Chapter: 7. Integrals

Exercise: Miscellaneous.

1. Find the integral of the function $\int \frac{1}{x-x^3}dx$

Solution: The given integrand is $\frac{1}{x-x^3}$

The given integrand can be written as

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}$$
$$=\frac{1}{x(1-x)(1+x)}$$

Use the concept of partial fractions, suppose that $\frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x}$

To get the values of constants, cancel out the common denominator.

Hence, $1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$

Plug in $x=1$

$$1=B(2)\Rightarrow B=\frac{1}{2}$$

Plug in $x=-1$

$$1=-C(2)\Rightarrow C=-\frac{1}{2}$$

Plug in $x=0$

$$A=1$$

Hence,

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

Integrate both sides

$$\int \frac{1}{x(1-x)(1+x)}dx = \int \frac{1}{x}dx + \int \frac{1}{2(1-x)}dx - \int \frac{1}{2(1+x)}dx$$

$$\int \frac{1}{x-x^3}dx = \log|x| - \frac{1}{2}\log|1-x| - \frac{1}{2}\log|1+x| + C$$

$$= \frac{1}{2}\log\left|x^2\right| - \frac{1}{2}\log\left|1-x^2\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$$

Therefore, $\int \frac{1}{x-x^3}dx = \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$

2. Find the integral of the function $\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}$

Solution:

$$\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}} = \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+b)-(x-b)} = \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow \int \frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}dx = \frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$= \frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right] = \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$$

3. $\frac{1}{x\sqrt{ax-x^2}}$ $\left[\text{Hint } x = \frac{a}{t}\right]$

Solution:

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t}\Rightarrow dx=-\frac{a}{t^2}dh$

$$\Rightarrow\int\frac{1}{x\sqrt{ax-x^2}}dx=\int\frac{1}{\frac{a}{t}\sqrt{a.\frac{a}{t}-\left(\frac{a}{t}\right)^2}}-\left(-\frac{a}{t^2}dt\right)$$

$$=-\int\frac{1}{at}\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt=-\frac{1}{a}\int\frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$=-\frac{1}{a}\int\frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$=-\frac{2}{a}\left[\frac{\sqrt{a-x}}{\sqrt{x}}\right]+C$$

$$=-\frac{2}{a}\left[\sqrt{\frac{a-x}{x}}\right]+C$$

4. $\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$

Solution:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we get

$$\frac{x^{-3}}{x^2.x^{-3}\left(x^4+1\right)^{\frac{3}{4}}}=\frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2-x^{-3}}$$

$$\frac{(x^4+1)^{\frac{-3}{4}}}{x^5.(x^4)^{-\frac{3}{4}}}=\frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{\frac{3}{4}}$$

$$=\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}$$

Let $\frac{1}{x^4}=t\Rightarrow-\frac{4}{x^5}dx=dt\Rightarrow\frac{1}{x^5}dx=-\frac{dt}{4}$

$$\therefore\int\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}dx=\int\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}=-\frac{1}{4}\int(1+t)^{-\frac{3}{4}}dt$$

$$=-\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C=-\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C}{\frac{1}{4}}$$

$$-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

5. $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\left[\text{Hint}:\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}\ Put\ x=t^6\right]$

Solution:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Let $x=t^6\Rightarrow dx=6t^5dt$

$$\therefore\int\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=\int\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx=\int\frac{6t^5}{t^2(1+t)}dt$$

$$=6\int \frac{t^3}{(1+t)}dt$$

On dividing, we get

$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=6\int\left\{\left(t^2-t+1\right)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1-x^{\frac{1}{6}}\right)+C$$

6. $\frac{5x}{(x+1)\left(x^2+9\right)}$

Solution:

Consider, $\frac{5x}{(x+1)\left(x^2+9\right)}=\frac{A}{(x+1)}+\frac{Bx+C}{\left(x^2+9\right)}$.........(1)

$$\Rightarrow 5x=A\left(x^2+9\right)+(Bx+C)(x+1)$$

$$\Rightarrow 5x=Ax^2+9A+Bx^2+Bx+Cx+C$$

Equating the coefficients of x^2, x and constant term, we get

$A+B=0$

$B+C=5$

$9A+C=0$

On solving these equations, we get

$A=-\frac{1}{2}, B=\frac{1}{2}\, and\, C=\frac{9}{2}$

From equation (1), we get

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int \frac{5x}{(x+1)(x+9)}dx = \int \left\{ \frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)} \right\} dx$$

$$= \frac{1}{2}\log|x+1| + \frac{1}{2}\int \frac{x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx = -\frac{1}{2}\log|x+1| + \frac{1}{4}\int \frac{2x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx$$

$$= \frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}.\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$= \frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$$

7. $\frac{\sin x}{\sin(x-a)}$

Solution:

$$\frac{\sin x}{\sin(x-a)}$$

Put, $x-a=t \Rightarrow dx = dt$

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t}dt = \int (\cos a + \cot t \sin a)dt$$

$$= t\cos a + \sin a \log|\sin t| + C_1$$

$$= (x-a)\cos a + \sin a \log|\sin(x-a)| + C_1$$

$$= x\cos a + \sin a \log|\sin(x-a)| - a\,\cos a + C_1$$

$$= \sin a \log|\sin(x-a)| + x\cos a + C$$

8. $\dfrac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$

Solution:

Let $\dfrac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}=\dfrac{e^{4\log x}\left(e^{\log x}-1\right)}{e^{2\log x}\left(e^{\log x}-1\right)}$

$=e^{2\log x}$

$=e^{\log x^2}$

$=x^2$

$\therefore\int\dfrac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}dx=\int x^2dx=\dfrac{x^3}{3}+C$

9. $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

Solution:

$\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

Put, $\sin x=t\Rightarrow\cos x\,dx=dt$

$\Rightarrow\int\dfrac{\cos x}{\sqrt{4-\sin^2 x}}dx=\int\dfrac{dt}{\sqrt{(2)^2-(t)^2}}$

$=\sin^{-1}\left(\dfrac{t}{2}\right)+C$

$=\sin^{-1}\left(\dfrac{\sin x}{2}\right)+C$

10. $\dfrac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

Solution:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}=\frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{\sin^2 x\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$$

$$=\frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{\left(\sin^2 x-\sin^2 x\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\cos^2 x\right)}$$

$$=\frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{\sin^2 x\left(1-\cos^2 x\right)+\cos^2 x\left(1-\sin^2 x\right)}$$

$$=\frac{-\left(\sin^4 x-\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$

$$=-\cos 2x$$

$$\therefore \int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=\int -\cos 2x\,dx=-\frac{\sin 2x}{2}+C$$

11. $$\frac{1}{\cos(x+a)\cos(x+b)}$$

Solution:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we get

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a).\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int \frac{1}{\cos(x+a)\cos(x+b)}dx = \frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log\left|\cos(x-a)\right| + \log\left|\cos(x+b)\right|\right] + C$$

$$= \frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right| + C$$

12. $\dfrac{x^3}{\sqrt{1-x^8}}$

Solution:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Put $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{4}\sin^{-1} t + C$$

$$= \frac{1}{4}\sin^{-1}\left(x^4\right) + C$$

13. $\dfrac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$

Solution:

Put, $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow \int \frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}dx = \int \frac{dt}{(t+1)(t+2)}$$

$$= \int\left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right]dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

14. $\dfrac{1}{(x^2+1)(x^2+4)}$

Solution:

$$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equation the coefficients of x^3, x^2, x and constant term, we get

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$

On solving these equations, we get

$A = 0, B = \frac{1}{3}, C = 0$ and $D = -\frac{1}{3}$

From equation (1), we get

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$$

$$\int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

15. $\cos^3 x e^{\log \sin x}$

Solution:

$$\cos^3 x e^{\log \sin x} = \cos^3 x \times \sin x$$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\int \cos^3 x\, e^{\log \sin x} dx = \int \cos^3 x \sin x\, dx$$

$$= -\int t.dt$$

$$= -\frac{t^4}{4} + C$$

$$-\frac{\cos^4 x}{4} + C$$

16. $e^{3\log x} = \left(x^4+1\right)^{-1}$

Solution:

$$e^{3\log x} = \left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

Let $x^4 + 1 = t \Rightarrow 4x^3 dx = dt$

$$\Rightarrow \int e^{3\log x} = \left(x^4+1\right)^{-1} dx = \int \frac{x^3}{\left(x^4+1\right)} dx$$

$$= \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log\left|x^4+1\right| + C$$

$$= \frac{1}{4}\log\left|x^4+1\right| + C$$

17. $f'(ax+b)\left[f(ax+b)\right]^n$

Solution:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Put, $f(ax+b)=t \Rightarrow a f'(ax+b)dx = dt$

$$\Rightarrow f'(ax+b)\left[f(ax+b)\right]^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right] = \frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1} + C$$

