$

Solution:

Equation of the tangent to curve is $y=mx+1$

Substituting $y=mx+1$ in $y^2=4x$

$\Rightarrow (mx+1)^2=4x$

$\Rightarrow m^2x^2+1+2mx-4x=0$

$\Rightarrow m^2x^2+x(2m-4)+1=0 \quad \text{........}(i)$

$(2m-4)^2-4(m^2)(1)=0$

$\Rightarrow 4m^2+16-16m-4m^2=0$

$\Rightarrow 16-16m=0$

$\Rightarrow m=1$

The required value of m is 1

The correct answer is A.

21. The normal at the point (1, 1) on the curve $2y+x^2=3$ is

$(A)\ x+y=0\ (B)\ x-y=0\ \ (C)\ x+y+1=0\ \ (D)\ x-y=1$

Solution:

$$\frac{2dy}{dx}+2x=0$$

$$\Rightarrow \frac{dy}{dx}=-x$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,1)}=-1$$

Slope of normal to curve at point (1, 1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}}=1$$

Equation of normal to given curve at (1, 1) is

$$\Rightarrow y-1=1(x-1)$$

$$\Rightarrow y-1=x-1$$

$$\Rightarrow x-y=0$$

The correct answer is B

22. The normal to the curve $x^2=4y$ passing (1, 2) is

$(A)\ x+y=3\ (B)\ x-y=3\ (C)\ x+y=1\ (D)\ x-y=1$

Solution:

$$2x=4\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{2}$$

Slope of normal to curve at point (h,k) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}}=-\frac{2}{h}$$

Equation of normal at point (h,k) is

$$y-k=\frac{-2}{h}(x-h)$$

Normal passes through the point $(1,2)$

$$2-k=\frac{-2}{h}(1-h)\ or\ k=2+\frac{2}{h}(1-h) \quad ......(i)$$

(h,k) lies on the curves $x^2=4y$, we have $h^2=4k$

$$\Rightarrow k=\frac{h^2}{4}$$

$$\frac{h^2}{4}=2+\frac{2}{h}(1-h)$$

$$\Rightarrow \frac{h^3}{4}=2h+2-2h=2$$

$$\Rightarrow h^3=8$$

$$\Rightarrow h=2$$

$$\therefore k=\frac{h^2}{4}\Rightarrow k=1$$

Equation of normal is

$$\Rightarrow y-1=\frac{-2}{2}(x-2)$$

$$\Rightarrow y-1=(x-2)$$

$$\Rightarrow x+y=3$$

The correct answer is A

23. The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

$(A)\left(4,\pm\frac{8}{3}\right)$ $(B)\left(4,\frac{-8}{3}\right)$ $(C)\left(4,\pm\frac{3}{8}\right)$ $(D)\left(\pm 4,\frac{8}{3}\right)$

Solution:

$$9(2y)\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

Slope of normal to given curve at point (x_1, y_1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1,y_1)}} = -\frac{6y_1}{x_1^2}$$

Equation of normal to curve at (x_1, y_1) is

$$(y - y_1) = \frac{-6y_1}{x_1^2}(x - x_1)$$

$$\Rightarrow x_1^2 y + x_1^2 y_1 = 6xy_1 + 6x_1 y_1$$

$$\Rightarrow 6x_1y_1 + x_1^2 y = 6x_1 y_1 + x_1^2 y_1$$

$$\Rightarrow \frac{6xy_1}{6x_1y_1 + x_1^2 y_1} = \frac{x^2 y}{6x_1 y_1 + x^2 y} = 1$$

$$\Rightarrow \frac{x}{\frac{x_1(6+x_1)}{6}} + \frac{y}{\frac{y_1(6+x_1)}{x_1}} = 1$$

Normal makes equal intercepts with axes

$$\therefore \frac{x_1(6+x_1)}{6} + \frac{y_1(6+x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow x_1^2 = 6y_1$$

(x_1, y_1) lies on the curve, so

$$9y_1^2 = x_1^3$$

$$9\left(\frac{x_1^2}{6}\right)^2 = x_1^3 \Rightarrow \frac{x_1^4}{4} = x_1^3 \Rightarrow x_1 = 4$$

$$9y_1^2 = (4)^3 = 64$$

$$\Rightarrow y_1^2 = \frac{64}{9}$$

$$\Rightarrow y_1 = \pm\frac{8}{3}$$

Required points are $\left(4, \pm\frac{8}{3}\right)$

The correct answer is A

Exercise 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when

   (a) r =3 cm (b) r = 4 cm

Solution:

We know that $A = \pi r^2$

$$\therefore \frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$$

(a) When r = 3 cm,

$$\frac{dA}{dr} = 2\pi(3) = 6\pi$$

The area is changing at $6\,cm^2/s$ when radius is 3 cm

(b) When r = 4 cm,

$$\frac{dA}{dr} = 2\pi(4) = 8\pi$$

The area is changing at $8\,cm^2/s$ when radius is 4 cm

2. The volume of a cube is increasing at the rate of $8\,cm^3/s$. How fast is the surface area increasing when the length of its edge is 12 cm?

Solution:

Let the side length, volume and surface area respectively be equal to x, V and S

$$V = x^3$$

$$S = 6x^2$$

$$\frac{dV}{dt} = 8cm^3/s$$

$$\therefore 8 = \frac{dV}{dt} = \frac{d}{dt}\left(x^3\right) = \frac{d}{dx}\left(x^3\right)\frac{dx}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$$

$$\frac{ds}{dt}=\frac{d}{dt}\left(6x^2\right)=\frac{d}{dx}\left(6x^2\right)\frac{dx}{dt}=12x\frac{dx}{dt}=12x\left(\frac{8}{3x^2}\right)=\frac{32}{x}$$

So, when $x=12cm, \frac{ds}{dt}=\frac{32}{12}cm^2/s=\frac{8}{3}cm^2/s$

3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm/s

Solution:

We know that $A=\pi r^2$

$$\therefore \frac{dA}{dt}=\frac{d}{dr}\left(\pi r^2\right)\frac{dr}{dt}=2\pi r\frac{dr}{dt}$$

$$\frac{dr}{dt}=3cm/s$$

$$\therefore \frac{dA}{dt}=2\pi r(3)=6\pi r$$

So, when r = 10 cm,

$$\frac{dA}{dt}=6\pi(10)=60\pi\ cm^2/s$$

4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

Solution:

Let the length and the volume of the cube respectively be x and V

$$V=x^3$$

$$\therefore \frac{dV}{dt}=\frac{d}{dt}\left(x^3\right)=\frac{d}{dx}\left(x^3\right)\frac{dx}{dt}=3x^2\frac{dx}{dt}$$

$$\frac{dx}{dt}=3cm/s$$

$$\therefore \frac{dV}{dt}=3x^2(3)=9x^2$$

So, when x = 10 cm,

$$\frac{dV}{dt} = 9(10)^2 = 900\ cm^3/s$$

5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Solution:

We know that $A = \pi r^2$

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2)\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$$

$$\frac{dr}{dt} = 5\,cm/s$$

So, when r = 8 cm,

$$\frac{dA}{dt} = 2\pi(8)(5) = 80\pi\ cm^2/s$$

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Solution:

We know that $C = 2\pi r$

$$\therefore \frac{dC}{dt} = \frac{dC}{dr}\frac{dr}{dt} = \frac{d}{dr}(2\pi r)\frac{dr}{dt} = 2\pi\frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.7\,cm/s$$

$$\therefore \frac{dC}{dt} = 2\pi(0.7) = 1.4\pi\ cm/s$$

7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x= 8 cm and y = 6cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

Solution:

It is given that $\frac{dx}{dt} = -5cm/\min, \frac{dx}{dt} = 4cm/\min, x = 8\,cm\ and\ \ y = 6cm,$

(a) The perimeter of a rectangle is given by $P = 2(x+y)$

$$\therefore \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4) = -2cm/\min$$

(b) The area of rectangle is given by $A = xy$

$$\therefore \frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} = -5y + 4x$$

When x = 8cm and y = 6 cm, $\frac{dA}{dt} = (-5\times 6 + 4\times 8)cm^2/\min = 2cm^2/\min$

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm

Solution:

We know that $V = \frac{4}{3}\pi r^3$

$$\therefore \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^2\right)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = 900\ cm^2/s$$

$$\therefore 900 = 4\pi r^2\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

So, when radius = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}cm/s$$

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is 10 cm

Solution:

We know that $V=\frac{4}{3}\pi r^{2}$

$$\therefore \frac{dV}{dr}=\frac{d}{dr}\left(\frac{4}{3}\pi r^{3}\right)=\frac{4}{3}\pi\left(3r^{2}\right)=4\pi r^{2}$$

So, when radius = 10 cm, $\frac{dV}{dr}=4\pi(10)^{2}=400\pi$

Thus, the volume of the balloon is increasing at the rate of $400\pi\ cm^{3}/s$

10. A ladder 5m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

Solution:

Let the height of the wall at which the ladder is touching it be y m and the distance of its foot from the wall on the ground be x m

$$\therefore x^{2}+y^{2}=5^{2}=25 \Rightarrow y=\sqrt{25-x^{2}}$$

$$\therefore \frac{dy}{dt}=\frac{d}{dt}\left(\sqrt{25-x^{2}}\right)=\frac{d}{dx}\left(\sqrt{25-x^{2}}\right)\frac{dx}{dt}=\frac{-x}{\sqrt{25-x^{2}}}\frac{dx}{dt}$$

$$\frac{dx}{dt}=2\ cm/s$$

$$\therefore \frac{dy}{dt}=\frac{-2x}{\sqrt{25-x^{2}}}$$

So, when x = 4 m,

$$\frac{dy}{dt}=\frac{-2\times 4}{\sqrt{25-16}}=-\frac{8}{3}$$

11. A particle is moving along the curve $6y=x^{3}+2$. Find the points on the curve at which the Y coordinate is changing 8 times as fast as the X coordinate

Solution:

The equation of the curve is $6y=x^{3}+2$

Differentiating with respect to time, we have,

$$6\frac{dy}{dt}=3x^{2}\frac{dx}{dt} \Rightarrow 2\frac{dy}{dt}=x^{2}\frac{dx}{dt}$$

According to the question, $\left(\frac{dy}{dt}=8\frac{dx}{dt}\right)$

$$\therefore 2\left(8\frac{dx}{dt}\right)=x^2\frac{dx}{dt}\Rightarrow 16\frac{dx}{dt}=x^2\frac{dx}{dt}\Rightarrow\left(x^2-16\right)\frac{dx}{dt}=0\Rightarrow x^2=16\Rightarrow x=\pm 4$$

When $x=4, y=\frac{4^3+2}{6}=\frac{66}{6}=11$

When $x=-4, y=\frac{\left(-4^3\right)+2}{6}=\frac{62}{6}=\frac{31}{3}$

Thus, the points on the curve are (4, 11) and $\left(-4,\frac{-31}{3}\right)$

12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}cm/s$. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Solution:

Assuming that the air bubble is a sphere,

$$V=\frac{4}{3}\pi r^2$$

$$\therefore \frac{dV}{dt}=\frac{d}{dt}\left(\frac{4\pi}{3}r^3\right)=\frac{d}{dr}\left(\frac{4\pi}{3}r^3\right)\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt}=\frac{1}{2}cm/s$$

So, when r = 1 cm, $\frac{dV}{dt}=4\pi(1)^2\left(\frac{1}{2}\right)=2\pi\ cm^3/s$

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x

Solution:

We know that $V=\frac{4}{3}\pi r^3$

$$d=\frac{3}{2}(2x+1)\Rightarrow r=\frac{3}{4}(2x+1)$$

$$\therefore V=\frac{4}{3}\pi\left(\frac{3}{4}\right)^3(2x+1)^3=\frac{9}{16}\pi(2x+1)^3$$

$$\therefore \frac{dV}{dx}=\frac{9}{116}\pi\frac{d}{dx}(2x+1)^3=\frac{27}{8}\pi(2x+1)^3$$

14. Sand is pouring from a pipe at the rate of $12\ cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one – sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Solution:

We know that $V=\frac{1}{3}\pi r^2 h$

$$h=\frac{1}{6}r\Rightarrow r=6h$$

$$\therefore V=\frac{1}{3}\pi(6h)^2 h=12\pi h^3$$

$$\therefore \frac{dV}{dt}=12\pi\frac{d}{dt}\left(h^3\right)\frac{dh}{dt}=12\pi\left(3h^2\right)\frac{dh}{dt}=36\pi h^2\frac{dh}{dt}$$

$$\frac{dV}{dt}=12\ cm^2/s$$

So, when h = 4 cm,

$$12=36\pi(4)^2\frac{dh}{dt}$$

$$\Rightarrow\frac{dh}{dt}=\frac{12}{36\pi(16)}=\frac{1}{48\pi}cm/s$$

15. The total cost C(x) in Rupees associated with the production of x units of an item is given by $C(x)=0.007x^3-0.003x^2+15x+4000$. Find the marginal cost when 17 units are produced

Solution:

Marginal cost is the rate of change of the total cost with respect to the output.

$\therefore$ Marginal cost $MC = \frac{dC}{dx} = 0.007\left(3x^2\right) - 0.003(2x) + 15 = 0.021x^2 - 0.006x + 15$

When $x = 17, MC = 0.021(17)^2 - 0.006(17) + 15$

$= 0.021(289) - 0.006(17) + 15$

$= 6.069 - 0.102 + 15$

$= 20.967$

So, when 17 units are produced, the marginal cost is Rs. 20.967.

16. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when x = 7

Solution:

Marginal revenue is the rate of change of the total revenue with respect to the number of

units sold.