18. $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Solution:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x(\sin x\cos\alpha + \cos x\sin\alpha)}}$$

$$= \frac{1}{\sqrt{\sin^4 x\cos\alpha + \sin^3 x\cos x\sin\alpha}}$$

$$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \text{cotxsin}\,\alpha}} = \frac{\cos ec^2 x}{\sqrt{\cos\alpha + \cot x\sin\alpha}}$$

Put, $\cos\alpha + \cot x\sin\alpha = \text{t} \Rightarrow -\text{cosec}^2 x\sin\alpha\,\text{dx} = \text{dt}$

$$\therefore \int \frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}dx = \int \frac{\cos ec^2 x}{\sqrt{\cos\alpha + \text{cotxsin}\,\alpha}}dx$$

$$= \frac{-1}{\sin\alpha}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{t}\right] + C$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha + \cot x\sin\alpha}\right] + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x\sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x\cos\alpha + \cos x\sin\alpha}{\sin x}} + C = \frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

19. $\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$

Solution:

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Let $I=\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}$

As we know that, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$

$$\Rightarrow I=\int\frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$=\frac{2}{\pi}\int\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

$$=\frac{2}{\pi}.\frac{\pi}{2}\int 1.dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx$$

$$=x-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx..........(1)$$

Let $I_1=2\int\cos^{-1}t.dt$

$$=2\left[\cos^{-1}t.\frac{t^2}{2}-\int\frac{-1}{\sqrt{1-t^2}}.\frac{t^2}{2}dt\right]$$

$$=t^2\cos^{-1}t+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\sqrt{1-t^2}dt+\int\frac{1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\frac{1}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t$$

From equation (1), we get

$$I=x-\frac{4}{\pi}\left[t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]$$

$$=x-\frac{4}{\pi}\left[x\cos^{-1}\sqrt{x}-\frac{\sqrt{x}}{2}\sqrt{1-x}+\frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-\frac{4}{\pi}\left[x\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)-\frac{\sqrt{x-x^2}}{2}+\frac{\pi}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-2x+\frac{4x}{\pi}\sin^{-1}\sqrt{x}+\frac{2}{\pi}\sqrt{x-x^2}-\frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

20. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$

Solution:

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Put $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta d\theta$

$$I = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta = -\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta d\theta$$

$$= -\int \tan\frac{\theta}{2}.2\sin\theta\cos\theta d\theta = -2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta d\theta$$

$$= -4\int\sin^2\frac{\theta}{2}\cos\theta d\theta$$

$$= -4\int\sin^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$= -4\int 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}d\theta$$

$$= -8\int\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}d\theta + 4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\sin^2\frac{\theta}{2}d\theta + 4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta + 4\int\frac{1-\cos\theta}{2}d\theta$$

$$= -2\int\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] + 4\left[\frac{\theta}{2} - \frac{\sin\theta}{2}\right] + C$$

$$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin\theta + C$$

$$=\theta+\frac{\sin 2\theta}{2}+2\sin\theta+C$$

$$=\theta+\frac{2\sin\theta\cos\theta}{2}+2\sin\theta+C$$

$$=\theta+\sqrt{1-\cos^2\theta}.\cos\theta-2\sqrt{1-\cos^2\theta}+C$$

$$=\cos^{-1}\sqrt{x}+\sqrt{1-x}.\sqrt{x}-2\sqrt{1-x}+C$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+C$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$$

21. $\frac{2+\sin 2x}{1+\cos 2x}e^x$

Solution:

$$I=\int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$$

$$=\int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$$

$$=\int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$$

$$=\int\left(\sec^2 x+\tan x\right)e^x$$

Let $f(x)=\tan x\Rightarrow f'(x)=\sec^2 x$

$$\therefore I=\int f(x)+f'(x)e^x dx$$

$$=e^x f(x)+C$$

$$=e^x\tan x+C$$

22. $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Solution:

Let $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+2)}$..........(1)

$$\Rightarrow x^2+x+1=A\left(x^2+3x+2\right)+B(x+2)+C\left(x^2+2x+1\right)$$

$$\Rightarrow x^2+x+1=(A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

Equating the coefficients of x^2, x and constant term, we get

$$A+C=1$$
$$3A+B+2C=1$$
$$2A+2B+C=1$$

On solving these equations, we get

From equation (1), we get

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}=-2\int\frac{1}{(x+1)}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

23. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Solution:

$$\mathrm{I}=\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$$

Let $x=\cos\theta\Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(\sin\theta d\theta)$$

$$=-\int\tan^{-1}\sqrt{\frac{3\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta=-\int\tan^{-1}\tan\frac{\theta}{2}.\sin\theta d\theta$$

$$= -\frac{1}{2}\int \theta.\sin\theta d\theta = -\frac{1}{2}\left[\theta.(-\cos\theta) - \int 1.(-\cos\theta)d\theta\right]$$

$$= -\frac{1}{2}[-\theta\cos\theta + \sin\theta]$$

$$= +\frac{1}{2}\theta\cos\theta - \frac{1}{2}\sin\theta$$

$$= \frac{1}{2}\cos^{-1}x.x - \frac{1}{2}\sqrt{1-x^2} + C = \frac{x}{2}\cos^{-1} - \frac{1}{2}\sqrt{1-x^2} + C$$

$$= \frac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C$$

24. $\dfrac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4}$

Solution:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right) - 2\log x\right]}{x^4} = \frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right) - \log x^2\right]$$

$$= \frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$= \frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$= \frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$= \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

Let $1+\frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3}dx = dt$

$$\therefore I = \int \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$= -\frac{1}{2}\int \sqrt{t}\log t dt = -\frac{1}{2}\int t^{\frac{1}{2}}.\log t dt$$

Using integration by parts, we get

$$I = -\frac{1}{2}\left[\log t.\int t^{\frac{1}{2}}dt - \left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$= -\frac{1}{2}\left[\log t\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{4}{9}t^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}}$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C$$

25. $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Solution:

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx = \int_{\frac{\pi}{2}}^{\pi}\left(\frac{\cos ec^2\frac{x}{2}}{2} - \cot\frac{x}{2}\right)dx$$

Let $f(x) = -\cot\frac{x}{2}$

$$\Rightarrow f'(x) = -\left(-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^{x}\left(f(x) + f'(x)\right)dx$$

$$= \left[e^{x}.f(x)\,dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^{x}.\cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^{x}\times\cot\frac{\pi}{2} - e^{\frac{\pi}{2}}\times\cot\frac{\pi}{4}\right]$$

$$= -\left[e^{\pi}\times 0 - e^{\frac{\pi}{2}}\times 1\right]$$

$$= e^{\frac{\pi}{2}}$$

26. $\int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x + \sin^4 x}dx$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x + \sin^4 x}dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\left(\frac{\sin x\cos x}{\cos^4 x}\right)}{\left(\frac{\cos^4 x + \sin^4 x}{\cos^4 x}\right)}dx$$

$$\Rightarrow I = \int_{\pi}^{\frac{\pi}{4}} \frac{\tan x\sec^2 x}{1+\tan^4 x}dx$$

Put, $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x dx = dt$

When $x=0, t=0$ and when $x=\frac{\pi}{4}, t=1$

$$\therefore I=\frac{1}{2}\int_0^4 \frac{dt}{1+t^2}=\frac{1}{2}\left[\tan^{-1} t\right]_0^1$$

$$=\frac{1}{2}\left[\tan^{-1}-\tan^{-1} 0\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}\right]$$

$$=\frac{\pi}{8}$$

27. $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\,dx}{\cos^2 x+4\sin^2 x}$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4}{4-3\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}} 1.dx+\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x-3}dx$$

$$\Rightarrow I=\frac{-1}{3}[x]_0^{\frac{\pi}{2}}+\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\left(1+\tan^2 x\right)-3}dx$$

$$\Rightarrow I=-\frac{\pi}{6}+\frac{2}{3}\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx\,.........(1)$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx$

Put, $2\tan x = t \Rightarrow 2\sec^2 x dx = dt$

When x=0, t = 0 and when $x = \frac{\pi}{2}, t = \infty$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x} dx = \int_0^{\infty} \frac{dt}{1+t^2}$$

$$= \left[\tan^{-1} t\right]_0^{\infty}$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{\pi}{2}$$

Therefore , from (1)we, get

$$I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

28. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

Solution:

Consider, $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx \Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin\cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x \cos^2 x - 2\sin x \cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin x - \cos x)^2}}$$