$\therefore$ Marginal Revenue $MR = \frac{dR}{dx} = 13(2x) + 26 = 26x + 26$

When $x = 7, MR = 26(7) + 26 = 182 + 26 = 208$

Thus, the marginal revenue is Rs 208

17. The rate of change of the area of a circle with respect to its radius r at r = 6 cm is

$(A) 10\pi$ $(B) 12\pi$ $(C) 8\pi$ $(D) 11\pi$

Solution:

We know that $A = \pi r^2$

$\therefore \frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$

So, when $r = 6\,cm, \dfrac{dA}{dr} = 2\pi \times 6 = 12\pi\ cm^2 / s$

Thus, the rate of change of the area of the circle is $12\pi\ cm^2 / s$

The correct answer is option B

18. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when x = 15 is

(A) 116 (B) 96 (C) 90 (D) 126

Solution:

Marginal revenue is the rate of change of the total revenue with respect to the number of units sold.

$\therefore$ Marginal Revenue $MR = \dfrac{dR}{dx} = 3(2x) + 36 = 6x + 36$

So, when $x = 15, MR = 6(15) + 36 = 90 + 36 = 126$

Hence, the marginal revenue is Rs 126.

The correct answer is option D.

Exercise 6.2

1. Show, that the function given by $f(x)=3x+17$ is strictly increasing on R.

Solution:

Let x_1 *and* x_2, be any two numbers in R.

$$x_1<x_2 \Rightarrow 3x_1+17<3x_2+17=f(x_1)<f(x_2)$$

Thus, f is strictly increasing on R.

Alternate Method

$f'(x)=3>0$ on R.

Thus, f is strictly increasing on R.

2. Show, that the function given by $f(x)=e^{2x}$ is strictly increasing on R.

Solution:

Let x_1 *and* x_2 be any two numbers in R.

$$x_1<x_2 \Rightarrow 2x_1<2x_2 \Rightarrow e^{2x_1}<e^{2x_2} \Rightarrow f(x_1)<f(x_2)$$

Thus, f is strictly increasing on R.

3. Show that the function given by $f(x)=\sin x$ is

(A) Strictly increasing in $\left(0,\frac{\pi}{2}\right)$ (B) Strictly decreasing $\left(\frac{\pi}{2},\pi\right)$

(C) Neither increasing nor decreasing in $(0,\pi)$

Solution:

$$f(x)=\sin x \Rightarrow f'(x)=\cos x$$

(A) $x\in\left(0,\frac{\pi}{2}\right)\Rightarrow \cos x>0 \Rightarrow f'(x)>0$

Thus, f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

(B) $x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \cos x < 0 \Rightarrow f'(x) < 0$

Thus, f is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(C) The results obtained in (A) and (B) are sufficient to state that f is neither increasing nor decreasing in $(0, \pi)$

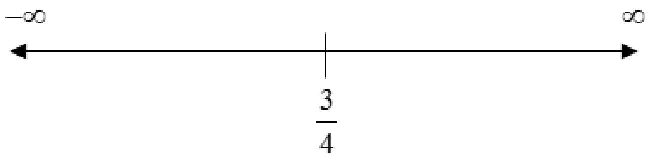
4. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is

(A) Strictly increasing (B) Strictly decreasing

Solution:

$f(x) = 2x^2 - 3x \Rightarrow f'(x) = 4x - 3$

$\therefore f'(x) = 0 \Rightarrow x = \frac{3}{4}$



In $\left(-\infty, \frac{3}{4}\right)$, $f'(x) = 4x - 3 < 0$

Hence, f is strictly decreasing in $\left(-\infty, \frac{3}{4}\right)$

In $\left(\frac{3}{4}, \infty\right)$, $f'(x) = 4x - 3 > 0$

Hence, f is strictly increasing in $\left(\frac{3}{4}, \infty\right)$

5. Find the intervals in which the function f given $f(x) = 2x^2 - 3x^2 - 36x + 7$ is

(A) Strictly increasing (B) Strictly decreasing

Solution:

$$f(x)=2x^2-3x^2-36x+7$$

$$f'(x)=6x^2-6x+36=6\left(x^2-x-6\right)=6(x+2)(x-3)$$

$$\therefore f'(x)=0 \Rightarrow x=-2,3$$

In $(-\infty,-2)$ *and* $(3,\infty)$, $f'(x)>0$

In $(-2,3)$, $f'(x)<0$

Hence, f is strictly increasing $(-\infty,-2)$ *and* $(3,\infty)$ and strictly decreasing in $(-2,3)$

6. Find the intervals in which the following functions are strictly increasing or decreasing

(a) x^2+2x-5 (b) $10-6x-2x^2$ (c) $-2x^3-9x^2-12x+1$

(d) $6-9x-x^2$ (e) $(x+1)^3(x-3)^3$

Solution:

$$f(x)=x^2+2x-5 \Rightarrow f'(x)=2x+2 \Rightarrow f'(x)=0 \Rightarrow x=1$$

$x=-1$ divides the number line into intervals $(-\infty,-1)$ *and* $(-1,\infty)$

In $(-\infty,-1)$, $f'(x)=2x+2<0$

$\therefore f$ is strictly decreasing in $(-\infty,-1)$

In $(-\infty,-1)$, $f'(x)=2x+2>0$, $\therefore f'(x)=2x+2>0$

$\therefore f$ is strictly decreasing in $(-1,\infty)$

(b) $f(x)=10-6x-2x^2 \Rightarrow f'(x)=-6-4x \Rightarrow f'(x)=0 \Rightarrow x=-\frac{3}{2}$

$x=-\frac{3}{2}$ divides the number line into two intervals $\left(-\infty,-\frac{3}{2}\right)$ *and* $\left(-\frac{3}{2},\infty\right)$

In $\left(-\infty,-\frac{3}{2}\right)$, $f'(x)=-6-4x<0$

$\therefore$ f is strictly increasing for $x<-\frac{3}{2}$

In $\left(-\frac{3}{2},\infty\right)$, $f'(x)=-6-4x>0$

$\therefore$ f is strictly increasing for $x>-\frac{3}{2}$

(c) $f(x)=-2x^3-9x^2-12x+1$

$\therefore f'(x)=-6x^2-18x-12=-6\left(x^2+3x+2\right)=-6(x+1)(x+2)$

$\therefore f'(x)=0 \Rightarrow x=-1,2$

$x=-1$ *and* $x=-2$ divide the number line into intervals $(-\infty,-2),(-2,-1)$ *and* $(-1,\infty)$

In $(-\infty,-2)$ *and* $(-1,\infty)$, $f'(x)=-6(x+1)(x+2)<0$

$\therefore$ f is strictly decreasing for $x<-2$ *and* $x>-1$

In $(-2,-1)$, $f'(x)=-6(x+1)(x+2)>0$

$\therefore$ f is strictly increasing for $-2<x<-1$

(d) $f(x)=6-9x-x^2 \Rightarrow f'(x)=-9-2x$

$f'(x)=0 \Rightarrow x=\frac{-9}{2}$

In $\left(-\infty,-\frac{9}{2}\right)$, $f'(x)>0$

$\therefore f$ is strictly increasing for $x < -\frac{9}{2}$

In $\left(-\frac{9}{2}, \infty\right)$, $f'(x) < 0$

$\therefore f$ is strictly decreasing for $x > -\frac{9}{2}$

(e) $f(x) = (x+1)^3 (x-3)^3$

$f'(x) = 3(x+1)^2 (x-3)^3 + 3(x-3)^2 (x+1)^3$

$= 3(x+1)^2 (x-3)^2 [x-3+x+1]$

$= 3(x+1)^2 (x-3)^2 (2x-2)$

$= 6(x+1)^2 (x-3)^2 (x-1)$

$f'(x) = 0 \Rightarrow x = -1, 3, 1$

$x = -1, 3, 1$ divides the number line into four intervals $(-\infty, -1), (-1, 1)(1, 3)$ *and* $(3, \infty)$

In $(-\infty, -1)$ *and* $(-1, 1)$, $f'(x) = 6(x+1)^2 (x-3)^2 (x-1) < 0$

$\therefore f$ is strictly decreasing in $(-\infty, -1)$ *and* $(-1, 1)$

In $(1, 3)$ *and* $(3, \infty)$, $f'(x) = 6(x+1)^2 (x-3)^2 (x-1) > 0$

$\therefore f$ is strictly increasing in $(1, 3)$ *and* $(3, \infty)$

7. Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$, is an increasing function throughout its domain

Solution:

$$y = \log(1+x) - \frac{2x}{2+x}$$

$$\therefore \frac{dy}{dx}=\frac{1}{1+x}-\frac{(2+x)(2)-2x(1)}{(2+x)^2}=\frac{1}{1+x}-\frac{4}{(2+x)^2}=\frac{x^2}{(1+x)(2+x)^2}$$

$$\frac{dy}{dx}=0$$

$$\Rightarrow \frac{x^2}{(2+x)^2}=0$$

$$\Rightarrow x^2=0$$

$$\Rightarrow x=0$$

Because $x>-1, x=0$ divides domain $(-1,\infty)$ in two intervals $-1<x<0$ *and* $x>0$

When $-1<x<0$,

$x<0\Rightarrow x^2>0$

$x>-1\Rightarrow(2+x)>0\Rightarrow(2+x)^2>0$

$$\therefore y'=\frac{x^2}{(2+x)^2}>0$$

When x > 0,

$$\therefore y'=\frac{x^2}{(2+x)^2}>0$$

Hence, f is increasing throughout the domain.

8. Find the values of x for which $y=[x(x-2)]^2$ is an increasing function

Solution:

$$y=[x(x-2)]^2=[x^2-2x]^2$$

$$\therefore \frac{dy}{dx}=y'=2(x^2-2x)(2x-2)=4x(x-2)(x-1)$$

$$\therefore \frac{dy}{dx}=0\Rightarrow x=0, x=2, x=1$$

$x=0, x=1$ *and* $x=2$ divide the number line into intervals $(-\infty,0),(0,1),(1,2)$ *and* $(2,\infty)$

In $(-\infty,0)$ *and* $(1,2), \dfrac{dy}{dx}<0$

$\therefore y$ is strictly decreasing in intervals $(-\infty,0)$ *and* $(1,2)$

In intervals $(0,1)$ *and* $(2,\infty), \dfrac{dy}{dx}>0$

$\therefore y$ is strictly increasing in intervals $(0,1)$ *and* $(2,\infty)$

9. Prove that $y=\dfrac{4\sin\theta}{(2+\cos\theta)}-\theta$ is an increasing function of θ in $\left[0,\dfrac{\pi}{2}\right]$

Solution:

$$y=\frac{4\sin\theta}{(2+\cos\theta)}-\theta$$

$$\therefore \frac{dy}{d\theta}=\frac{(2+\cos\theta)(4\cos\theta)-4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4\cos^2\theta+4\sin^2\theta}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4}{(2+\cos\theta)}-1$$

$$\frac{dy}{d\theta}=0$$

$$\Rightarrow \frac{8\cos\theta+4}{(2+\cos\theta)^2}=1$$

$$\Rightarrow 8\cos\theta+4=4+\cos^2\theta+4\cos\theta$$

$$\Rightarrow \cos^2\theta-4\cos\theta=0$$

$$\Rightarrow \cos\theta(\cos\theta-4)=0$$

$$\Rightarrow \cos\theta=0 \text{ or } \cos\theta=4$$

Because $\cos\theta\neq 4, \cos\theta=0$

$$\cos\theta=0\Rightarrow \theta=\frac{\pi}{2}$$

$$\frac{dy}{d\theta}=\frac{8\cos\theta+4-\left(4+\cos^2\theta+4\cos\theta\right)}{\left(2+\cos\theta\right)^2}=\frac{4\cos\theta-\cos^2\theta}{\left(2+\cos\theta\right)^2}=\frac{\cos\theta\left(4-\cos\theta\right)}{\left(2+\cos\theta\right)^2}$$

In $\left[0,\frac{\pi}{2}\right],\cos\theta>0,$

$4>\cos\theta\Rightarrow 4-\cos\theta>0$

$\therefore\cos\theta\left(4-\cos\theta\right)>0$

$\left(2+\cos\theta\right)^2>0$

$$\Rightarrow\frac{\cos\theta\left(4-\cos\theta\right)}{\left(2+\cos\theta\right)^2}>0$$

$$\Rightarrow\frac{dy}{d\theta}>0$$

So, y is strictly increasing in $\left(0,\frac{\pi}{2}\right)$

The function is continuous at $x=0$ *and* $x=\frac{\pi}{2}$

So, y is increasing in $\left[0,\frac{\pi}{2}\right]$

10. Prove that the logarithmic function is strictly increasing on $\left(0,\infty\right)$

Solution:

$f\left(x\right)=\log x$

$\therefore f'\left(x\right)=\frac{1}{x}$

For $x>0,\ f'\left(x\right)=\frac{1}{x}>0$

Thus, the logarithmic function is strictly increasing in interval $\left(0,\infty\right)$

11. Prove that the function f given by $f\left(x\right)=x^2-x+1$is neither strictly increasing nor stricty decreasing on $\left(-1,1\right)$

Solution:

$f(x)=x^2-x+1$

$\therefore f'(x)=2x-1$

$\therefore f'(x)=0 \Rightarrow x=\frac{1}{2}$

$x=\frac{1}{2}$ divides $(-1,1)$ into $\left(-1,\frac{1}{2}\right)$ *and* $\left(\frac{1}{2},1\right)$

In $\left(-1,\frac{1}{2}\right)$, $f'(x)=2x-1<0$

So, f is strictly decreasing in $\left(-1,\frac{1}{2}\right)$

In $\left(\frac{1}{2},1\right)$, $f'(x)=2x-1>0$

So, f is strictly increasing in interval $\left(\frac{1}{2},1\right)$

Thus, f is neither strictly increasing nor strictly decreasing in interval $(-1,1)$

12. Which of the following functions are strictly decreasing on $\left(0,\frac{\pi}{2}\right)$?

$(A)\cos x$ $\quad (B)\cos 2x$ $\quad (C)\cos 3x$ $\quad (D)\tan x$

Solution:

(A) $f_1(x)=\cos x$

$\therefore f_1'(x)=-\sin x$

In $\left(0,\frac{\pi}{2}\right)$, $f_1'(x)=-\sin x<0$

$\therefore f_1'(x)=\cos x$ is strictly decreasing in $\left(0,\frac{\pi}{2}\right)$

(B) $f_2(x)=\cos 2x$

$\therefore f_2'(x) = -2\sin 2x$

$$0 < x < \frac{\pi}{2} \Rightarrow 0 < 2x < \pi \Rightarrow \sin 2x > 0 \Rightarrow -2\sin 2x < 0$$

$\therefore f_2'(x) = -2\sin 2x < 0$ in $\left(0, \frac{\pi}{2}\right)$

$\therefore f_2(x) = \cos 2x$ is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$

$(C)\ f_3(x) = \cos 3x$

$\therefore f_3'(x) = -3\sin 3x$

$f_3'(x) = 0$

$\Rightarrow \sin 3x = 0 \Rightarrow 3x = \pi, as\ x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow x = \frac{\pi}{3}$

$x = \frac{\pi}{3}$ *divides* $\left(0, \frac{\pi}{2}\right)$ into $\left(0, \frac{\pi}{3}\right)$ *and* $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

In $\left(0, \frac{\pi}{3}\right)$, $f_3(x) = -3\sin 3x < 0$ $\quad \left[0 < x < \frac{\pi}{3} \Rightarrow 0 < 3x < \pi\right]$

$\therefore f_3$ is strictly decreasing in $\left(0, \frac{\pi}{3}\right)$

In $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, $f_3(x) = -3\sin 3x > 0$ $\quad \left[\frac{\pi}{3} < x < \frac{\pi}{2} \Rightarrow \pi < 3x < \frac{3\pi}{2}\right]$

$\therefore f_3$ is strictly increasing in $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

So, f_3 is neither increasing nor decreasing in interval $\left(0, \frac{\pi}{2}\right)$

$(D)\ f_4(x) = \tan x$

$\therefore f_4'(x) = \sec^2 x$

In $\left(0, \frac{\pi}{2}\right)$, $f_4'(x) = \sec^2 x > 0$

$\therefore f_4$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

So, the correct answer are A and B

13. On which of the following intervals is the function f is given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?

$A.(0,1)$ $B.\left(\frac{\pi}{2}, \pi\right)$ $C.\left(0, \frac{\pi}{2}\right)$ D. None of these

Solution:

$f(x) = x^{100} + \sin x - 1$

$\therefore f'(x) = 100x^{99} + \cos x$

In $(0,1)$, $\cos x > 0$ *and* $100x^{99} > 0$

$\therefore f'(x) > 0$

So, f is strictly increasing in (0, 1)

In $\left(\frac{\pi}{2}, x\right)$, $\cos x < 0$ *and* $100x^{99} > 0$

$100x^{99} > \text{cosx}$

$\therefore f'(x) > 0$ *in* $\left(\frac{\pi}{2}, x\right)$

So, f is strictly increasing in interval $\left(\frac{\pi}{2}, x\right)$

In interval $\left(0, \frac{\pi}{2}\right)$, $\cos x > 0$ *and* $100x^{99} > 0$

$\therefore 100x^{99} + \cos x > 0$

$\Rightarrow f'(x) > 0$ *on* $\left(0, \frac{\pi}{2}\right)$

$\therefore$ f is strictly increasing in interval $\left(0, \frac{\pi}{2}\right)$

Hence, f is strictly decreasing in none of the intervals.

The correct answer is D.

14. Find the least value of a such that the function f given $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2)

Solution:

$f(x) = x^2 + ax + 1$

$\therefore f'(x) = 2x + a$

$\therefore f'(x) > 0$ *in* $(1, 2)$

$\Rightarrow 2x + a > 0$

$\Rightarrow 2x > -a$

$\Rightarrow x > \frac{-a}{2}$

So, we need to find the smallest value of a such that

$x > \frac{-a}{2}$, when $x \in (1, 2)$

$\Rightarrow x > \frac{-a}{2}$ $(when\, 1 < x < 2)$

$\frac{-a}{2} = 1 \Rightarrow a = -2$

Hence, the required value of a is – 2

15. Let I be any interval disjoint from (-1, 1), prove that the function f given by

$f(x) = x + \frac{1}{x}$ is strictly increasing on I

Solution:

$$f(x) = x + \frac{1}{x}$$

$$\therefore f'(x) = 1 - \frac{1}{x^2}$$

$$\therefore f'(x) = 0 \Rightarrow \frac{1}{x^2} \Rightarrow x = \pm 1$$

x = 1 and x = - 1 divide the real line in intervals $(-\infty, 1), (-1, 1)$ *and* $(1, \infty)$

In $(-1, 1)$,

$$-1 < x < 1$$

$$\Rightarrow x^2 < 1$$

$$\Rightarrow 1 < \frac{1}{x^2}, x \neq 0$$

$$\Rightarrow 1 - \frac{1}{x^2} < 0, x \neq 0$$

$$\therefore f'(x) = 1 - \frac{1}{x^2} < 0 \text{ on } (-1, 1) \sim \{0\}$$

$\therefore f$ is strictly decreasing on $(-1, 1) \sim \{0\}$

In $(-\infty, -1)$ *and* $(1, \infty)$

$$x < -1 \text{ or } 1 < x$$

$$\Rightarrow x^2 > 1$$

$\Rightarrow 1 > \frac{1}{x^2}$

$\Rightarrow 1 - \frac{1}{x^2} > 0$

$\therefore f'(x) = 1 - \frac{1}{x^2} > 0 \; on \; (-\infty, -1) \; and \; (1, \infty)$

$\therefore f$ is strictly increasing on $(-\infty, 1) \; and \; (1, \infty)$

Hence, f is strictly increasing in $I - (-1, 1)$

16. Prove that the function f given by $f(x) = \log \sin x$ strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$

Solution:

$f(x) = \log \sin x$

$f'(x) = \frac{1}{\sin x} \cos x = \cot x$

In $\left(0, \frac{\pi}{2}\right)$, $f'(x) = \cot x > 0$

$\therefore f$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

In $\left(\frac{\pi}{2}, \pi\right)$, $f'(x) = \cot x < 0$

$\therefore f$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$

17. Prove that the function f given by $f(x) = \log \cos x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$

Solution:

$f(x) = \log \cos x$

$\therefore f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

In $\left(0, \frac{\pi}{2}\right)$, $\tan x > 0 \Rightarrow -\tan x < 0$

$\therefore f'(x) =< 0 \; on \left(0, \frac{\pi}{2}\right)$

$\therefore f$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

In $\left(\frac{\pi}{2}, \pi\right)$, $\tan x < 0 \Rightarrow -\tan x > 0$

$\therefore f'(x) > 0 \; on \left(\frac{\pi}{2}, \pi\right)$

$\therefore f$ is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x = 100$ is increasing in R

Solution:

$f(x) = x^3 - 3x^2 + 3x = 100$

$f'(x) = 3x^2 - 6x + 3$

$= 3(x^2 - 2x + 1)$

$= 3(x-1)^2$

For $x \in R (x-1)^2 \geq 0$

So $f'(x)$ is always positive in R

So, the f is increasing in R

19. The interval in which $y = x^2e^{-x}$ is increasing is

$A.(-\infty,\infty)$ $B.(-2,0)$ $C.(2,\infty)$ $D.(0,2)$

Solution:

$$y = x^2e^{-x}$$

$$\therefore \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$$

$$\frac{dy}{dx} = 0$$

$$\Rightarrow x = 0 \; and \; x = 2$$

In $(-\infty,0)$ and $(2,\infty)$, $f'(x) < 0$ as e^{-x} is always positive

$\therefore$ f is decreasing on $(-\infty,0)$ and $(2,\infty)$

In $(0,2)$, $f'(x) > 0$

$\therefore$ f is strictly increasing on (0, 2)

So, f is strictly increasing in (0, 2)

The correct answer is D.

Exercise 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at x = 4

Solution:

$$\left.\frac{dy}{dx}\right]_{x=4} = \frac{d}{dx}\left(3x^4 - 4x\right) = 12x^3 - 4\Big]_{x=4} = 12(4)^3 - 4 = 12(64) - 4 = 764$$

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at x =10

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x-1}{x-2}\right) = \frac{(x-2)(1)-(x-1)(1)}{(x-2)^2} = \frac{x-2-x+1}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

$$\therefore \left.\frac{dy}{dx}\right]_{x=10} = \left.\frac{-1}{(x-2)^2}\right]_{x=10} = \frac{-1}{(10-2)^2} = \frac{-1}{64}$$

3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose X- coordinate is 2

Solution:

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^3 - x + 1\right) = 3x^2 - 1$$

$$\therefore \left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2} = 3(2)^2 - 1 = 12 - 1 = 11$$

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose X- coordinate is 3.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 3x + 2\right) = 3x^2 - 3$$

$$\therefore \left.\frac{dy}{dx}\right]_{x=3} = 3x^2 - 3\Big]_{x=3} = 3(3)^2 - 3 = 27 - 3 = 24$$

5. Find the slope of the normal to the curve $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$

Solution:

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos^3\theta\right) = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(a\sin^3\theta\right) = 3a\sin^2\theta(\cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$\therefore \left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\tan\theta\Big]_{\theta=\frac{\pi}{4}} = -\tan\frac{\pi}{4} = -1$$

Slope of normal at $\theta = \frac{\pi}{4} = \frac{-1}{-1} = 1$

6. Find the slope of the normal to the curve $x = 1 - a\sin\theta$ *and* $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$

Solution:

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(1 - a\sin\theta) = -a\cos\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos^2\theta\right) = -2b\sin\theta\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\theta\Big]_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$$

Slope of normal at $\theta = \frac{\pi}{2} = \frac{-1}{\left(\frac{2b}{a}\right)} = -\frac{a}{2b}$

7. Find the points at which tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the X-axis

Solution:

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 3x^2 - 9x + 7\right) = 3x^2 - 6x - 9$$

Since tangent is parallel to the X- axis, slope = 0

$$\therefore 3x^2 - 6x - 9 = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x-3)(x+1) = 0$$

$$\Rightarrow x = 3 \; or \; x = -1$$

$$x = 3, y = (3)^3 - 9(3) + 7 = 27 - 27 - 27 - 7 = -20$$

$$x = 1, y = (-1)^3 - 3(-1)^2 + 7 = -1 - 3 + 9 + 7 = 12$$

Hence, the required points are $(3,-20)$ *and* $(-1,12)$

8. Find a point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $(2,0)$ *and* $(4,4)$

Solution:

Slope of chord = $\frac{4-0}{4-2} = \frac{4}{2} = 2$

Slope of tangent = $\frac{dy}{dx} = 2(x-2)$

$$\therefore 2(x-2) = 2 \Rightarrow x - 2 = 1 \Rightarrow x = 3$$

When $x=3, y=(3-2)^2=1$

Hence, the point is (3,1)

9. Find the point on the curve $y=x^3-11x+5$ at which the tangents is $y=x-11$

Solution:

Equation of tangent is y = x – 11

$\therefore$ Slope of the tangent = 1

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^3-11x+5\right)=3x^2-11$$

$$\therefore 3x^2-11=1\Rightarrow 3x^2=12\Rightarrow x^2=4\Rightarrow x=\pm2$$

$$x=2, y=(2)^3-11(2)+5=8-22+5=-9$$

$$x=-2, y=(-2)^3-11(-2)+5=-8+22+5=19$$

So, the points are $(2,-9)$ *and* $(-2,19)$

10. Find the equation of all lines having slope – 1 that are tangents to the curve

$$y=\frac{1}{x-1}, x\neq1$$

Solution:

$$\frac{dy}{dx}=\frac{-1}{(x-1)^2}$$

$$\frac{-1}{(x-1)^2}=-1\Rightarrow(x-1)^2\Rightarrow x-1=\pm1\Rightarrow x=2,0$$

$$x=0, y=-1 \text{ and } x=2, y=1$$

$$y-(-1)=-1(x-0)$$

$$\Rightarrow y+1=-x$$

$$\Rightarrow y+x+1=0$$

$$y-1=-1(x-2)$$

$$\Rightarrow y-1=x+2$$

$$\Rightarrow y+x-3=0$$

So, the equations of the required lines are $y+x+1=0$ *and* $y+x-3=0$

11. Find the equation of all lines having slope 2 which are tangents to the curve

$$y=\frac{1}{x-3}, x\neq 3$$

Solution:

$$\frac{dy}{dx}=\frac{-1}{(x-3)^2}$$

$$\frac{-1}{(x-3)^2}=2\Rightarrow 2(x-3)^2=-1\Rightarrow (x-3)^2=\frac{-1}{2}$$

Which is not possible

So, there is no tangent to the curve of slope 2.