Let $(\sin x - \cos x) = t - (\sin x + \cos x) dx = dt$

When $x=\frac{\pi}{6}, t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when $x=\frac{\pi}{3}, t=\left(\frac{\sqrt{3}-1}{2}\right)$

$$I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I=\int_{-\left(\frac{1-\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}}=\frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function

$$\int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx$$

We know that if f(x) is an even function, then

$$\Rightarrow I=2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$=\left[2\sin^{-1}t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

29. $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

Solution:

Consider, $I=\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$I=\int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)}\times\frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)}dx$$

$$=\int_0^1 \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{1+x-x}dx$$

$$=\int_0^1 \sqrt{1+x}dx+\int_0^1 \sqrt{x}dx$$

$$=\left[\frac{2}{3}(1+x)^{\frac{2}{3}}\right]_0^1\left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{3}\left[(2)^{\frac{2}{3}}-1\right]+\frac{2}{3}[1]$$

$$=\frac{2}{3}(2)^{\frac{2}{3}}=\frac{2.2\sqrt{2}}{3}$$

$$=\frac{4\sqrt{2}}{3}$$

30. $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

Solution:

Consider, $I=\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

Put, $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)dx=dt$

Where $x=0, t=-1$ and when $x=\frac{\pi}{4}, t=0$

$$\Rightarrow (\sin x-\cos x)^2=t^2$$

$$\Rightarrow \sin^2+\cos^2-2\sin x\cos x=t^2$$

$$\Rightarrow 1-\sin 2x=t^2$$

$$\Rightarrow \sin 2x=1-t^2$$

$$\therefore I=\int_{-1}^{0} \frac{dt}{9+16(1-t^2)}$$

$$=\int_{-1}^{0} \frac{dt}{9+16-16t^2}$$

$$= \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \int_{-1}^{0} \frac{dt}{(5)^2 - (4t)^2}$$

$$= \frac{1}{4}\left[\frac{1}{2(5)} \log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log(1) - \log\left|\frac{1}{9}\right|\right]$$

$$= \frac{1}{40}\log 9$$

31. $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\, dx$

Put, $\sin x = t \Rightarrow \cos x dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$$\Rightarrow I = \int_0^1 t \tan^{-1}(t)\, dt ....(1)$$

Consider, $\int t \tan^{-1} t dt = \tan^{-1} t . \int t dt - \int \left\{ \frac{d}{dt}(\tan^{-1} t) t dt \right\} dt$

$$= \tan^{-1} t . \frac{t^2}{2} - \int \frac{1}{1+t^2} . \frac{t^2}{2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}\int \frac{t^2 + 1 - 1}{1+t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}\int 1.dt + \frac{1}{2}\int \frac{1}{1+t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}.t + \frac{1}{2}\int \frac{1}{1+t^2} dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}.t+\frac{1}{2}\tan^{-1}t$$

$$\Rightarrow \int_0^1 t.\tan^{-1}t dt=\left[\frac{t^2\tan^{-1}t}{2}-\frac{t}{2}+\frac{1}{2}\tan^{-1}t\right]_0^1$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-1+\frac{\pi}{4}\right]=\frac{1}{2}\left[\frac{\pi}{2}-1\right]=\frac{\pi}{4}-\frac{1}{2}$$

From equation (1), we get

$$I=2\left[\frac{\pi}{2}-\frac{1}{2}\right]=\frac{\pi}{2}-1$$

32. $\int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx$

Solution:

Let $\int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx........(1)$

$$I=\int_0^\pi\left\{\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}\right\}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^\pi\left\{\frac{-(\pi-x)\tan x}{-(\sec x+\tan x)}\right\}dx$$

$$\Rightarrow I=\int_0^\pi \frac{(\pi-x)\tan x}{(\sec x+\tan x)}dx..........(2)$$

Adding (1) and (2), we get

$$2I=\int_0^\pi \frac{\pi\tan x}{\sec x+\tan x}dx \Rightarrow 2I=\pi\int_0^\pi \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx$$

$$\Rightarrow 2I=\pi\int_0^\pi \frac{\sin x+1-1}{1+\sin x}dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} 1.dx = \pi\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$\Rightarrow 2I = \pi[x]_0^{\pi} - \pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$\Rightarrow 2I = \pi^2 - \pi\int_0^{\pi}\left(\sec^2 x - \tan x\sec x\right)dx$$

$$\Rightarrow 2I = \pi^2 - \pi[\tan x - \text{secx}]_0^{\pi}$$

$$\Rightarrow 2I = \pi^2 - \pi[\tan x - \text{secx} - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi^2 - \pi[0-(-1)-0+1]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi(\pi-2)$$

$$\Rightarrow I = \frac{\pi}{2}(\pi-2)$$

33. $\int_1^4 \left[|x-1|+|x-2|+|x-3|\right]dx$

Solution:

Consider, $I = \int_1^4 \left[|x-1|+|x-2|+|x-3|\right]dx$

$$\Rightarrow I = \int_1^4 |x-1|dx + \int_1^4 |x-2|dx + \int_1^4 |x-3|dx$$

$$I = I_1 + I_2 + I_3.........(1)$$

Where, $I_1 = \int_1^4 |x-1|dx$

$$(x-1)\ge 0 \; for\, 1\le x\le 4$$

$$\therefore I_1 = \int_1^4 (x-1)dx$$

$$\Rightarrow I_1 = \left[\frac{x^2}{x} - x\right]_1^4$$

$$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1\right] = \frac{9}{2} \ldots\ldots\ldots(2)$$

$$I_2 = \int_1^4 |x-2|\, dx$$

$x - 2 \geq 0$ for $2 \leq x \leq 4$ and $x - 2 \leq 0$ for $1 \leq x \leq 2$

$$\therefore I_2 = \int_1^2 (2-x)\, dx + \int_2^4 (x-2)\, dx$$

$$\Rightarrow I_2 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 \Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2}\right] + [8 - 8 - 2 + 4]$$

$$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2} \ldots\ldots\ldots(3)$$

$$I_3 = \int_1^4 |x-3|\, dx$$

$x - 3 \geq 0$ for $3 \leq x \leq 4$ and $x - 3 \leq 0$ for $1 \leq x \leq 3$

$$\therefore I_3 = \int_1^3 (3-x)\, dx + \int_3^4 (x-3)\, dx$$

$$\Rightarrow I_3 = \left[3 - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$$

$$\Rightarrow I_3 = [6 - 4] + \left[\frac{1}{5}\right] = \frac{5}{2} \ldots\ldots\ldots(4)$$

From equations (1), (2), (3), and (4), we get

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

34. $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Solution:

Consider, $I = \int_1^3 \frac{dx}{x^2(x+1)}$

Let, $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$

$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$

Equating the coefficients of x^2, x an constant term, we get

$A + C = 0$
$A + B = 0$
$B = 1$

On solving these equations, we get

$A = -1, C = 1$, and $B = 1$

$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$

$\Rightarrow I = \int_1^3 \left\{\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}\right\} dx = \left[-\log x - \frac{1}{x} + \log(x+1)\right]_1^3$

$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x}\right]_1^3 = \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$

$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$

$= \log 2 - \log 3 + \frac{2}{3}$

$= \log\left(\frac{2}{3}\right)+\frac{2}{3}$

Hence proved

35. $\int_0^4 xe^x dx = 1$

Solution:

Let $I = \int_0^4 xe^x dx$

Using integration by pars, we get

$I = \int_0^4 xe^x dx - \int_0^1 \left\{\left(\frac{d}{dx}(x)\right)\int e^x dx\right\} dx$

$= \left[xe^x\right]_0^1 - \int_0^1 e^x dx$

$= \left[xe^x\right]_0^1 - \left[e^x\right]_0^1$

$= e - e + 1$

$= 1$

Hence proved

36. $\int_{-1}^1 x^{17}\cos^4 x\,dx = 0$

Solution:

Consider, $I = \int_{-1}^1 x^{17}\cos^4 x\,dx$

Let $f(x) = x^{17}\cos^4 x$

$\Rightarrow f(x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

$f(x)$ is an odd function

We know that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

$$\therefore \int_{-1}^{1} x^{17}\cos^4 x\,dx = 0$$

Hence proved

37. $\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \sin^3 x dx$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x.\sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left(1-\cos^2 x\right)\sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x\,dx - \int_0^{\frac{\pi}{2}} \cos^2 x.\sin x\,dx$$

$$= \left[-\cos x\right]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$$

$$1+\frac{1}{3}[-1] = 1-\frac{1}{3} = \frac{2}{3}$$

Hence proved

38. $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1-\log 2$

Solution:

Consider, $I = \int_0^{\frac{\pi}{4}} 2\tan^3 x dx$

$$I = \int_0^{\frac{\pi}{4}} 2\tan^2 x \tan x\,dx = 2\int_0^{\frac{\pi}{4}} \left(\sec^2 x - 1\right)\tan x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \, dx - 2\int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log \cos x\right]_0^{\frac{\pi}{4}} = 1 + 2\left[\log \cos \frac{\pi}{4} - \log \cos 0\right]$$

$$= 1 + 2\left[\log \frac{1}{\sqrt{2}} - \log 1\right] = 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence proved