12. Find the equations of all lines having slope 0 which are tangent to the curve

$$y=\frac{1}{x^2-2x+3}$$

Solution:

$$\frac{dy}{dx}=\frac{-(2x-2)}{(x^2-2x+3)^2}=\frac{-2(x-1)}{(x^2-2x+3)^2}$$

$$\frac{-2(x-1)}{(x^2-2x+3)^2}=0\Rightarrow -2(x-1)=0\Rightarrow x=1$$

When $x=1, y=\frac{1}{1-2+3}=\frac{1}{2}$

$$y-\frac{1}{2}=0(x-1)$$

$$\Rightarrow y-\frac{1}{2}=0$$

$$\Rightarrow y=\frac{1}{2}$$

So, the equation of the line is $y = \frac{1}{2}$

13. Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

i. Parallel to x – axis ii. Parallel to y – axis

Solution:

$$\frac{2x}{9} + \frac{2y}{16}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-16x}{9y}$$

$$(i)\ \frac{dy}{dx} = \frac{-16x}{9y} = 0 \Rightarrow x = 0$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1\ for\ x = 0 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

So, the points are (0, 4) and (0, -4)

$$(ii)\ \frac{dx}{dy} = 0 \Rightarrow \frac{-1}{\left(\frac{-16x}{9y}\right)} = \frac{9y}{16x} = 0 \Rightarrow y = 0$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1\ for\ y = 0 \Rightarrow x = \pm 3$$

So, the points are $(3,0)$ *and* $(-3,0)$

14. Find the equations of the tangents and normal to the given curves at the indicated points.

$I.\ \ y = x^4 - 6x^3 + 13x^2 - 10x + 5\ at\ (0,5)$

$II.\ \ y = x^4 - 6x^3 + 13x^2 - 10x + 5\ at\ (1,3)$

$III.\ \ y = x^3\ at\ (1,1)$

$IV.\ \ y = x^2\ at\ (0,0)$

$V.\ \ x = \cos t, y = \sin t\ at\ t = \frac{\pi}{4}$

Solution:

$$I.\ \frac{dy}{dx}=4x^3-18x^2+26x-10$$

$$\left.\frac{dy}{dx}\right]_{(0,5)}=-10$$

Slope of tangent at (0, 5) is – 10

$$y-5=-10(x-0)$$

$$\Rightarrow y-5=-10x$$

$$\Rightarrow 10x+y=5$$

Slope of normal at (0, 5) is $\frac{-1}{-10}=\frac{1}{10}$

$$y-5=\frac{1}{10}(x-0)$$

$$\Rightarrow 10y-50=x$$

$$\Rightarrow x-10y+50=0$$

$$II.\frac{dy}{dx}=4x^3-18x^2+26x-10$$

$$\left.\frac{dy}{dx}\right]_{(1,3)}=4-18+26-10=2$$

Slope of tangent at (1, 3) is 2

$$y-3=2(x-1)$$

$$\Rightarrow y-3=2x-2$$

$$\Rightarrow y=2x+1$$

Slope of normal at (1, 3) is $-\frac{1}{2}$

$$y-3=\frac{1}{2}(x-1)$$

$\Rightarrow 2y-6=x+1$

$\Rightarrow x+2y-7=0$

$III.\ \frac{dy}{dx}=3x^2$

$\left.\frac{dy}{dx}\right]_{(1,1)}=3(1)^2=3$

Slope of tangent at (1, 1) is 3

$y-1=3(x-1)$

$\Rightarrow y=3x-2$

Slope of normal at (1, 1) is $-\frac{1}{3}$

$y-1=\frac{-1}{3}(x-1)$

$\Rightarrow 3y-3=-x+1$

$\Rightarrow x+3y-4=0$

$IV.\ \frac{dy}{dx}=2x$

$\left.\frac{dy}{dx}\right]_{(0,0)}=0$

Slope of tangent at (0, 0) is 0

$y-0=0(x-0)$

$\Rightarrow y=0$

Slope of normal at (0, 0) is $-\frac{1}{0}$, which is undefined

$\therefore x=0$

$V.\ x=\cos t \quad y=\sin t$

$$\therefore \frac{dx}{dt}=-\sin t \quad \frac{dy}{dt}=\cos t$$

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\cos t}{-\sin t}=-\cot t$$

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}}=-\cot t=-1$$

Slope of tangent at $t=\frac{\pi}{4}$ is -1

$$t=\frac{\pi}{4}, x=\frac{1}{\sqrt{2}}, y=\frac{1}{\sqrt{2}}$$

$$y-\frac{1}{\sqrt{2}}=-1\left(x-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow x+y-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$$

$$\Rightarrow x+y-\sqrt{2}=0$$

Slope of normal at $t=\frac{\pi}{4}$ is $\frac{-1}{-1}=1$

$$y-\frac{1}{\sqrt{2}}=1\left(x-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow x=y$$

15. Find the equation of the tangent line to the curve $y=x^2-2x+7$ which is

a. Parallel to the line $2x-y+9=0$

b. Perpendicular to the line $5y-15x=13$

Solution:

$$\frac{dy}{dx}=2x-2$$

$$(a)\ 2x-y+9=0\Rightarrow y=2x+9$$

Slope of line = 2

$$\therefore 2=2x-2\Rightarrow 2x=4\Rightarrow x=2$$

$$x=2\Rightarrow y=4-4+7=7$$

Equation of tangent through (2, 7) is

$$y-7=2(x-2)\Rightarrow y-2x-3=0$$

$$(b)\ 5y-15x=13\Rightarrow y=3x+\frac{13}{5}$$

Slope of line = 3

$$\therefore 2x-2=\frac{-1}{3}$$

$$\Rightarrow 2x=\frac{-1}{3}+2$$

$$\Rightarrow 2x=\frac{5}{3}$$

$$\Rightarrow x=\frac{5}{6}$$

$$\Rightarrow y=\frac{25}{36}+\frac{10}{6}+7=\frac{25-60+252}{36}=\frac{217}{36}$$

Equation of tangent through $\left(\frac{5}{6},\frac{217}{36}\right)$ is

$$y-\frac{217}{36}=\frac{1}{3}\left(x-\frac{5}{6}\right)$$

$$\Rightarrow \frac{36y-217}{36}=\frac{-1}{18}(6x-5)$$

$\Rightarrow 36y - 217 = -2(6x - 5)$

$\Rightarrow 36y - 217 = -12x + 10$

$\Rightarrow 36y + 12x - 227 = 0$

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ *and* $x = -2$ are parallel

Solution:

$$\therefore \frac{dy}{dx} = 21x^2$$

$$\left.\frac{dy}{dx}\right]_{x=2} = 21(2)^2 = 84$$

$$\left.\frac{dy}{dx}\right]_{x=-2} = 21(-2)^2 = 84$$

Clearly, the tangents are parallel

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point

Solution:

$$\therefore \frac{dy}{dx} = 3x^2$$

According to the equation, $y = \frac{dy}{dx} = 3x^2$

Also, $y = x^3$

$\therefore 3x^2 = x^3$

$x^2(x - 3) = 0$

$x = 0, x = 3$

$x = 0, y = 0$ *and* $x = 3, y = 3(3)^2 = 27$

So the points are (0, 0) and (3, 27)

18. For the curve $y=4x^3-2x^5$, find all the points at which the tangents passes through the origin

Solution:

$$\frac{dy}{dx}=12x^2-10x^4$$

Equation of tangent through (X, Y) is

$$Y-y=\left(12x^2-10x^4\right)\left(X-x\right)$$

For passing through origin, X = 0, Y= 0

$$-y=\left(12x^2-10x^4\right)\left(-x\right)$$

$$y=12x^3-10x^5$$

Also, $y=4x^3-2x^5$

$$\therefore 12x^3-10x^5=4x^3-2x^5$$

$$\Rightarrow 8x^5-8x^3=0$$

$$\Rightarrow x^5-x^3=0$$

$$\Rightarrow x^3\left(x^2-1\right)=0$$

$$\Rightarrow x=,\pm 1$$

$$x=0, y=4(0)^3-2(0)^5=0$$

$$x=1, y=4(1)^3-2(1)^5=2$$

$$x=-1, y=4(-1)^3-2(-1)^5=-2$$

So, the points are (0, 0), (1, 2) and $(-1,-2)$

19. Find the points on the curve $x^2+y^2-2x-3=0$ at which the tangents are parallel to the x- axis

Solution:

$$2x+2y\frac{dy}{dx}-2=0$$

$$\Rightarrow y\frac{dy}{dx}=1-x$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-x}{y}$$

For parallel to X axis,

$$\therefore \frac{1-x}{y}=0\Rightarrow 1-x=0\Rightarrow x=1$$

$$x^2+y^2-2x-3=0$$

$$\Rightarrow y^2=4, y=\pm 2$$

So, the points are (1, 2) and (1, -2)

20. Find the equation of the normal at the point $\left(am^2, am^3\right)$ for the curve $ay^2=x^3$

Solution:

$$2ay\frac{dy}{dx}=3x^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{3x^2}{2ay}$$

Slope of tangent at $\left(am^2, am^3\right)$ is

$$\left.\frac{dy}{dx}\right]_{\left(am^2, am^3\right)}=\frac{3\left(am^2\right)^2}{2a\left(am^3\right)}=\frac{3a^2m^4}{2a^2m^3}=\frac{3m}{2}$$

Slope of normal $= \frac{-2}{3m}$

$$y - am^3 = \frac{-2}{3m}\left(x - am^2\right)$$

$$\Rightarrow 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow 2x + 3my - am^2\left(2 + 3m^2\right) = 0$$

21. Find the equation of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$

Solution:

$$\frac{dy}{dx} = 3x^2 + 2$$

Slope of the normal $= \frac{-1}{3x^2 + 2}$

$$x + 14y + 4 = 0 \Rightarrow y = -\frac{1}{14}x - \frac{4}{14}$$

$$\therefore \frac{-1}{3x^2 + 2} = \frac{-1}{14}$$

$$\Rightarrow 3x^2 + 2 = 14$$

$$\Rightarrow 3x^2 = 12$$

$$\Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

$$x = 2, y = 8 + 4 + 6 = 18$$

$$x = -2, y = -8 - 4 + 6 = -6$$

Equation of normal through (2, 18) is

$$y - 18 = \frac{-1}{14}(x - 2)$$

$\Rightarrow 14y - 252 = x + 2$

$\Rightarrow x + 14y - 254 = 0$

Equation of normal through $(-2, -6)$ is

$y - (-6) = \frac{-1}{14}[x - (-2)]$

$\Rightarrow y + 6 = \frac{-1}{14}(x + 2)$

$\Rightarrow 14y + 84 = -x - 2$

$\Rightarrow x + 14y + 86 = 0$

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$

Solution:

$2y\frac{dy}{dx} = 4a$

$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

$\left.\frac{dy}{dx}\right]_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$

Slope of tangent $= \frac{1}{t}$

Equation of tangent is

$y - 2at = \frac{1}{t}(x - at^2)$

$\Rightarrow ty - 2at^2 = x - at^2$

$\Rightarrow ty = x + at^2$

Slope of normal $= -\frac{1}{\left(\frac{1}{t}\right)} = -t$

Equation of normal is

$y - 2at = -t\left(x - at^2\right)$

$\Rightarrow y - 2at = -tx + at^3$

$\Rightarrow y = -tx + 2at + at^3$

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$. [Hint: Tow curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other]

Solution:

The curves are $x = y^2 \text{ and } xy = k$

Putting $x = y^2 \text{ and } xy = k,$

$y^3 = k \Rightarrow y = k^{\frac{1}{3}}$

$\therefore x = k^{\frac{2}{3}}$

So, the point of intersection is $\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$

Differentiating $x = y^2$

$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

So, slope of tangent to $x = y^2 at \left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$ is

$\left.\frac{dx}{dy}\right]_{\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)} = \frac{1}{2k^{\frac{1}{3}}}$

Differentiating xy = k,

$$x\frac{dy}{dx}+y=0\Rightarrow\frac{dy}{dx}=\frac{-y}{x}$$

Slope of tangent to $xy=k\ at\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)$ is

$$\left.\frac{dx}{dy}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}=\left.\frac{-y}{x}\right]_{\left(k^{\frac{2}{3}},k^{\frac{1}{3}}\right)}=-\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}}=\frac{-1}{k^{\frac{1}{3}}}$$

$\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(\frac{-1}{k^{\frac{1}{3}}}\right)=-1$ for perpendicularity condition

$$\Rightarrow 2k^{\frac{2}{3}}=1$$

$$\Rightarrow\left(2k^{\frac{2}{3}}\right)^3=(1)^3$$

$$\Rightarrow 8k^2=1$$

So, the curves intersect at right angles if $8k^2=1$

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_0,y_0)

Solution:

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow\frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Slope of tangent at (x_0, y_0) is $\left.\frac{dy}{dx}\right]_{(x_0,y_0)} = \frac{b^2x_0}{a^2y_0}$

Equation of tangent at (x_0, y_0) is

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)$$

$$\Rightarrow a^2yy_0 - a^2y_0^2 = b^2xx_0 - b^2x_0^2$$

$$\Rightarrow b^2xx_0 - a^2yy_0 - b^2x_0^2 + a^2y_0^2 = 0$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right) = 0$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} - 1 = 0$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$$

Slope of normal at $(x_0, y_0) = \frac{-a^2y_0}{b^2x_0}$

Equation of normal at (x_0, y_0) is

$$y - y_0 = \frac{-a^2y_0}{b^2x_0}(x - x_0)$$

$$\Rightarrow \frac{y - y_0}{a^2y_0} = \frac{-(x - x_0)}{b^2x_0}$$

$$\Rightarrow \frac{y - y_0}{a^2y_0} - \frac{(x - x_0)}{b^2x_0} = 0$$

25. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$

Solution:

Slope of tangent at (x, y) is

$$\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$$

The given line is $4x-2y+5=0$

$$4x-2y+5=0 \Rightarrow y=2x+\frac{5}{2}$$

Slope of line = 2

$$\frac{3}{2\sqrt{3x-2}}=2$$

$$\Rightarrow \sqrt{3x-2}=\frac{3}{4}$$

$$\Rightarrow 3x-2=\frac{9}{16}$$

$$\Rightarrow 3x=\frac{9}{16}+2=\frac{41}{16}$$

$$\Rightarrow x=\frac{41}{48}$$

$$x=\frac{41}{48}, y=\sqrt{3\left(\frac{41}{48}\right)-2}=\sqrt{\frac{41}{16}-2}=\sqrt{\frac{41-32}{16}}=\sqrt{\frac{9}{16}}=\frac{3}{4}$$

Equation of tangent through $\left(\frac{41}{48}, \frac{3}{4}\right)$ is

$$y-\frac{3}{4}=2\left(x-\frac{41}{48}\right)$$

$$\Rightarrow \frac{4y-3}{4}=2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow 4y-3=\left(\frac{48x-41}{6}\right)$$

$\Rightarrow 24y - 18 = 48x - 41$

$\Rightarrow 48x - 24y = 23$

26. The slope of the normal to the curve $y = 2x^2 + 3\sin x$ at x= 0 is

$(A) = 3, (B) = \frac{1}{3}, (C) = -3, (D) = -\frac{1}{3}$

Solution:

$$\left.\frac{dy}{dx}\right]_{x=0} = 4x + 3\cos x\Big]_{x=0} = 0 + 3\cos 0 = 3$$

Slope of normal $= \frac{-1}{3}$

The correct answer is D.

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

$(A)(1,2), (B)(2,1), (C)(1,-2), (D)(-1,2)$

Solution:

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

Given line is $y = x + 1$

Slope of line = 1

$$\frac{2}{y} = 1$$

$\Rightarrow y = 2$

$y = x + 1 \Rightarrow x = y - 1 \Rightarrow x = 2 - 1 = 1$

So, line $y = x + 1$ is tangent to curve at point (1, 2)

The correct answer if A

Exercise 6.4

1. Using differentials, find the approximate value of each of the following up to 3 places of decimal

$$(i)\sqrt{25.3},(ii)\sqrt{49.5},(iii)\sqrt{0.6},(iv)(0.009)^{\frac{1}{3}},(v)(0.999)^{\frac{1}{10}},(vi)(15)^{\frac{1}{4}},(vii)(26)^{\frac{1}{3}}$$

$$(viii)(255)^{\frac{1}{4}},(ix)(82)^{\frac{1}{4}},(x)(401)^{\frac{1}{2}},(xi)(0.0037)^{\frac{1}{2}},(xii)(26.57)^{\frac{1}{3}},(xiii)(81.5)^{\frac{1}{4}},(xiv)(3.968)^{\frac{3}{2}}$$

$$(xv)(32.15)^{\frac{1}{5}}$$

Solution:

$$(i)\sqrt{25.3}$$

$$y=\sqrt{x}.\ Let\ x=25\ and\ \Delta x=0.3$$

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{25.3}-\sqrt{25}=\sqrt{25.3}-5$$

$$\Rightarrow\sqrt{25.3}=\Delta y+5$$

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{2\sqrt{x}}(0.3)$$

$$=\frac{2}{2\sqrt{25}}(0.3)=0.03$$

$$\sqrt{25.3}\approx 0.03+5=5.03$$

$$(ii)\sqrt{49.5}$$

$$y=\sqrt{x}.\ Let\ x=49\ and\ \Delta x=0.5$$

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{49.5}-\sqrt{49}=\sqrt{49.5}-7$$

$$\Rightarrow\sqrt{49.5}=7+\Delta y$$

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{2\sqrt{x}}(0.5)$$

$$=\frac{1}{2\sqrt{49}}(0.5)=\frac{1}{14}(0.5)=0.035$$

$$\sqrt{49.5}\approx 7+0.035=7.035$$

$$(iii)\sqrt{0.6}$$

$$y=\sqrt{x}.Let\ x=1\ and\ \Delta x=-0.4$$

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{0.6}-1$$

$$\Rightarrow \sqrt{0.6}=1+\Delta y$$

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{2\sqrt{x}}(\Delta x)$$

$$=\frac{1}{2}(-0.4)=-0.2$$

$$\sqrt{0.6}\approx 1+(-0.2)=1-0.2=0.8$$

$$(iv)(0.009)^{\frac{1}{3}}$$

$$y=x^{\frac{1}{3}}.Let x=0.008\ and\ \Delta x=0.001$$

$$\Delta y=(x+\Delta x)^{\frac{1}{3}}-(x)^{\frac{1}{3}}=(0.009)^{\frac{1}{3}}-(0.008)^{\frac{1}{3}}=(0.009)^{\frac{1}{3}}-0.2$$

$$\Rightarrow (0.009)^{\frac{1}{3}}=0.2+\Delta y$$

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{3(x)^{\frac{2}{3}}}(\Delta x)$$

$$=\frac{1}{3\times 0.04}(0.001)=\frac{0.001}{0.12}=0.008$$

$(0.009)^{\frac{1}{3}}$ *is* $0.2+0.008=0.208$

$(v)(0.999)^{\frac{1}{10}}$

$y=(x)^{\frac{1}{10}}$. Let $x=1$ and $\Delta x=-0.001$

$$\Delta y=(x+\Delta x)^{\frac{1}{10}}-(x)^{\frac{1}{10}}=(0.999)^{\frac{1}{10}}-1$$

$$\Rightarrow(0.999)^{\frac{1}{10}}=1+\Delta y$$

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{10(x)^{\frac{9}{10}}}(\Delta x)$$

$$=\frac{1}{10}(-0.001)=-0.0001$$

$(0.999)^{\frac{1}{10}}$ *is* $1+(-0.0001)=0.9999$

$(vi)(15)^{\frac{1}{4}}$

$y=x^{\frac{1}{4}}$. Let $x=16$ and $\Delta x=-1$

$$\Delta y=(x+\Delta x)^{\frac{1}{4}}-x^{\frac{1}{4}}=(15)^{\frac{1}{4}}-(16)^{\frac{1}{4}}=(15)^{\frac{1}{4}}-2$$

$$\Rightarrow(15)^{\frac{1}{4}}=2+\Delta y$$

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{4(x)^{\frac{3}{4}}}(\Delta x)$$

$$=\frac{1}{4(16)^{\frac{3}{4}}}(-1)=\frac{-1}{4\times 8}=\frac{-1}{32}=-0.03125$$

$(15)^{\frac{1}{4}}$ *is* $2+(-0.03125)=1.96875$

$(vii)(26)^{\frac{1}{3}}$

$y=(x)^{\frac{1}{3}}.Let\ x=27\ and\ \Delta x=-1$

$\Delta y=(x+\Delta x)^{\frac{1}{3}}-(x)^{\frac{1}{3}}=(26)^{\frac{1}{3}}-(27)^{\frac{1}{3}}=(26)^{\frac{1}{3}}-3$

$\Rightarrow(26)^{\frac{1}{3}}=3+\Delta y$

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{3(x)^{\frac{2}{3}}}(\Delta x)$$

$$=\frac{1}{3(27)^{\frac{2}{3}}}(-1)=\frac{-1}{27}=-0.0\overline{370}$$

$(26)^{\frac{1}{3}}\ is\ 3+(-0.0370)=2.9629$

$(viii)(255)^{\frac{1}{4}}$

$y=(x)^{\frac{1}{4}}.Let\ x=256\ and\ \Delta x=-1$

$\Delta y=(x+\Delta x)^{\frac{1}{4}}-(x)^{\frac{1}{4}}=(255)^{\frac{1}{4}}-(256)^{\frac{1}{4}}=(255)^{\frac{1}{4}}-4$

$\Rightarrow(255)^{\frac{1}{4}}=4+\Delta y$

$$dy\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{4(x)^{\frac{3}{4}}}(\Delta x)$$

$$=\frac{1}{4(256)^{\frac{3}{4}}}(-1)=\frac{-1}{4\times4^{3}}=-0.0039$$

$(255)^{\frac{1}{4}}\ is\ 4+(-0.0039)=3.9961$

$(ix)\ (82)^{\frac{1}{4}}$

$y = x^{\frac{1}{4}}. Let\ x = 81\ and\ \Delta x = 1$

$\Delta y = (x+\Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - (81)^{\frac{1}{4}} = (82)^{\frac{1}{4}} - 3$

$\Rightarrow (82)^{\frac{1}{4}} = \Delta y + 3$

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x)$

$= \frac{1}{4(81)^{\frac{3}{4}}}(1) = \frac{1}{4(3)^3} = \frac{1}{108} = 0.009$

$(82)^{\frac{1}{4}}\ is\ 3 + 0.009 = 3.009$

$(x)(401)^{\frac{1}{2}}$

$y = x^{\frac{1}{2}}. Let\ x = 400\ and\ \Delta x = 1$

$\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{401} - \sqrt{400} = \sqrt{401} - 20$

$\Rightarrow \sqrt{401} = 20 + \Delta y$

$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x)$

$= \frac{1}{2\times 20}(1) = \frac{1}{40} = 0.025$

$\sqrt{401}\ is\ 20 + 0.025 = 20.025$

$(xi)(0.037)^{\frac{1}{2}}$

$y = x^{\frac{1}{2}}. Let\ x = 0.0036\ and\ \Delta x = 0.0001$

$\Delta y = (x+\Delta x)^{\frac{1}{2}} - (x)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - (0.0036)^{\frac{1}{2}} = (0.0037)^{\frac{1}{2}} - 0.06$

$$\Rightarrow (0.0037)^{\frac{1}{2}} = 0.06 + \Delta y$$

$$dy\left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x)$$

$$= \frac{1}{2\times 0.06}(0.0001)$$

$$= \frac{0.0001}{0.12} = 0.00083$$

$$(0.0037)^{\frac{1}{2}}\ is\ 0.06 + 0.00083 = 0.6083$$

$$(xii)\ (26.57)^{\frac{1}{3}}$$

$$y = x^{\frac{1}{3}}.Let\ x = 27\ and\ \Delta x = -0.43$$

$$\Delta y = (x+\Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (26.57)^{\frac{1}{3}} - 3$$

$$\Rightarrow (26.57)^{\frac{1}{3}} = 3 + \Delta y$$

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x)$$

$$= \frac{1}{3(9)}(-0.43)$$

$$= \frac{-0.43}{27} = -0.015$$

$$(26.57)^{\frac{1}{3}}\ is\ 3 + (-0.015) = 2.984$$

$$(xiii)\ (81.5)^{\frac{1}{4}}$$

$$y = x^{\frac{1}{4}}.\ Let x = 81\ and\ \Delta x = 0.5$$

$$\Delta y=(x+\Delta x)^{\frac{1}{4}}-(x)^{\frac{1}{4}}=(81.5)^{\frac{1}{4}}-(81)^{\frac{1}{4}}=(81.5)^{\frac{1}{4}}-3$$

$$\Rightarrow (81.5)^{\frac{1}{4}}=3+\Delta y$$

$$dy=\left(\frac{dy}{dx}\right)\Delta x=\frac{1}{4(x)^{\frac{3}{4}}}(\Delta x)$$

$$=\frac{1}{4(3)^{3}}(0.5)=\frac{0.5}{108}=0.0046$$

$(81.5)^{\frac{1}{4}}$ *is* $3+0.0046=3.0046$

$(xiv)\ (3.968)^{\frac{3}{2}}$

$y=x^{\frac{3}{2}}$. *Let* $x=4$ *and* $\Delta x=-0.032$

$$\Delta y=(x+\Delta x)^{\frac{3}{2}}-x^{\frac{2}{3}}=(3.968)^{\frac{3}{2}}-(4)^{\frac{2}{3}}=(3.968)^{\frac{3}{2}}-8$$

$$\Rightarrow (3.968)^{\frac{3}{2}}=8+\Delta y$$

$$dy\left(\frac{dy}{dx}\right)\Delta x=\frac{3}{2}(x)^{\frac{1}{2}}(\Delta x)$$

$$=\frac{3}{2}(2)(-0.032)$$

$$=-0.096$$

$(3.968)^{\frac{3}{2}}$ *is* $8+(-0.096)=7.904$

$(xv)(32.15)^{\frac{1}{15}}$

$y=x^{\frac{1}{15}}$.*Let* $x=32$ *and* $\Delta x=-0.15$

$$\Delta y=(x+\Delta x)^{\frac{1}{5}}-x^{\frac{1}{5}}=(32.15)^{\frac{1}{5}}-(32)^{\frac{1}{5}}=(32.15)^{\frac{1}{5}}-2$$

$$\Rightarrow (32.15)^{\frac{1}{15}} = 2 + \Delta y$$

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{5(x)^{\frac{4}{5}}}.(\Delta x)$$

$$= \frac{1}{5\times(2)^4}(0.15)$$

$$= \frac{0.15}{80} = 0.00187$$

$$(32.15)^{\frac{1}{15}}\ is 2 + 0.00187 = 2.00187$$

2. Find the approximate value of (2.01), where $f(x) = 4x^2 + 5x + 2$

Solution:

$$x = 2\ and \Delta x = 0.01$$

$$f(2.01) = f(x+\Delta x) = 4(x+\Delta x)^2 + 5(x+\Delta x) + 2$$

$$\Delta y = f(x+\Delta x) - f(x)$$

$$f(x+\Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x)\Delta x$$

$$\Rightarrow f(2.01) \approx (4x^2 + 5x + 2) + (8x+5)\Delta x$$

$$= \left[4(2)^2 + 5(2) + 2\right] + \left[8(2) + 5\right](0.01)$$

$$= (16 + 10 + 2) + (16 + 5)(0.01)$$

$$= 28 + (21)(0.01)$$

$$= 28 + 0.21$$

$$= 28.21$$

$f(2.01)$ *is* 28.21

3. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 4x^2 + 15$

Solution:

$x = 5$ *and* $\Delta x = 0.001$

$f(5.001) = f(x + \Delta x) = (x + \Delta x)^3 - 7(x + \Delta x)^2 + 15$

$\Delta y = f(x + \Delta x) - f(x)$

$\therefore f(x + \Delta x) = f(x) + \Delta y$

$\approx f(x) + f'(x)\Delta x$

$\Rightarrow f(5.001) \approx (x^3 - 7x^2 + 15) + (3x^2 - 14x)\Delta x$

$= \left[(5)^3 - 7(5)^2 + 15\right] + \left[3(5)^2 - 14(5)\right](0.001)$

$= (125 - 175 + 15) + (75 - 70)(0.001)$

$= -35 + (5)(0.001)$

$= -35 + 0.005$

$= -34.995$

$f(5.001)$ *is* -34.995

4. Find the approximate change in the volume V of a cube side x meters caused by increasing side by 1%

Solution:

$V = x^3$

$\therefore dV = \left(\frac{dV}{dx}\right)\Delta x$

$$= \left(3x^2\right)\Delta x$$

$$= \left(3x^2\right)(0.01x)$$

$$= 0.03x^3$$

So, the approx change in the volume of the cube is $0.03x^3 m^3$

5. Find the approximate change in the surface area of a cube of side x meters caused by decreasing the side by 1%

Solution:

$$S = 6x^2$$

$$\therefore \frac{dS}{dx} = \left(\frac{dS}{dx}\right)\Delta x$$

$$= (12x)\Delta x$$

$$= (12x)(0.01x)$$

$$= 0.12x^2$$

So, the approx change in volume of cube is $0.12x^2 m^2$

6. If the radius of a sphere is measured as 7m with an error of 0.02m, then find the approximate error in calculating its volume