39. $\int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - 1$

Solution:

Let $\int_0^1 \sin^{-1} x \, dx$

$$\Rightarrow I = \int_0^1 \sin^{-1} x.1.dx$$

Using integration by parts, we get

$$I = \left[\sin^{-1} x.x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} x \, dx$$

$$= \left[\sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} dx$$

Put $1 - x^2 = t \Rightarrow -2xdx = dt$

When $x = 0, t = 1$ and when x= 1 , t = 0

$$I = \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t}}$$

$$I = \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$= \sin^{-1}(1) + \left[-\sqrt{1}\right]$$

$= \frac{\pi}{2} - 1$

Hence proved

40. Evaluate $\int_0^1 e^{2-3x} dx$ as a limit of a sum

Solution:

Let $I = \int_0^1 e^{2-3x} dx$

We know that,

$$\int_a^b f(x)dx = (b-c)\lim_{x\to n}\frac{1}{n}\left[f(a)+f(a+h)+......+f\left(a+(n-1)h\right)\right]$$

Where, $h = \frac{b-a}{n}$

Here, $a=0, b=1,$ and $f(x) = e^{2-3x}$

$$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x} dx = (1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+.......+f\left(0+(n-1)h\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2+e^{2-3x}+......+e^{2-3(n-1)h}\right] = \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+......+e^{-3(n-1)h}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)}{1-\left(e^{-3h}\right)}\right\}\right] = \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{e^2-\left(1-e^3\right)}{1-e^{-\frac{3}{n}}}\right\}\right] = e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\left\{\frac{1}{e^{-\frac{3}{n}}-1}\right\}\right]$$

$$e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right] = \frac{e^2\left(e^{-3}-1\right)}{3}\lim_{n\to 8}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$\frac{-e^2\left(e^{-3}-1\right)}{3}(1)$ $\left[\lim_{x\to\infty}\frac{x}{e^x-1}\right]$

$=\frac{e^{-1}+e^2}{3}$

$=\frac{1}{3}\left(e^2-\frac{1}{e}\right)$

41. $\int\frac{dx}{e^x+e^{-x}}$ is equal to

A) $\tan^{-1}\left(e^x\right)+C$ B) $\tan^{-1}\left(e^{-x}\right)+C$ C) $\log\left(e^x-e^{-x}\right)+C$ D) $\log\left(e^x+e^{-x}\right)+C$

Solution:

Consider, $I=\int\frac{dx}{e^x+e^{-x}}dx=\int\frac{e^x}{e^{2x}+1}dx$

Put, $e^x=t\Rightarrow e^x dx=dt$

$\therefore I=\int\frac{dx}{1+t^2}$

$=\tan^{-1}t+C$

$=\tan^{-1}\left(e^x\right)+C$

Thus, the correct answer is A

42. $\frac{\cos 2x}{(\sin x+\cos x)^2}dx$ is

A) $\frac{-1}{\sin x+\cos x}+C$ B) $\log\left|\sin x+\cos x\right|+C$

C) $\log\left|\sin x-\cos x\right|+C$ D) $\frac{1}{(\sin x+\cos x)}+C$ is equal to

Solution:

Consider, $I=\int\frac{\cos 2x}{(\sin x+\cos x)^2}dx$ $I=\int\frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)^2}dx$

$$\int\frac{(\cos x+\sin x)(\cos x-\sin x)}{(\sin x+\cos x)^2}dx=\int\frac{\cos x-\sin x}{\cos x+\sin x}dx$$

Let $\cos x+\sin x=t\Rightarrow(\cos x-\sin x)dx=dt$

$$\therefore I=\int\frac{dt}{t}$$

$$=\log|t|+C$$

$$=\log|\cos x+\sin x|+C$$

Thus, the correct answer is B

43. If $f(a+b-x)=f(x)$, then $\int_a^b x\,f(x)\,dx$ is equal to

A) $\frac{a+b}{2}\int_a^b f(b-x)\,dx$

B) $\frac{a+b}{2}\int_a^b f(b+x)\,dx$

C) $\frac{b-a}{2}\int_a^b f(x)\,dx$

D) $\frac{a+b}{2}\int_a^b f(x)\,dx$

Solution:

Consider, $I=\int_a^b x\,f(x)\,dx$.........(1)

$I=\int_a^b(a+b-x)\,f(a+b-x)\,dx$ $\left(\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right)$

$$\Rightarrow I=\int_a^b(a+b-x)\,f(x)\,dx$$

$\Rightarrow I=(a+b)\int_a^b f(x)\,dx-I$ (Using (1))

$$\Rightarrow I+I=(a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow 2I = (a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow I = \left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx$$

Thus, the correct answer is D

44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A) 1 B) 0 C) -1 D) $\frac{\pi}{4}$

Solution:

Consider, $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$$\Rightarrow I = \int_0^1 \tan^{-1}\left(\frac{x(1-x)}{1+x(1-x)}\right)dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1} x - \tan^{-1}(1-x)\right]dx.........(1)$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(1-1+x)\right]dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx.........(2)$$

Adding (1) and (2) we get

$$\Rightarrow 2I = \int_0^1 \left(\tan^{-1} x - \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x\right)dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Thus, the correct answer is B

Exercise 7.1

1. Find an antiderivative of the function $f(x)=\sin 2x$ by inspection method.

Solution:

Recall the functions whose derivative is $\sin 2x$

$$\frac{d}{dx}(\cos 2x)=-2\sin 2x \qquad \bullet\frac{d}{dx}(\cos ax)=-a\sin ax$$

It implies that

$$\begin{aligned}\sin 2x &= -\frac{1}{2}\frac{d}{dx}(\cos 2x)\\ &= \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)\end{aligned}$$

Therefore, antiderivative of $f(x)=\sin 2x$ is $-\frac{1}{2}\cos 2x$

2. Find the antiderivative of the function $f(x)=\cos 3x$ by inspection method.

Solution:

Recall the functions whose derivative is $\cos 3x$

$$\frac{d}{dx}(\sin 3x)=3\cos 3x \qquad \bullet\frac{d}{dx}(\sin ax)=a\cos ax$$

It implies that

$$\begin{aligned}\cos 3x &= \frac{1}{3}\frac{d}{dx}(\sin 3x)\\ &= \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)\end{aligned}$$

Therefore, antiderivative of $f(x)=\cos 3x$ is $\frac{1}{3}\sin 3x$

3. Find the antiderivative of the function $f(x) = e^{2x}$ by inspection method.

Solution:

Recall the functions whose derivative is e^{2x}

$$\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x} \qquad \bullet \frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

It implies that

$$e^{2x} = \frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, antiderivative of $f(x) = e^{2x}$ is $\frac{1}{2}e^{2x}$

4. Find the antiderivative of the function $f(x) = (ax+b)^2$ by inspection method.

Solution:

Recall the functions whose derivative is $f(x) = (ax+b)^2$

$$\frac{d}{dx}\left[(ax+b)^3\right] = 3(ax+b)^2 \qquad \bullet \frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}$$

It implies that

$$(ax+b)^2 = \frac{1}{3}\frac{d}{dx}(ax+b)^3$$
$$= \frac{d}{dx}\left(\frac{1}{3}(ax+b)^3\right)$$

Therefore, antiderivative of $f(x) = (ax+b)^2$ is $\frac{1}{3}(ax+b)^3$

5. Find the antiderivative of the function $\sin 2x - 4e^{3x}$ by inspection method.

Solution:

Recall the functions whose derivative is $f(x) = \sin 2x$

$$\frac{d}{dx}\cos 2x = -2\sin 2x \qquad \bullet \frac{d}{dx}\cos ax = -a\sin ax$$

It implies that

$$\sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$
$$= \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Recall the functions whose derivative is $f(x) = 4e^{3x}$

$$\frac{d}{dx}e^{3x} = 3e^{3x} \qquad \bullet \frac{d}{dx}e^{ax} = ae^{ax}$$

It implies that

$$e^{3x} = \frac{1}{3}\frac{d}{dx}\left(e^{3x}\right)$$
$$4e^{3x} = \frac{4}{3}\frac{d}{dx}\left(e^{3x}\right) \qquad \bullet \text{Multiply 4 on both sides}$$
$$= \frac{d}{dx}\left(\frac{4}{3}e^{3x}\right)$$

Therefore, antiderivative of $\sin 2x - 4e^{3x}$ is $-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}$

6. Find $\int\left(4e^{3x} + 1\right)dx$

Solution:

Consider the given integral $\int\left(4e^{3x} + 1\right)dx$

$$\int\left(4e^{3x}+1\right)dx=4\int e^{3x}dx+\int dx$$

$$=4\left(\frac{1}{3}e^{3x}\right)+x+C \qquad \bullet\int e^{ax}dx=\frac{1}{a}e^{ax}+c$$

$$=\frac{4}{3}e^{3x}+x+C$$

Therefore, $\int\left(4e^{3x}+1\right)dx=\frac{4}{3}e^{3x}+x+C$

7. Find $\int x^2\left(1-\frac{1}{x^2}\right)dx$

Solution:

Consider the given integral $\int x^2\left(1-\frac{1}{x^2}\right)dx$

$$\int x^2\left(1-\frac{1}{x^2}\right)dx=\int x^2dx-\int dx$$

$$=\frac{x^3}{3}-x+C \qquad \bullet\int x^n dx=\frac{x^{n+1}}{n+1}+c$$

Therefore, $\int x^2\left(1-\frac{1}{x^2}\right)dx=\frac{x^3}{3}-x+C$

8. Find $\int\left(ax^2+bx+c\right)dx$

Solution:

Consider the given integral $\int\left(ax^2+bx+c\right)dx$

$$\int\left(ax^2+bx+c\right)dx=a\int x^2dx+b\int xdx+c\int dx$$

$$=a\left(\frac{x^3}{3}\right)+b\left(\frac{x^2}{2}\right)+cx+D$$

$$=\frac{ax^3}{3}+\frac{bx^2}{2}+cx+D$$

Therefore, $\int\left(ax^2+bx+c\right)dx=\frac{ax^3}{3}+\frac{bx^2}{2}+cx+D$

9. Find $\int\left(2x^2+e^x\right)dx$

Solution:

The given integral is $\int\left(2x^2+e^x\right)dx$

$$\int\left(2x^2+e^x\right)dx=2\int x^2dx+\int e^xdx \qquad \bullet\int x^ndx=\frac{x^{n+1}}{n+1}+c$$

$$=2\frac{x^3}{3}+e^x+C$$

Therefore, $\int\left(2x^2+e^x\right)dx=\frac{2x^3}{3}+e^x+C$

10. Find $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$

Solution:

The given integral is $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx=\int\left(x+\frac{1}{x}-2\right)dx$$

$$=\int xdx+\int\frac{1}{x}dx-\int 2dx$$

$$=\frac{x^2}{2}+\ln x-2x+C \qquad \bullet\int\frac{1}{x}dx=\ln x+c$$

Therefore, $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx=\frac{x^2}{2}+\ln x-2x+C$

11. Find $\int\frac{x^3+5x^2-4}{x^2}dx$

Solution:

Consider the integral $\int \frac{x^3+5x^2-4}{x^2}dx$

$$\int \frac{x^3+5x^2-4}{x^2}dx = \int \frac{x^3+5x^2-4}{x^2}dx$$
$$= \int x+5-\frac{4}{x^2}dx$$
$$= \frac{x^2}{2}+5x+\frac{8}{x}+C$$

Therefore, $\int \frac{x^3+5x^2-4}{x^2}dx = \frac{x^2}{2}+5x+\frac{8}{x}+C$

12. Find $\int \frac{x^3+3x+4}{\sqrt{x}}dx$

Solution:

The given integral is $\int \frac{x^3+3x+4}{\sqrt{x}}dx$

$$\int \frac{x^3+3x+4}{\sqrt{x}}dx = \int \frac{x^3}{\sqrt{x}}+\frac{3x}{\sqrt{x}}+\frac{4}{\sqrt{x}}dx$$
$$= \int x^{\frac{5}{2}}dx + 3\int x^{\frac{1}{2}}dx + 4\int x^{\frac{1}{2}}dx$$
$$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 3\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 4\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C$$
$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + 3\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 4\frac{x^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 3\left(\frac{2}{3}\right)x^{\frac{5}{2}} + 4\left(\frac{2}{1}\right)x^{\frac{1}{2}}+C$$
$$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{5}{2}} + 8x^{\frac{1}{2}}+C$$

Therefore, $\int \frac{x^3+3x+4}{\sqrt{x}}dx = \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{5}{2}} + 8\sqrt{x}+C$

13. Find , $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

Solution:

The given integral is $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} dx = \int \frac{x^2(x-1) + (x-1)}{x-1} dx$$
$$= \int \frac{(x-1)(x^2+1)}{x-1} dx$$
$$= \int (x^2 + 1) dx$$
$$= \frac{x^3}{3} + x + C$$

Therefore, $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx = \frac{x^3}{3} + x + C$

14. Find $\int (1-x)\sqrt{x} dx$

Solution:

The given integral is $\int (1-x)\sqrt{x} dx$

$$\int (1-x)\sqrt{x} dx = \int \sqrt{x} - x\sqrt{x} dx$$
$$= \int \sqrt{x} dx - \int x^{\frac{3}{2}} dx$$
$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$$
$$= \frac{2}{3} x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}} + C$$

Therefore, $\int (1-x)\sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}} + C$

15. Find $\int \sqrt{x}\left(3x^2+2x+3\right)dx$

Solution:

The given integral is $\int \sqrt{x}\left(3x^2+2x+3\right)dx$

$$\int \sqrt{x}\left(3x^2+2x+3\right)dx = \int 3x^2\sqrt{x}+2x\sqrt{x}+3\sqrt{x}dx$$

$$= 3\int x^{\frac{5}{2}}dx + 2\int x^{\frac{3}{2}}dx + 3\int x^{\frac{1}{2}}dx$$

$$= 3\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 2\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 3\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= 3\left(\frac{2}{7}\right)x^{\frac{7}{2}} + 2\left(\frac{2}{5}\right)x^{\frac{5}{2}} + 3\left(\frac{3}{2}\right)x^{\frac{3}{2}} + C$$

$$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + \frac{9}{2}x^{\frac{3}{2}} + C$$

Therefore, $\int \sqrt{x}\left(3x^2+2x+3\right)dx = \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + \frac{9}{2}x^{\frac{3}{2}} + C$ Adf

16. Find $\int \left(2x-3\cos x+e^x\right)dx$

Solution:

The given integral is $\int \left(2x-3\cos x+e^x\right)dx$

$$\int \left(2x-3\cos x+e^x\right)dx = 2\int xdx - \int 3\cos xdx + \int e^x dx$$

$$= 2\frac{x^2}{2} - 3(\sin x) + e^x + C$$

$$= x^2 - 3\sin x + e^x + C$$

Therefore, $\int \left(2x-3\cos x+e^x\right)dx = x^2 - 3\sin x + e^x + C$

17. Find $\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$

Solution:

The given integral is $\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx=2\int x^2dx-3\int \sin xdx+5\int x^{\frac{1}{2}}dx$$

$$=2\frac{x^3}{3}-3\left(-\cos x\right)+5\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C$$

$$=\frac{2x^3}{3}+3\cos x+\frac{10}{3}x^{\frac{3}{2}}+C$$

Therefore, $\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx=\frac{2x^3}{3}+3\cos x+\frac{10}{3}x^{\frac{3}{2}}+C$

18. Find $\int \sec x\left(\sec x+\tan x\right)dx$

Solution:

The given integral is $\int \sec x\left(\sec x+\tan x\right)dx$

$$\int \sec x\left(\sec x+\tan x\right)dx=\int \sec^2 xdx+\int \sec x\tan xdx$$

$$=\tan x+\sec x+C$$

Therefore, $\int \sec x\left(\sec x+\tan x\right)dx=\tan x+\sec x+C$

19. Find $\int\frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$

Solution:

The given integral is $\int\frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx = \int \tan^2 x dx$$
$$= \int \sec^2 x - 1 dx$$
$$= \int \sec^2 x dx - \int dx$$
$$= \tan x - x + C$$

Therefore, $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx = \tan x - x + C$

20. Find $\int \frac{2 - 3\sin x}{\cos^2 x} dx$

Solution:

The given integral is $\int \frac{2 - 3\sin x}{\cos^2 x} dx$

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx = 2\int \sec^2 x dx - 3\int \tan x \sec x dx$$
$$= 2\tan x - 3\sec x + C$$

Therefore, $\int \frac{2 - 3\sin x}{\cos^2 x} dx = 2\tan x - 3\sec x + C$

21. The antiderivative of $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

1) $\frac{1}{3} x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$

2) $\frac{2}{3} x^{\frac{2}{3}} + \frac{1}{2} x^{\frac{1}{2}} + C$

3) $\frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

4) $\frac{3}{2} x^{\frac{3}{2}} + \frac{1}{2} x^{\frac{1}{2}} + C$

Solution:

The given integral is $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

$$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=\int x^{\frac{1}{2}}dx+\int x^{-\frac{1}{2}}dx$$

$$=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C$$

$$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+C$$

$$=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$$

Hence, $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$

Therefore, the option 3 is correct.