Solution:

$$r = 7m \; and \; \Delta r = 0.02m$$

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\therefore dV = \left(\frac{dV}{dr}\right)\Delta r$$

$$=\left(4\pi r^2\right)\Delta r$$

$$=4\pi(7)^2(0.02)m^3=3.92\pi m^3$$

So, the approx error in calculating the volume is $3.92\pi m^3$

7. If the radius of a sphere is measured as 9m with an error of 0.03m, then find the approximate error in calculating in surface area

Solution:

$$r=9m \text{ and } \Delta r=0.03m$$

$$\therefore \frac{dS}{dr}=\frac{d}{dr}\left(4\pi r^2\right)=8\pi r$$

$$\therefore dS=\left(\frac{dS}{dr}\right)\Delta r$$

$$=(8\pi r)\Delta r$$

$$=8\pi(9)(0.03)m^2$$

$$=2.16\pi m^2$$

So, the approx error in calculating the surface area is $2.16\pi m^2$

8. If $f(x)=3x^2+15x+5$, then the approximate value of (3.02) is

$(A)47.66,\ (B)57.66,\ (C)67.66\ \ (D)77.66$

Solution:

$$f(3.02)=f(x+\Delta x)=3(x+\Delta x)^2+15(x+\Delta x)+5$$

$$\Delta y=f(x+\Delta x)-f(x)$$

$$\Rightarrow f(x+\Delta x)=f(x)+\Delta y$$

$$\approx f(x)+f'(x)\Delta x$$

$$\Rightarrow f(3.02) \approx (3x^2 + 15x + 5) + (6x + 15)\Delta x$$

$$= \left[3(3^2) + 15(3) + 5\right] + \left[6(3) + 15\right](0.02)$$

$$= (27 + 45 + 5) + (18 + 15)(0.02)$$

$$= 77 + (33)(0.02)$$

$$= 77 + 0.66$$

$$= 77.66$$

So, approx value of (3.02) is 77.66

The correct answer is D

9. The approximate change in the volume of a cube of side x meters caused by increasing the side by 3% is

$(A) 0.06x^3m^3$ $(B) 0.6x^3m^3$ $(C) 0.09x^3m^3$ $(D) 0.9x^3m^3$

Solution:

$$V = x^3$$

$$\therefore dV = \left(\frac{dV}{dx}\right)\Delta x$$

$$= (3x^2)\Delta x$$

$$= (3x^2)(0.03x)$$

$$= 0.09x^3m^3$$

So, the approx change in the volume of the cube is $0.09x^3m^3$

The correct answer is C

Exercise 6.5

1. Find the maximum and minimum values, if any, of the following given by

$(i)\ f(x)=(2x-1)^2+3 \quad (ii)\ f(x)=9x^2+12x+2 \quad (iii)\ f(x)=-(x-1)^2+10$

$(iv)\ g(x)=x^3+1$

Solution:

$(i)\ f(x)=(2x-1)^2+3$

$(2x-1)^2 \geq 0$ for every $x \in R$

$f(x)=(2x-1)^2+3 \geq 3 \ for\ x \in R$

The minimum value of f occurs when $2x-1=0$

$2x-1=0, x=\frac{1}{2}$

Min value of $f\left(\frac{1}{2}\right)=\left(2.\frac{1}{2}-1\right)^2+3=3$

The function f does not have a maximum value

$(ii)\ f(x)=9x^2+12x+2=(3x^2+2)^2-2$

$(3x^2+2)^2 \geq 0\ for\ x \in R$

$f(x)=(3x^2+2)^2-2 \geq -2\ for\ x \in R$

Minimum value of f is when $3x+2=0$

$3x+2=0=0, x=\frac{-2}{3}$

Minimum value of $f\left(\frac{-2}{3}\right)=\left(3\left(\frac{-2}{3}\right)+2\right)^2-2=-2$

f does not have a maximum value

$(iii)\ f(x)=-(x-1)^2+10$

$(x-1)^2\geq 0\ for\ x\in R$

$f(x)=-(x-1)^2+10\leq 0\ for\ x\in R$

maximum value of f is when $(x-1)=0$

$(x-1)=0, x=0$

Maximum value of $f=f(1)=-(1-1)^2+10=10$

F does not have a minimum value

$(iv)\ g(x)=x^3+1$

g neither has a maximum value nor a minimum value.

2. Find the maximum and minimum values, if any, of the following functions given by

$(i) f(x)=|x+2|-1$ $\quad (ii) g(x)=-|x+1|+3$ $\quad (iii)\ h(x)=\sin(2x)+5$

$(iv)\ f(x)=|\sin 4x+3|$ $\quad (v)\ h(x)=x+4, x\in(-1,1)$

Solution:

$(i)\ f(x)=|x+2|-1$

$|x+2|\geq 0\ for\ x\in R$

$f(x)=|x+2|-1\geq -1\ for\ x\in R$

minimum value of f is when $|x+2|=0$

$|x+2|=0$

$\Rightarrow x=-2$

minimum value of $f = f(-2) = |-2+2|-1=1$

f does not have a maximum value

$(ii)\ g(x) = -|x+1|+3$

$-|x+1| \le 0 \ for\ x \in R$

$g(x) = -|x+1|+3 \le 3 \ for\ x \in R$

maximum value of g is when $|x+1|=0$

$|x+1|=0$

$\Rightarrow x=-1$

Maximum value of $g = g(-1) = -|1+1|+3=3$

g does not have a minimum value

$(iii)\ h(x) = \sin 2x+5$

$-1 \le \sin 2\mathrm{x} \le 1$

$-1+5 \le \sin 2\mathrm{x}+5 \le 1+5$

$4 \le \sin 2\mathrm{x}+5 \le 6$

maximum and minimum values of h are 6 and 4 respectively

$(iv) f(x) = |\sin 4x+3|$

$-1 \le \sin 4\mathrm{x} \le 1$

$2 \le \sin 4\mathrm{x}+3 \le 4$

$2 \le |\sin 4\mathrm{x}+3| \le 4$

maximum and minimum values of f are 4 and 2 respectively

$(v)\ h(x) = x+4, x \in (-1,1)$

Here, if a point x_0 is closest to – 1, then we find $\frac{x_0}{2}+1 < x_0+1$ for $x_0 \in (-1,1)$

Also if x_1 is closet to – 1, then we find $x_1+1 < \frac{x_1+1}{2}+1$ for all $x_0 \in (-1,1)$

Function has neither maximum nor minimum value in $(-1,1)$

3. Find the local maxima and minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be

$(i)\ f(x)=x^2 \quad (ii)\ g(x)=x^3-3x \quad (iii)\ h(x)=\sin x+\cos .0<x<\frac{\pi}{2}$

$(iv)\ f(x)=\sin x-\cos x, 0<x<2\pi \quad (v)\ f(x)=x^3-6x^2+9x+15 \quad (vi)\ g(x)=\frac{x}{2}+\frac{2}{x}, x>0$

$(vii)\ g(x)=\frac{1}{x^2+2} \quad (vii)\ f(x)=x\sqrt{1-x}, x>0$

Solution:

$(i)\ f(x)=x^2$

$\therefore f'(x)=2x$

$f'(x)=0 \Rightarrow x=0$

We have $f'(0)=2,$

by second derivative test, $x=0$ is a point of local minima and local minimum value of

f

at x= 0 is $f(0)=0$

$(ii)\ g(x)=x^3-3x$

$\therefore g'(x)=3x^2-3$

$\therefore g'(x)=0 \Rightarrow 3x^2=3 \Rightarrow x=\pm 1$

$g'(x)=6x$

$g'(1)=6>0$

$g'(-1)=-6>0$

By second derivative test, x = 1 is a point of local minima and local minimum value of g

At $x=1$ is $g(1)=1^3-3=1-3=-2$

x= -1 is a point of local maxima and local maximum value of g at

x = -1 is $g(1)=(-1)^3-3(-1)=-1+3=-2$

$(iii)\ h(x)=\sin x+\cos .0<x<\frac{\pi}{2}$

$\therefore h'(x)=\cos x+\sin x$

$\therefore h'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}\in\left(0,\frac{\pi}{2}\right)$

$h'(x)=\sin x-\cos x=-(\sin x+\cos x)$

$h'\left(\frac{\pi}{4}\right)=-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)=-\frac{2}{\sqrt{2}}=-\sqrt{2}<0$

Therefore, by second derivative test, $x=\frac{\pi}{4}$ is a point of local maxima and the local

Maximum value of h at $x=\frac{\pi}{4}$ is $h\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$

$(iv)\ f(x)=\sin x-\cos x, 0<x<2\pi$

$\therefore f'(x)=\cos x+\sin x$

$\therefore f'(x)=0 \Rightarrow \cos x=-\sin x \Rightarrow \tan x=-1 \Rightarrow x=\frac{3\pi}{4},\frac{7\pi}{4}\in(0,2\pi)$

$f''(x)=-\sin x+\cos x$

$$f''\left(\frac{3\pi}{4}\right)=-\sin\frac{3\pi}{4}+\cos\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\sqrt{2}>0$$

$$f''\left(\frac{7\pi}{4}\right)=-\sin\frac{7\pi}{4}+\cos\frac{7\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}>0$$

Therefore, by second derivative test, $x=\frac{3\pi}{4}$ is a point of local maxima and the local maximum value of f at $x=\frac{3\pi}{4}$ is

$$f\left(\frac{3\pi}{4}\right)=\sin\frac{3\pi}{4}+\cos\frac{3\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$$

$x=\frac{7\pi}{4}$ is a point of local minima and the local minimum value of f at $x=\frac{7\pi}{4}$ is

$$f\left(\frac{7\pi}{4}\right)=\sin\frac{7\pi}{4}-\cos\frac{7\pi}{4}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\sqrt{2}$$

$(v)\ f(x)=x^3-6x+9x+15$

$\therefore f'(x)=3x^2-12x+9$

$f(x)=0\Rightarrow 3(x^2-4x+3)=0$

$\Rightarrow 3(x-1)(x-3)=0$

$\Rightarrow x=1,3$

$f''(x)=6x-12=6(x-2)$

$f''(1)=6(1-2)=-6<0$

$f''(3)=6(3-2)=6<0$

By second derivative test, x = 1 is a point of local maxima and the local maximum value of f at x= 1 is $f(1)=1-6+9+15=19$

x =3 is a point of local minima and the local minimum value of f at x = 3 is $f(3)=27-54+27+15=15$

$$(vi)\ g(x)=\frac{x}{2}+\frac{2}{x}, x>0$$

$$\therefore g'(x)=\frac{1}{2}-\frac{2}{x^2}$$

$$\therefore g'(x)=0 \Rightarrow \frac{2}{x^2}=\frac{1}{2} \Rightarrow x^3=4 \Rightarrow x=\pm$$

$$x>0, x=2$$

$$g''(x)=\frac{4}{x^3}$$

$$g''(2)=\frac{4}{2^3}=\frac{1}{2}>0$$

By second derivative test, x = 2 is a point of local minima and the local minimum value of g at x = 2 is $g(2)=\frac{2}{2}+\frac{2}{2}=1+1=2$

$$(vii)\ g(x)=\frac{1}{x^2+2}$$

$$\therefore g'(x)=\frac{-(2x)}{(x^3+2)^2}$$

$$g'(x)=0 \Rightarrow \frac{-2x}{(x^3+2)^2}=0 \Rightarrow x=0$$

For value close to x = 0 and left of 0, $g^1(x)>0$

For value close to x= 0 and to right of 0 $g^1(x)<0$

By first derivative test x = 0 is a point of local maxima and the local maximum value of g(0) is $\frac{1}{0+2}=\frac{1}{2}$

$(viii)\ f(x)=x\sqrt{1-x}, x>0$

$$\therefore f'(x)=x\sqrt{1-x}+x.\frac{1}{2\sqrt{1-x}}(-1)=\sqrt{1-x}-\frac{x}{2\sqrt{1-x}}$$

$$=\frac{2(1-x)-x}{2\sqrt{1-x}}-\frac{2-3x}{2\sqrt{1-x}}$$

$$f'(x)=0\Rightarrow\frac{2-3x}{2\sqrt{1-x}}=0\Rightarrow 2-3x=0\Rightarrow x=\frac{2}{3}$$

$$f''(x)=\frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$

$$=\frac{\sqrt{1-x}(-3)+2(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$=\frac{-6(1-x)+2(2-3x)}{4(1-x)^{\frac{3}{2}}}$$

$$=\frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f'\left(\frac{2}{3}\right)=\frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}}=\frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}}=\frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}}<0$$

By second derivative test, $x=\frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x=\frac{2}{3}$ is

$$f\left(\frac{2}{3}\right)=\frac{2}{3}\sqrt{1-\frac{2}{3}}=\frac{2}{3}\sqrt{\frac{1}{3}}=\frac{2}{3\sqrt{3}}=\frac{2\sqrt{3}}{9}$$

4. Prove that the following functions do not have maxima or minima

$(i)\ f(x)=e^x \quad (ii)\ g(x)=\log x \quad (iii)\ h(x)=x^3+x^2+x+1$

Solution:

$(i)\ f(x)=e^x$

$\therefore f'(x)=e^x$

If $f'(x)=0, e^x=0$. But exponential function can never be 0 for any value of x

There is no $c\in R$ such that $f'(c)=0$

f does not have maxima or minima

(ii) We have, $g(x)=\log x$

$\therefore g'(x)=\frac{1}{x}$

$\log x$ is defined for positive x, $g'(x)>0$ for any x

there does not exist $c\in R$ such that $g'(c)=0$

function g does not have maxima or minima

(iii) We have, $h(x)=x^3+x^2+x+1$

$\therefore h'(x)=3x^2+2x+1$

there does not exist $c\in R$ such that $h'(c)=0$

function h does not have maxima or minima

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals

$(i)\ f(x)=x^3, x\in[-2,2] \qquad (ii)\ f(x)=\sin x+\cos x, x\in[0,\pi]$

$(iii)\ f(x)=4x-\frac{1}{2}x^2, x\in\left[-2,\frac{9}{2}\right] \qquad (iv)\ f(x)=(x-1)^2+3, x\in[-3,1]$