22. If $\frac{d}{dx}\left(f\left(x\right)\right)=4x^3-\frac{3}{x^4}$ such that $f\left(2\right)=0$. Then $f\left(x\right)$ is

1) $x^4+\frac{1}{x^3}-\frac{129}{8}$

2) $x^3+\frac{1}{x^4}+\frac{129}{8}$

3) $x^4+\frac{1}{x^3}+\frac{129}{8}$

4) $x^3+\frac{1}{x^4}-\frac{129}{8}$

Solution:

Given $\frac{d}{dx}\left(f\left(x\right)\right)=4x^3-\frac{3}{x^4}$

It implies that

$$f\left(x\right)=\int 4x^3-\frac{3}{x^4}dx$$

$$=4\int x^3dx-3\int\frac{1}{x^4}dx$$

$$=4\frac{x^{3+1}}{3+1}-3\frac{x^{-4+1}}{-4+1}+C$$

$$=x^4+\frac{1}{x^3}+C$$

Given $f(2)=0$

$$f(2)=2^{4}+2^{-3}+C$$

$$0=16+\frac{1}{8}+C$$

$$C=\frac{128+1}{8}$$

$$=-\frac{129}{8}$$

Hence, $f(x)=x^{4}+\frac{1}{x^{4}}-\frac{129}{8}$

Therefore, option 1 is correct.

Exercise: 7.2

1. Integrate the function $\frac{2x}{1+x^2}$

Solution:

The given integral is $\int \frac{2x}{1+x^2} dx$

Put $t = 1 + x^2$ so that $dt = 2xdx$

Hence,

$$\begin{aligned}\int \frac{2x}{1+x^2} dx &= \int \frac{1}{t} dt \\ &= \log|t| + C \\ &= \log\left|1+x^2\right| + C\end{aligned}$$

Therefore, $\int \frac{2x}{1+x^2} dx = \log\left|1+x^2\right| + C$

2. Integrate the function $\frac{(\log x)^2}{x}$

Solution: The given integral is $\int \frac{(\log x)^2}{x} dx$

Put $t = \log x$ so that $dt = \frac{1}{x} dx$

Hence,

$$\begin{aligned}\int \frac{(\log x)^2}{x} dx &= \int t^2 dt \\ &= \frac{t^3}{3} + C \\ &= \frac{(\log x)^3}{3} + C\end{aligned}$$

Therefore, $\int \frac{(\log x)^2}{x} dx = \frac{|(\log x)|^3}{3} + C$

3. Integrate the function $\frac{1}{x + x\log x}$

Solution:

The given integral is $\int \frac{1}{x + x\log x} dx = \int \frac{1}{x(1 + \log x)} dx$

Put $t = 1 + \log x$ so that $dt = \frac{1}{x} dx$

Hence,

$$\begin{aligned}\int \frac{1}{x + x\log x} dx &= \int \frac{1}{x(1 + \log x)} dx \\ &= \int \frac{1}{t} dx \\ &= \log t + C \\ &= \log(1 + \log x) + C\end{aligned}$$

Therefore, $\int \frac{1}{x + x\log x} dx = \log|(1 + \log x)| + C$

4. Integrate the function $\sin x \sin(\cos x)$

Solution:

The given integral is $\int \sin x \sin(\cos x) dx$

Put $t = \cos x$ so that $dt = -\sin x dx$

Hence,

$$\begin{aligned}\int \sin x \sin(\cos x) dx &= -\int \sin t dt \\ &= \cos t + C \\ &= \cos(\cos x) + C\end{aligned}$$

Therefore, $\int \sin x \sin(\cos x)\,dx = \cos(\cos x) + C$

5. Integrate the function $\sin(ax+b)\cos(ax+b)$

Solution:

The given integral is $\int \sin(ax+b)\cos(ax+b)\,dx$

Put $t = \sin(ax+b)$ so that $dt = a\cos(ax+b)\,dx$

Hence,

$$\begin{aligned}\int \sin(ax+b)\cos(ax+b)\,dx &= \frac{1}{a}\int t\,dt \\ &= \frac{1}{a}\frac{t^2}{2} + C \\ &= \frac{1}{2a}\sin^2(ax+b) + C\end{aligned}$$

Therefore, $\int \sin(ax+b)\cos(ax+b)\,dx = \frac{1}{2a}\sin^2(ax+b) + C$

6. Integrate the function $\sqrt{ax+b}$

Solution:

The given integral is $\int \sqrt{ax+b}\,dx$

Put $t = (ax+b)$ so that $dt = a\,dx$

Hence,

$$\begin{aligned}\int \sqrt{ax+b}\,dx &= \frac{1}{a}\int \sqrt{t}\,dt \\ &= \frac{1}{a}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \\ &= \frac{2}{3a}(ax+b)^{\frac{3}{2}}\end{aligned}$$

Therefore, $\int \sqrt{ax+b}dx = \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$

7. Integrate the function $x\sqrt{x+2}$

Solution:

The given integral is $\int x\sqrt{x+2}dx$

Put $t = x+2$ so that $dt = dx$

Hence,

$$\int x\left(\sqrt{x+2}\right)dx = \int (t-2)\sqrt{t}dt$$
$$= \int t^{\frac{3}{2}}dt - 2\int t^{\frac{1}{2}}dt$$
$$= \frac{2}{5}t^{\frac{5}{2}} - 2\left(\frac{2}{3}\right)t^{\frac{3}{2}} + C$$
$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Therefore, $\int x\left(\sqrt{x+2}\right)dx = \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$

8. Integrate the function $x\sqrt{1+2x^2}$

Solution: The given integral is $\int x\sqrt{1+2x^2}dx$

Put $t = 1+2x^2$ so that $dt = 4xdx$

Hence,

$$\int x\sqrt{1+2x^2}dx = \frac{1}{4}\int 4x\sqrt{1+2x^2}dx$$
$$= \frac{1}{4}\int \sqrt{t}dt$$

$$=\frac{1}{4}\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+C$$

$$=\frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}}+C$$

Therefore, $\int x\sqrt{1+2x^2}dx=\frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}}+C$

9. Integrate the function $(4x+2)\sqrt{x^2+x+1}$

Solution:

The given integral is $\int(4x+2)\sqrt{x^2+x+1}dx=2\int(2x+1)\sqrt{x^2+x+1}dx$

Put $t=x^2+x+1$ so that $dt=(2x+1)dx$

Hence,

$$\int(4x+2)\sqrt{x^2+x+1}dx=2\int(2x+1)\sqrt{x^2+x+1}dx$$
$$=2\int\sqrt{t}dt$$
$$=2\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+C$$
$$=\frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}}+C$$

Therefore, $\int(4x+2)\sqrt{x^2+x+1}dx=\frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}}+C$

10. Integrate the function $\frac{1}{x-\sqrt{x}}$

Solution:

The given integral is $\int\frac{1}{x-\sqrt{x}}\cdot dx=\int\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}\cdot dx$

Put $t=\sqrt{x}-1$ so that $dt=\frac{1}{2\sqrt{x}}dx$

Hence,

$$\int \frac{1}{x-\sqrt{x}}\cdot dx=\int \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}\cdot dx$$
$$=2\int \frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{x}-1}\right)dx$$
$$=2\int \frac{1}{t}dt$$
$$=2\log(t)+C$$
$$=2\log\left|\left(\sqrt{x}-1\right)\right|+C$$

Therefore, $\int \frac{1}{x-\sqrt{x}}\cdot dx=2\log\left|\left(\sqrt{x}-1\right)\right|+C$

11. Integrate the function $\frac{x}{\sqrt{x+4}}$

Solution:

The given integral is $\int \frac{x}{\sqrt{x+4}}dx$

$$\int \frac{x}{\sqrt{x+4}}\cdot dx=\int \frac{x+4-4}{\sqrt{x+4}}\cdot dx$$
$$=\int \frac{x+4}{\sqrt{x+4}}-\frac{4}{\sqrt{x+4}}\cdot dx$$
$$=\int \sqrt{x+4}dx-4\int \frac{1}{\sqrt{x+4}}dx$$
$$=\frac{(x+4)^{\frac{1}{2}+1}}{\frac{3}{2}}-4\frac{(x+4)^{-\frac{1}{2}+1}}{\frac{1}{2}}+C$$
$$=\frac{2}{3}(x+4)^{\frac{3}{2}}-8\sqrt{x+4}+C$$

Therefore, $\int \frac{x}{\sqrt{x+4}}dx=\frac{2}{3}\sqrt{x+4}\,(x-8)+C$

12. Integrate the function $\left(x^3-1\right)^{\frac{1}{3}} x^5$

Solution:

The given integral is $\int\left(x^3-1\right)^{\frac{1}{3}} x^5 dx$

Put $t=x^3-1$ so that $dt=3x^2 dx$

Hence,

$$\begin{aligned}
\int\left(x^3-1\right)^{\frac{1}{3}} x^5 dx &= \frac{1}{3}\int\left(x^3-1\right)^{\frac{1}{3}}\left(3x^2\right) x^3 dx \\
&= \frac{1}{3}\int t^{\frac{1}{3}}(t+1) dt \\
&= \frac{1}{3}\int t^{\frac{4}{3}}+t^{\frac{1}{3}} dt \\
&= \frac{1}{3}\left(\frac{t^{\frac{7}{3}}}{\frac{7}{3}}\right)+\frac{1}{3}\left(\frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right) \\
&= \frac{1}{7} t^{\frac{7}{3}}+\frac{1}{4} t^{\frac{4}{3}}+C \\
&= \left(x^3-1\right)\left(\frac{1}{7}\left(x^3-1\right)^{\frac{4}{3}}+\frac{1}{4}\left(x^3-1\right)^{\frac{1}{3}}\right)+C
\end{aligned}$$

Therefore, $\int\left(x^3-1\right)^{\frac{1}{3}} x^5 dx=\left(x^3-1\right)\left(\frac{1}{7}\left(x^3-1\right)^{\frac{4}{3}}+\frac{1}{4}\left(x^3-1\right)^{\frac{1}{3}}\right)+C$

13. Integrate the function $\frac{x^2}{\left(2+3x^3\right)^3}$

Solution:

The given integral is $\int \frac{x^2}{\left(2+3x^3\right)^3} dx$

Put $t=2+3x^3$ so that $dt=9x^2 dx$

Hence,

$$\int \frac{x^2}{\left(2+3x^3\right)^3}dx = \frac{1}{9}\int \frac{9x^2}{\left(2+3x^3\right)^3}dx$$

$$= \frac{1}{9}\int t^{-3}dt$$

$$= \frac{1}{9}\left(\frac{t^{-3+1}}{-3+1}\right)$$

$$= -\frac{1}{18t^2}$$

$$= -\frac{1}{18\left(2+3x^3\right)^2} + C$$

Therefore, $\int \frac{x^2}{\left(2+3x^3\right)^3}dx = -\frac{1}{18\left(2+3x^3\right)^2} + C$

14. Find the integral of the function $\frac{1}{x(\log x)^m}$

Solution:

The given integral is $\int \frac{1}{x(\log x)^m}dx$

Put $t = \log x$ so that $dt = \frac{1}{x}dx$

Hence,

$$\int \frac{1}{x(\log x)^m}dx = \int \frac{1}{t^m}dt$$

$$= \frac{t^{-m+1}}{-m+1} + C$$

$$= \frac{(\log x)^{-m+1}}{-m+1} + C$$

Therefore, $\int \frac{1}{x(\log x)^m}dx = \frac{(\log x)^{-m+1}}{-m+1} + C$

15. Integrate the function $\frac{x}{9-4x^2}$

Solution:

The given integral is $\int \frac{x}{9-4x^2}dx$

Put $t = 9-4x^2$, so that $dt = -8x$

Hence, the integral can be rewrite as

$$\begin{aligned}\int \frac{x}{9-4x^2}dx &= -\frac{1}{8}\int \frac{-8x}{9-4x^2}dx \\ &= -\frac{1}{8}\int \frac{dt}{t} \\ &= -\frac{1}{8}\log(t)+C \\ &= -\frac{1}{8}\log\left(9-4x^2\right)+C\end{aligned}$$

Therefore, $\int \frac{x}{9-4x^2}dx = -\frac{1}{8}\log\left(9-4x^2\right)+C$

16. Find the integral of the function e^{2x+3}

Solution:

The given integral is $\int e^{2x+3}dx$

Put $t = 2x+3$, so that $dt = 2dx$

Hence, the given integral can be rewrite as

$$\begin{aligned}\int e^{2x+3}dx &= \frac{1}{2}\int e^{2x+3}(2dx) \\ &= \frac{1}{2}\int e^{t}dt \\ &= \frac{1}{2}e^{t}+C \\ &= \frac{1}{2}e^{2x+3}+C\end{aligned}$$

Therefore, $\int e^{2x+3}dx = \frac{1}{2}e^{2x+3}+C$

17. Find the integral of the function $\frac{x}{e^{x^2}}$

Solution:

The given integral is $\int \frac{x}{e^{x^2}} dx$

Put $t = x^2$, so that $dt = 2xdx$

Hence the integral can be rewrite as

$$\int \frac{x}{e^{x^2}} dx = \frac{1}{2}\int \frac{2x}{e^{x^2}} dx$$
$$= \frac{1}{2}\int \frac{dt}{e^t}$$
$$= \frac{1}{2}\int e^{-t} dt$$
$$= -\frac{1}{2}e^{-t} + C$$
$$= -\frac{1}{2}e^{-x^2} + C$$

Therefore, $\int \frac{x}{e^{x^2}} dx = -\frac{1}{2}e^{-x^2} + C$

18. Find the integral of the function $\frac{e^{\tan^{-1}x}}{1+x^2}$

Solution:

The given integral is $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$

Put $t = \tan^{-1}x$, so that $dt = \frac{1}{1+x^2} dx$

Hence the integral can be rewrite as

$$\int \frac{e^{\tan^{-1}x}}{1+x^2} dx = \int e^t dt$$
$$= e^t + C$$
$$= e^{\tan^{-1}x} + C$$

Therefore, $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx = e^{\tan^{-1}x} + C$

19. Find the integral of the function $\frac{e^{2x}-1}{e^{2x}+1}$

Solution:

The given integral is

$$\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$$, by dividing both numerator and denominator with e^x

Put $t = e^x + e^{-x}$ so that $dt = \left(e^x - e^{-x}\right)dx$

Hence the integral can be rewrite as

$$\begin{aligned}\int \frac{e^{2x}-1}{e^{2x}+1}dx &= \int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx \\ &= \int \frac{dt}{t} \\ &= \ln(t) + C \\ &= \ln\left(e^x + e^{-x}\right) + C\end{aligned}$$

Therefore, $\int \frac{e^{2x}-1}{e^{2x}+1}dx = \ln\left(e^x + e^{-x}\right) + C$

20. Find the integral of the function $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Solution:

The given integral is

$$\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx$$, by dividing both numerator and denominator with e^x

Put $t = e^{2x} + e^{-2x}$ so that $dt = 2\left(e^{2x} - e^{-2x}\right)dx$

Hence the integral can be rewrite as

$$\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx = \frac{1}{2}\int \frac{2\left(e^{2x}-e^{-2x}\right)}{e^{2x}+e^{-2x}}dx$$
$$= \frac{1}{2}\int \frac{dt}{t}$$
$$= \frac{1}{2}\log(t)+C$$
$$= \frac{1}{2}\log\left(e^{2x}+e^{-2x}\right)+C$$

Therefore, $\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}dx = \frac{1}{2}\log\left(e^{2x}+e^{-2x}\right)+C$

21. Find the integral of the function $\tan^2(2x-3)$

Solution:

The given integral is $\int \tan^2(2x-3)\,dx = \int\left(\sec^2(2x-3)-1\right)dx$

Put $t = 2x-3$ so that $dt = 2dx$

Hence the integral can be rewrite as

$$\int \tan^2(2x-3)\,dx = \int\left(\sec^2(2x-3)-1\right)dx$$
$$= \frac{1}{2}\int\left(\sec^2(2x-3)-1\right)2dx$$
$$= \frac{1}{2}\int\left(\sec^2(t)-1\right)dt$$
$$= \frac{1}{2}\left(\tan(t)-t\right)+c$$
$$= \frac{1}{2}\tan(2x-3)-(2x-3)+c$$
$$= \frac{1}{2}\tan(2x-3)-x+C$$

Therefore, $\int \tan^2(2x-3)\,dx = \frac{1}{2}\tan(2x-3)-x+C$

22. Find the integral of the function $\sec^2(7-4x)$

Solution:

The given integral is $\int \sec^2(7-4x)\,dx$

Put $t = 7-4x$ so that $dt = -4dx$

Hence the integral can be rewrite as

$$\begin{aligned}\int \sec^2(7-4x)\,dx &= -\frac{1}{4}\int \sec^2(7-4x)(-4dx)\\ &= -\frac{1}{4}\int \sec^2 t\,dt\\ &= -\frac{1}{4}\tan(t)+C\\ &= -\frac{1}{4}\tan(7-4x)+C\end{aligned}$$