Solution:

$(i)\ f(x)=x^3$

$\therefore f'(x)=3x^2$

$f'(x)=0 \Rightarrow x=0$

$f(0)=0$

$f(-2)=(-2)^3=-8$

$f(2)=(2)^3=8$

Hence, the absolute maximum of f on $[-2,2]$ is 8 at $x=-2$

The absolute minimum of f on $[-2,2]$ is -8 at $x=-2$

$(ii)\ f(x)=\sin x+\cos x$

$\therefore f'(x)=\cos x-\sin x$

$f'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$

$f(0)=\sin 0+\cos 0=0+1=1$

$f(\pi)=\sin\pi+\cos\pi=0-1=1$

The absolute maximum of f on $[0,\pi]$ is $\sqrt{2}$ at $x=\frac{\pi}{4}$

The absolute minimum of f on $[0,\pi]$ is -1 at $x=\pi$

$(iii)\ f(x)=4x-\frac{1}{2}x^2$

$\therefore f'(x)=4x-\frac{1}{2}(2x)=4-x$

$\therefore f'(x)=0 \Rightarrow x=4$

$f(4)=16-\frac{1}{2}(16)=16-8=8$

$f(-2)=-8-\frac{1}{2}(4)=-8-2=-10$

$f\left(\frac{9}{2}\right)=4\left(\frac{9}{2}\right)-\frac{1}{2}\left(\frac{9}{2}\right)^2=18-\frac{81}{8}=18-10.125=7.875$

The absolute maximum of f on $\left[-2,\frac{9}{2}\right]$ is 8 at x = 4

The absolute minimum of f on $\left[-2,\frac{9}{2}\right]$ is -10 at x = - 2

$(iv)\ f(x)=(x-1)^2+3$

$\therefore f'(x)=2(x-1)$

$f'(x)=0 \Rightarrow 2(x-1)=0, x=1$

$f(1)=(1-1)^2+3=0+3=3$

$f(-3)=(-3-1)^2+3=16+3=19$

Absolute maximum value of f on $[-3,1]$ is 19 at x= - 3

Minimum value of f on $[-3,1]$ is at x = 1

6. Find the maximum profit that a company can make, if the profit function is given by $p(x)=41-24x-18x^2$

Solution:

$p(x)=41-24x-18x^2$

$\therefore p'(x)=-24-36x$

$$p''(x) = -36$$

$$p'(x) = 0 \Rightarrow \frac{-24}{36} = -\frac{2}{3}$$

$$p'\left(\frac{-2}{3}\right) = -36 < 0$$

By second derivative test, $x = -\frac{2}{3}$ is the point of local maximum of p

$\therefore$ Maximum profit $= p = \left(-\frac{2}{3}\right)$

$$= 41 - 24\left(-\frac{2}{3}\right) - 18\left(-\frac{2}{3}\right)^2$$

$$= 41 + 16 - 8$$

$$= 49$$

7. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is

(i) Increasing (ii) Decreasing

Solution:

$$f(x) = x^3 + \frac{1}{x^3}$$

$$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

$$f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = 1 \Rightarrow x \pm 1$$

In $(-\infty, -1)$ and $(1, \infty)$, $f'(x) > 0$

when $x < -1$ *and* $x > 1$, f is increasing

In $(-1,1)$, $f(x) < 0$

when $-1 < x < 1$, f is decreasing

8. At which point in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain, its maximum value?

Solution:

$$\mathrm{f}(x) = \sin 2x$$

$$\therefore f'(x) = 2\cos 2x$$

$$f'(x) = 0 \Rightarrow \cos 2x = 0$$

$$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1, \mathrm{f}\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = -1$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{2\pi}{2} = 1, \mathrm{f}\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{2} = -1$$

$$f(0) = \sin 0 = 0,\ f(2\pi) = \sin 2\pi = 0$$

Absolute maximum value of $f[0, 2\pi]$ is at $x = \frac{\pi}{4}$ *and* $x = \frac{5\pi}{4}$

9. What is the maximum value of the function $\sin x + \cos x$?

Solution:

$$\mathrm{f}(x) = \sin x + \cos x$$

$$\therefore f'(x) = \cos x - \sin x$$

$$f'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \ldots\ldots,$$

$$f'(x) = -\sin x - \cos x = -(\sin x + \cos x)$$

$f''(x)$ will be negative when $(\sin x + \cos x)$ is positive

We know that sin x and cos x are positive in the first quad5rant

$f''(x)$ will be negative when $x \in \left(0, \frac{\pi}{2}\right)$

Consider $x = \frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = -\left(\frac{2}{\sqrt{2}}\right) = -\sqrt{2} < 0$$

By second derivative test, f will be that maximum at $x = \frac{\pi}{4}$ and the maximum value of f is

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in $[-3, -1]$

Solution:

$$f(x) = 2x^3 - 24x + 107$$

$$f'(x) = 6x^2 - 24 = 6(x^2 - 4)$$

$$f'(x) = 0 \Rightarrow 6(x^2 - 4) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

Consider $[1, 3]$

$$f(2) = 2(8) - 24(2) + 107 = 16 - 48 + 107 = 75$$

$$f(1) = 2(1) - 24(1) + 107 = 2 - 24 + 107 = 85$$

$$f(3) = 2(27) - 24(3) + 107 = 54 - 72 + 107 = 89$$

Absolute maximum of $f(x)$ in the $[1, 3]$ is 89 at x = 3

Consider $[-3, -1]$

$$f(-3) = 2(-27) - 24(-3) + 107 = 54 + 72 + 107 = 125$$

$$f(-1) = 2(-1) - 24(-1) + 107 = 2 + 24 + 107 = 129$$

$$f(-2) = 2(-8) - 24(-2) + 107 = -16 + 48 + 107 = 139$$

Absolute maximum of $f(x)$ in $[-3, -1]$ is 139 at x= - 2

11. It is given that at x=1, the function x^4-62x^2+ax+9 attains its maximum value, on the interval $[0,2]$. Find the value of a

Solution:

$$f(x)=x^4-62x^2+ax+9$$

$$\therefore f'(x)=4x^2-124x+a$$

$$\therefore f'(1)=0$$

$$\Rightarrow 4-124+a=0$$

$$\Rightarrow a=120$$

The value of a is 120

12. Find the maximum and minimum values of $x+\sin 2x$ on $[0,2\pi]$

Solution:

$$f(x)=x+\sin 2x$$

$$\therefore f'(x)=1+2\cos 2x$$

$$f'(x)=0\Rightarrow \cos 2x=-\frac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\frac{2\pi}{3}$$

$$2x=2\pi\pm\frac{2\pi}{3}n\in Z$$

$$\Rightarrow x=\pi\pm\frac{\pi}{3},n\in Z$$

$$\Rightarrow x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}\in[0,2\pi]$$

$$f\left(\frac{\pi}{3}\right)=\frac{\pi}{3}+\sin\frac{2\pi}{3}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}+\sin\frac{4\pi}{3}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right)=\frac{4\pi}{3}+\sin\frac{8\pi}{3}=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sin\frac{10\pi}{3}=\frac{5\pi}{3}-\frac{\sqrt{3}}{2}$$

$$f(0)=0+\sin 0=0$$

$$f(2\pi)=2\pi+\sin 4\pi=2\pi+0=2\pi$$

Absolute maximum value of $f(x)$ in $[0,2\pi]$ is at $x=2\pi$

Absolute minimum value of $f(x)$ in $[0,2\pi]$ is at x = 0

13. Find two numbers whose sum is 24 and whose product is as large as possible

Solution:

Let number be x

The other number is $(24-x)$

$p(x)$ denote the product of the two numbers

$$P(x)=x(24-x)=24x-x^2$$

$$\therefore P'(x)=24-2x$$

$$P''(x)=-2$$

$$P'(x)=0 \Rightarrow x=12$$

$$P''(12)=-2<0$$

x =12 is point of local maxima of P

Product of the numbers is the maximum when numbers are 12 and $24-12=12$

14. Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum

Solution:

Numbers are x and y such that $x+y=60$

$y=60-x$

$f(x)=xy^3$

$\Rightarrow f(x)=x(60-x)^3$

$\therefore f'(x)=(60+x)^3-3x(60-x)^2$

$=(60+x)^3[60-x-3x]$

$=(60+x)^3[60-4x]$

$f''(x)=-2(60-x)(60-4x)-4(60-x)^2$

$=-2(60-x)[60-4x+2(60-x)]$

$=-2(60-x)(180-6x)$

$=-12(60-x)(30-x)$

$f'(x)=0 \Rightarrow x=60 \; or \; x=15$

$x=60, \; f''(x)=0$

$x=15, f''(x)=-12(60-15)(30-15)=12\times45\times15<0$

x =15 is a point of local maxima of f

function xy^3 is maximum when $x=15 \; and \; y=60-15=45$

required numbers are 15 and 45

15. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum

Solution:

One number be x

Other number is $y=(35-x)$

$p(x)=x^2y^5$

$P(x)=x^2(35-x)^5$

$\therefore P'(x)=2x(35-x)^5-5x^2(35-x)^4$

$=x(35-x)^4[2(35-x)-5x]$

$=x(35-x)^4(70-7x)$

$=7x(35-x)^4(10-x)$

And $P''(x)=7(35-x)^4(10-x)+7x\left[-(35-5)^4-4(35-x)^3(10-x)\right]$

$=7(35-x)^4(10-x)-7x(35-x)^4-28x(35-x)^3(10-x)$

$=7(35-x)^3\left[(35-x)(10-x)-x(35-x)-4x(10-x)\right]$

$=7(35-x)^3\left[350-45x+x^2-35x+x^2-40x+4x^2\right]$

$=7(35-x)^3(6x^2-120x+350)$

$P'(x)=0\Rightarrow x=0, x=35, x=10$

$x=35, f'(x)=f(x)=0 \text{ and } y=35-35=0$

$x=0, y=35-0=34$ and product x^2y^2 will be 0

$x=0 \text{ and } x=35$ cannot be the possible values of x

$x = 10,$

$$P''(x) = 7(35-10)^3(6\times100-120\times10+350)$$

$$= 7(25)^3(-250) < 0$$

$P(x)$ will be the maximum when $x = 10$ *and* $y = 35-10 = 25$

The numbers are 10 and 25

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum

Solution:

One number be x

The other number is $(16-x)$

Sum of cubes of these numbers be denote by $S(x)$

$$\therefore S'(x) = 3x^2 - 3(16-x)^2, S''(x) = 6x + 6(16-x)$$

$$S'(x) = 0 \Rightarrow 3x^2 - 3(16-x^2) = 0$$

$$\Rightarrow x^2 - (16-x)^2 = 0$$

$$\Rightarrow x^2 - 256 - x^2 + 32x = 0$$

$$\Rightarrow x = \frac{256}{32} = 8$$

$$S''(8) = 6(8) + 6(16-8) = 48 + 48 = 96 > 0$$

By second derivative test, x = 8 is point of local minima of S.

Sum of the cubes of the numbers is minimum when the numbers are 8 and 16 – 8 = 8

17. A square piece of tin od side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution:

Side of the square to be cut off be x cm

Length and breath of the box will be $(18-2x)cm$ each and the height of the box is x cm.

$V(x)=x(18-2x)^2$

$\therefore V'(x)=(18-2x)^2-4x(18-2x)$

$=(18-2x)[18-2x-4x]$

$=(18-2x)(18-6x)$

$=6\times 2(9-x)(3-x)$

$=12(9-x)(3-x)$

$V''(x)=12[-(9-x)-(3-x)]$

$=-12(9-x+3-x)$

$=-12(12-2x)$

$=-24(6-x)$

$v'(x)=0\Rightarrow x=9 \; or \; x=3$

$x=9,$ then the length and the breadth will become 0

$\therefore x\neq 9$

$\Rightarrow x=3$

$V''(3) = -24(6-3) = -72 < 0$

$\therefore$ By second derivative test, x = 3 is the point of maxima of V

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution:

Side of the square to be cut be x cm

Height of the box is x, the length is $45-2x$,

Breadth is $24-2x$

$V(x) = x(45-2x)(24-2x)$

$= x(1080 - 90x - 48x + 4x^2)$

$= 4x^3 - 138x^2 + 1080x$

$\therefore V'(x) = 12x^2 - 276x + 1080$

$= 12(x^2 - 23x + 90)$

$= 12(x-18)(x-5)$

$V''(x) = 24x - 276 = 12(2x-23)$

$V'(x) = 0 \Rightarrow x = 18 \text{ and } x = 5$

Not possible to cut a square of side 18cm from each corner of rectangular sheet, x cannot be equal to 18

$x = 15$

$V''(5) = 12(10-23) = 12(-13) = -156 < 0$

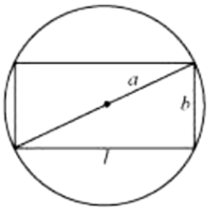
x = 5 is the point of maxima

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area

Solution:

A rectangle of length l and breadth b be inscribed in the given circle of radius a.

The diagonal passes through the centre and is of length 2a cm



$$(2a)^2 = l^2 + b^2$$

$$\Rightarrow b^2 = 4a^2 - l^2$$

$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

$$A = l\sqrt{4a^2 - l^2}$$

$$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l) = \sqrt{4a^2 - l^2} - \frac{1}{\sqrt{4a^2 - l^2}}$$

$$= \frac{4a^2 - l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)}$$

$$= \frac{(4a^2 - l^2)(-4l) + 1(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-12a^2l + 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}} = \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$\frac{dA}{dl}=0 \; gives \; 4a^2=2l^2 \Rightarrow l=\sqrt{2}a$$

$$\Rightarrow b=\sqrt{4a^2-2a^2}=\sqrt{2a^2}=\sqrt{2}a$$

when $l=\sqrt{2}a$,

$$\frac{d^2A}{dl^2}=\frac{-2\left(\sqrt{2}a\right)\left(6a^2-2a^2\right)}{2\sqrt{2a^3}}=\frac{-8\sqrt{2a^3}}{2\sqrt{2a^3}}=-4<0$$

when $l=\sqrt{2}a$, then area of rectangle is maximum

Since $l=b=\sqrt{2}a$, rectangle is a square

20. Show that the right circular cylinder of given surface and maximum volume is such that is heights is equal to the diameter of the base

Solution:

$$S=2\pi r^2+2\pi rh$$

$$\Rightarrow h=\frac{S-2\pi r^2}{2\pi r}$$

$$=\frac{S}{2\pi}\left(\frac{1}{r}\right)-r$$

$$V=\pi r^2 h=\pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right)-r\right]=\frac{Sr}{2}=-\pi r^3$$

$$\frac{dV}{dr}=\frac{S}{2}-3\pi r^2, \frac{d^2V}{dr^2}=-6\pi r$$

$$\frac{dV}{dr}=0 \Rightarrow \frac{S}{2}=3\pi r^2 \Rightarrow r^2=\frac{S}{6\pi}$$

$$r^2=\frac{S}{6\pi}, \frac{d^2V}{dr^2}=-6\pi\left(\sqrt{\frac{S}{6\pi}}\right)<0$$

Volume is maximum when $r^2=\dfrac{S}{6\pi}$

When $r^2 = \dfrac{S}{6\pi}$, then $h = \dfrac{6\pi r^2}{2\pi}\left(\dfrac{1}{r}\right) - r = 3r - r = 2r$

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area?

Solution:

$$V = \pi r^2 h = 100$$

$$\therefore h = \frac{100}{\pi r^2}$$

$$S = 2\pi^2 + 2\pi rh = 2\pi r^2 + \frac{200}{r}$$

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

When $r = \left(\dfrac{50}{\pi}\right)^{\frac{1}{3}}, \dfrac{d^2S}{dr^2} > 0$

The surface area is the minimum when the radius of the cylinder is $\left(\dfrac{50}{\pi}\right)^{\frac{1}{3}} cm$

$$r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, h = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm$$

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the circle is minimum?

Solution:

Piece of length l be cut from wire to make square

Other piece of wire to be made into circle is $(28-l)m$

Side of square = $\frac{l}{4}$

$$2\pi r = 28-l \Rightarrow r = \frac{1}{2\pi}(28-l)$$

$$A = \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28-l)\right]^2$$

$$= \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$

$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28-l)$$

$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

$$\frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{l}{2\pi}(28-l) = 0$$

$$\Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0$$

$$(\pi+4)l - 112 = 0$$

$$\Rightarrow l = \frac{112}{\pi+4}$$

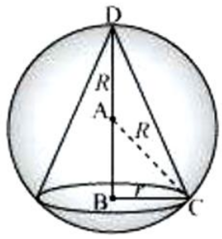
When $l = \frac{112}{\pi+4}, \frac{d^2A}{dl^2} > 0$

The area is minimum when $l = \frac{112}{\pi+4}$

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere

Solution:

Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.



$$V=\frac{1}{3}\pi r^2 h$$

$$h=R+AB=R+\sqrt{R^2-r^2}$$

$$\therefore V=\frac{1}{3}\pi r^2\left(R+\sqrt{R^2-r^2}\right)$$

$$=\frac{1}{3}\pi r^2 R+\frac{1}{3}\pi r^2\sqrt{R^2+r^2}$$

$$\therefore \frac{dV}{dr}=\frac{2}{3}\pi rR+\frac{2}{3}\pi r\sqrt{R^2-r^2}+\frac{1}{3}\pi r^2\frac{(-2r)}{2\sqrt{R^2-r^2}}$$

$$=\frac{2}{3}\pi rR+\frac{2}{3}\pi r\sqrt{R^2-r^2}-\frac{1}{3}\pi\frac{r^3}{\sqrt{R^2-r^2}}$$

$$=\frac{2}{3}\pi rR+\frac{2\pi r\left(R^2-r^2\right)-\pi r^3}{3\sqrt{R^2-r^2}}$$

$$=\frac{2}{3}\pi rR+\frac{2\pi rR^2-3\pi r^3}{3\sqrt{R^2-r^2}}$$

$$\frac{d^2V}{dr^2}=\frac{2\pi R}{3}+\frac{3\sqrt{R^2-r^2}\left(2\pi R^2-9\pi r^2\right)-\left(2\pi rR^2-3\pi r^3\right)\frac{(-2r)}{6\sqrt{R^2-r^2}}}{9\left(R^2-r^2\right)}$$

$$=\frac{2}{3}\pi rR+\frac{9\left(R^2-r^2\right)\left(2\pi R^2-9\pi r^2\right)+2\pi r^2R^2+3\pi r^4}{27\left(R^2-r^2\right)^{\frac{3}{2}}}$$

$$\frac{dV}{dr}=0\Rightarrow \pi\frac{2}{3}rR=\frac{3\pi r^3-2\pi R^2}{3\sqrt{R^2-r^2}}$$

$$\Rightarrow 2R=\frac{3\pi r^3-2\pi R^2}{\sqrt{R^2-r^2}}=2R\sqrt{R^2-r^2}=3r^2-2R^2$$

$$\Rightarrow 4R^2\left(R^2-r^2\right)=\left(3r^2-2R^2\right)^2$$

$$\Rightarrow 4R^4-4R^2r^2=9r^4+4R^4-12r^2R^2$$

$$\Rightarrow 9r^4=8R^2r^2$$

$$\Rightarrow r^2=\frac{8}{9}R^2$$

$$r^2=\frac{8}{9}R^2,\frac{d^2V}{dr^2}<0$$

Volume of the cone is the maximum when $r^2=\frac{8}{9}R^2$

$$r^2=\frac{8}{9}R^2,h=R+\sqrt{R^2-\frac{8}{9}R^2}=R+\sqrt{\frac{1}{9}R^2}=R+\frac{R}{3}=\frac{4}{3}R$$

$$=\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$$

$$=\frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$=\frac{8}{27}\times\left(volume\ of\ the\ sphere\right)$$

24. Show that he right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base

Solution:

$$V=\frac{1}{3\pi}\pi r^2 h \Rightarrow h=\frac{3V}{r^2}$$

$$S=\pi rl$$

$$=\pi r\sqrt{r^2+h^2}$$

$$=\pi r\sqrt{r^2+\frac{9V^2}{\pi^2 r^4}}\pi=\frac{r\sqrt{9^2 r^6+V^2}}{\pi r^2}$$

$$=\frac{1}{r}\sqrt{\pi^2 r^6+9V^2}$$

$$\therefore \frac{dS}{dr}=\frac{r.\frac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6+9V^2}}-\sqrt{\pi^2 r^6+9V^2}}{r^2}$$

$$=\frac{3\pi^2 r^6-\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

$$=\frac{2\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

$$\frac{dS}{dr}=0\Rightarrow 2\pi^2 r^6=9V^2\Rightarrow r^6=\frac{9V^2}{2\pi^2}$$

$$r^6=\frac{9V^2}{2\pi^2},\frac{d^2S}{dr^2}>0$$

Surface area of the cone is least when $r^6=\frac{9V^2}{2\pi^2}$

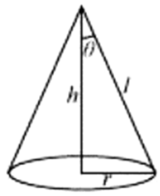
$$r^6=\frac{9V^2}{2\pi^2},h=\frac{3V}{\pi r^2}=\frac{3V}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}}=\frac{3}{\pi r^2}\frac{\sqrt{2\pi r^3}}{3}=\sqrt{2}r$$

25. Show that the semi – vertical angle of the cone of the maximum volume and of the given slant height is $\tan^{-1}\sqrt{2}$

Solution:

Let θ be semi – vertical angle of cone

$\theta \in \left[0, \frac{\pi}{2}\right]$



$r = l \sin\theta$ *and* $h = l \cos\theta$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(l^2 \sin^2\theta\right)\left(l\cos\theta\right)$$

$$= \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

$$\therefore \frac{dV}{d\theta} = \frac{l^3\pi}{3}\left[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)\right]$$

$$= \frac{l^3\pi}{3}\left[-\sin^3 + 2\sin\theta\cos^2\theta\right]$$

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta\right]$$

$$= \frac{l^3\pi}{3}\left[2\cos^2\theta - 7\sin^2\theta\cos\theta\right]$$

$$\frac{dV}{d\theta} = 0$$

$$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta$$

$$\Rightarrow \tan^2\theta = 2$$

$$\Rightarrow \tan\theta = \sqrt{2}$$

$$\Rightarrow \theta = \tan^{-1}\sqrt{2}$$

When $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2 \; or \; \sin^2\theta = 2\cos^2\theta$

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[2\cos^3\theta - 14\cos^3\theta\right] = -4\pi l^3\cos^3\theta < 0 \;\; for \; \theta \in \left[0, \frac{\pi}{2}\right]$$

Volume is the maximum when $\theta = \tan^{-1}\sqrt{2}$

26. The point on the curve $x^2 = 2y$ which is nearest to the point $(0,5)$ is

$(A)\left(2\sqrt{2},4\right),(B)\left(2\sqrt{2},0\right),(C)=(0,0),(D)=(2,2)$

Solution:

Position of point is $\left(x, \frac{x^2}{2}\right)$

Distance $d(x)$ between points $\left(x, \frac{x^2}{2}\right)$ and $(0,5)$ is

$$d(x) = \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 5\right)^2} = \sqrt{x^2 + \frac{x^4}{4} + 25 - 5x^2} = \sqrt{\frac{x^4}{4} - 4x^2 + 25}$$

$$\therefore d'(x) = \frac{\left(x^3 - 8x\right)}{2\sqrt{\frac{x^4}{4} - 4x^2 + 25}} = \frac{\left(x^3 - 8x\right)}{\sqrt{x^4 - 16x^2 + 100}}$$

$$d'(x) = 0 \Rightarrow x^3 - 8x = 0$$

$$\Rightarrow x\left(x^2 - 8\right) = 0$$

$$\Rightarrow x = 0, \pm 2\sqrt{2}$$

$$d''(x) = \frac{\sqrt{x^2 - 16x^2 + 100}\left(3x^2 - 8\right) - \left(x^3 - 8x\right)\frac{4x^3 - 32x}{2\sqrt{x^4 - 16x^2 + 100}}}{\left(x^2 - 16x^2 + 100\right)}$$

$$= \frac{\left(x^4 - 16x^2 + 100\right)\left(3x^2 - 8\right) - 2\left(x^3 - 8x\right)\left(x^3 - 8x\right)}{\left(x^2 - 16x^2 + 100\right)^{\frac{3}{2}}}$$

$$=\frac{\left(x^4-16x^2+100\right)\left(3x^2-8\right)-2\left(x^3-8x\right)^2}{\left(x^2-16x^2+100\right)^{\frac{3}{2}}}$$

$$x=0, then\ d''(x)=\frac{36(-8)}{6^3}<0$$

$$x=\pm2\sqrt{2}, d''(x)>0$$

d(x) is the minimum at $x=\pm2\sqrt{2}$

$$x=\pm2\sqrt{2}, y=\frac{\left(2\sqrt{2}\right)^2}{2}=4$$

The correct answer is A

27. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

$(A)0, (B)1, (C)3, (D)\frac{1}{3}$

Solution:

$$f(x)=\frac{1-x+x^2}{1+x+x^2}$$

$$\therefore f'(x)=\frac{\left(1-x+x^2\right)(-1+2x)-\left(1-x+x^2\right)(1+2x)}{\left(1+x+x^2\right)^2}$$

$$=\frac{-1+2x-x+2x^2-x^2+2x^2-1-2x+x+2x^2-x^2-2x^3}{\left(1+x+x^2\right)^2}$$

$$=\frac{2x^2-2}{\left(1+x+x^2\right)^2}=\frac{2\left(x^2-1\right)}{\left(1+x+x^2\right)^2}$$

$$\therefore f'(x)=0\Rightarrow x^2=1\Rightarrow x=\pm1$$

$$f''(x)=\frac{2\left[\left(1+x+x^2\right)(2x)-\left(x^2-1\right)(2)\left(1+x+x^2\right)(1+2x)\right]}{\left(1+x+x^2\right)^4}$$

$$=\frac{4\left(1+x+x^2\right)\left[\left(1+x+x^2\right)x-\left(x^2-1\right)\left(1+2x\right)\right]}{\left(1+x+x^2\right)^4}$$

$$=4\frac{\left[x+x^2+x^3-x^2-2x^3+1+2x\right]}{\left(1+x+x^2\right)^3}$$

$$=\frac{4\left(1+x+x^2\right)^3}{\left(1+x+x^2\right)^3}$$

$$f''(1)=\frac{4(1+3-1)}{(1+1+1)^3}=\frac{4(3)}{(3)^3}=\frac{4}{9}>0$$

$$f''(-1)=\frac{4(1+3-1)}{(1+1+1)^3}=4(-1)=4<0$$

f is the minimum at x = 1 and the minimum value is given by

$$f(1)=\frac{1-1+1}{1+1+1}=\frac{1}{3}$$

The correct answer is D

28. The maximum value of $\left[x(x+1)+1\right]^{\frac{1}{3}}, 0\le x\le 1$ is

$(A)\left(\frac{1}{3}\right)^{\frac{1}{3}}, (B)\frac{1}{2}, (C)1, (D)0$

Solution:

$$f(x)=\left[x(x+1)+1\right]^{\frac{1}{3}}$$

$$\therefore f'(x)=\frac{2x-1}{3\left[x(x+1)+1\right]^{\frac{2}{3}}}$$

$$f'(x)=0\Rightarrow x=\frac{1}{2}$$

$$f(0)=\left[0(0-1)+1\right]^{\frac{1}{3}}=1$$

$$f(2)=[1(1-1)+1]^{\frac{1}{3}}=1$$

$$f\left(\frac{1}{2}\right)=\left[\frac{1}{2}\left(\frac{-1}{2}\right)+1\right]^{\frac{1}{3}}=\left(\frac{3}{4}\right)^{\frac{1}{3}}$$

Maximum value of f in $[0,1]$ is 1

The correct answer is C