Therefore, $\int \sec^2(7-4x)\,dx = -\frac{1}{4}\tan(7-4x)+C$

23. Find the integral of the function $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

Solution:

The given integral is $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Put $t=\sin^{-1}x$, so that $dt = \frac{1}{\sqrt{1-x^2}}dx$

Hence the integral can be rewrite as

$$\begin{aligned}\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx &= \int t\,dt\\ &= \frac{t^2}{2}+C\\ &= \frac{(\sin^{-1}x)^2}{2}+C\end{aligned}$$

Therefore, $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \frac{\left(\sin^{-1} x\right)^2}{2} + C$

24. Find the integral of the function $\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

Solution:

The given integral is $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx$

Put $t = (3\cos x + 2\sin x)$ so that $dt = (2\cos x - 3\sin x) dx$

Hence the integral can be rewrite as

$$\begin{aligned}\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx &= \frac{1}{2}\int \frac{2\cos x - 3\sin x}{3\cos x + 2\sin x} dx \\ &= \frac{1}{2}\int \frac{dt}{t} \\ &= \frac{1}{2}\ln(t) + C \\ &= \frac{1}{2}\ln|3\cos x + 2\sin x| + C\end{aligned}$$

Therefore, $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx = \frac{1}{2}\ln|3\cos x + 2\sin x| + C$

25. Find the integral of the function $\frac{1}{\cos^2 x(1-\tan x)^2}$

Solution:

The given integral is $\int \frac{1}{\cos^2 x(1-\tan x)^2} dx = \int \frac{\sec^2 x}{(1-\tan x)^2} dx$

Put $t = 1 - \tan x$ so that $dt = -\sec^2 x dx$

Hence the integral can be rewrite as

$$\int \frac{1}{\cos^2 x(1-\tan x)^2}dx = -\int \frac{-\sec^2 x}{(1-\tan x)^2}dx$$
$$= -\int \frac{dt}{t^2}$$
$$= -\frac{t^{-1}}{-1}$$
$$= \frac{1}{t}$$
$$= \frac{1}{1-\tan x}$$

Therefore, $\int \frac{1}{\cos^2 x(1-\tan x)^2}dx = \frac{1}{1-\tan x}$

26. Find the integral of the function $\frac{\cos\sqrt{x}}{\sqrt{x}}$

Solution:

The given integral is $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

Put $t = \sqrt{x}$ so that $dt = \frac{1}{2\sqrt{x}}dx$

Hence the integral can be rewrite as

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \frac{\cos\sqrt{x}}{2\sqrt{x}}dx$$
$$= 2\int \cos t dt$$
$$= 2(\sin t) + C$$
$$= 2\sin\left(\sqrt{x}\right) + C$$

Therefore, $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\sin\left(\sqrt{x}\right) + C$

27. Find the integral of the function $\sqrt{\sin 2x}\cos 2x$

Solution:

The given integral is $\int \sqrt{\sin 2x}\cos 2x dx$

Put $t = \sin 2x$, so that $dt = 2\cos 2x$

The integral can be rewrite as

$$\int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{2}\int \sqrt{\sin 2x}\, 2\cos 2x dx$$
$$= \frac{1}{2}\int \sqrt{t} dt$$
$$= \frac{1}{2}\cdot\frac{2}{3}t^{\frac{3}{2}} + C$$
$$= \frac{1}{3}t^{\frac{3}{2}} + C$$
$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

Therefore, $\int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$

28. Find the integral of the function $\dfrac{\cos x}{\sqrt{1+\sin x}}$

Solution:

The given integral is $\int \frac{\cos x}{\sqrt{1+\sin x}} dx$

Put $t = 1+\sin x$, so that $dt = \cos x dx$

The integral can be rewrite as

$$\int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{1+\sin x} + C$$

Therefore, $\int \frac{\cos x}{\sqrt{1+\sin x}} dx = 2\sqrt{1+\sin x} + C$

29. Find the integral of the function $\cot x \log \sin x$

Solution:

The given integral is $\int \cot x \log \sin x dx$

Put $t = \log \sin x$, so that $dt = \frac{1}{\sin x} \cos x dx$

The integral can be rewrite as

$$\begin{aligned}\int \cot x \log \sin x dx &= \int \log \sin x \frac{1}{\sin x} \cos x dx \\ &= \int t dt \\ &= \frac{t^2}{2} + C \\ &= \frac{1}{2}(\log \sin x)^2 + C\end{aligned}$$

Therefore, $\int \cot x \log \sin x dx = \frac{1}{2}(\log \sin x)^2 + C$

30. Find the integral of the function $\frac{\sin x}{1+\cos x}$

Solution:

The given integral is $\int \frac{\sin x}{1+\cos x} dx$

Put $t = 1+\cos x$, so that $dt = -\sin x$

Hence, the given integral becomes

$$\begin{aligned}\int \frac{\sin x}{1+\cos x} dx &= -\int \frac{-\sin x}{1+\cos x} dx \\ &= -\int \frac{dt}{t} \\ &= -\ln(t) + C \\ &= -\ln(1+\cos x) + C\end{aligned}$$

Therefore, $\int \frac{\sin x}{1+\cos x} dx = -\ln(1+\cos x) + C$

31. Find the integral of the function $\frac{\sin x}{(1+\cos x)^2}$

Solution:

The given integral is $\int \frac{\sin x}{(1+\cos x)^2} dx$

Put $t = 1+\cos x$, so that $dt = -\sin x$

Hence, the given integral becomes

$$\int \frac{\sin x}{(1+\cos x)^2} dx = -\int \frac{-\sin x}{(1+\cos x)^2} dx$$
$$= -\int \frac{dt}{t^2}$$
$$= \frac{1}{t} + C$$
$$= \frac{1}{1+\cos x} + C$$

Therefore, $\int \frac{\sin x}{(1+\cos x)^2} dx = \frac{1}{1+\cos x} + C$

32. Find the integral of the function $\frac{1}{1+\cot x}$

Solution:

The given integral is $\int \frac{1}{1+\cot x} dx$

$$\int \frac{1}{1+\cot x} dx = \int \frac{1}{1+\frac{\cos x}{\sin x}} dx$$
$$= \int \frac{\sin x}{\sin x+\cos x} dx$$
$$= \frac{1}{2}\int \frac{2\sin x}{\sin x+\cos x} dx$$
$$= \frac{1}{2}\int \frac{(\sin x+\cos x)+(\sin x-\cos x)}{\sin x+\cos x} dx$$

Put $sinx + cosx = t$, so that $(\cos x - \sin x)dx = dt$

The above integral becomes

$$\frac{1}{2}\int 1 - \left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)dx = \frac{x}{2} - \frac{1}{2}\int \frac{dt}{t} + C$$

$$= \frac{x}{2} - \frac{1}{2}\ln(t) + C$$

$$= \frac{x}{2} - \frac{1}{2}\ln(\sin x + \cos x) + C$$

Therefore, $\int \frac{1}{1+\cot x}dx = \frac{x}{2} - \frac{1}{2}\ln(\sin x + \cos x) + C$

33. Find the integral of the function $\frac{1}{1-\tan x}$

Solution:

The given integral is $\int \frac{1}{1-\tan x}dx$

$$\int \frac{1}{1-\tan x}dx = \int \frac{1}{1-\frac{\sin x}{\cos x}}dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x - \sin x}dx$$

Put $cosx - \sin x = t$, so that $(-\sin x - \cos x)dx = dt$

The above integral becomes

$$\frac{1}{2}\int 1 - \left(\frac{-\cos x - \sin x}{\cos x - \sin x}\right)dx = \frac{x}{2} - \frac{1}{2}\int \frac{dt}{t} + C$$

$$= \frac{x}{2} - \frac{1}{2}\ln(t) + C$$
$$= \frac{x}{2} - \frac{1}{2}\ln(\cos x - \sin x) + C$$

Therefore, $\int \frac{1}{1-\tan x}dx = \frac{x}{2} - \frac{1}{2}\ln(\cos x - \sin x) + C$

34. Find the integral of the function $\frac{\sqrt{\tan x}}{\sin x \cos x}$

Solution:

The given integral is $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$

The above integral rewrite and simplify as below

$$\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx = \int \frac{\sqrt{\tan x}}{\frac{\sin x}{\cos x}\cos^2 x}dx$$
$$= \int \frac{\sqrt{\tan x}\sec^2 x}{\tan x}dx$$
$$= \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$$

Put $\tan x = t$, so that $\sec^2 x dx = dt$

$$\int \frac{\sec^2 x}{\sqrt{\tan x}}dx = \int \frac{dt}{\sqrt{t}}$$
$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{\tan x} + C$$

Therefore, $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx = 2\sqrt{\tan x} + C$

35. Find the integral of the function $\frac{(1+\log x)^2}{x}$

Solution:

The given integral is $\int \frac{(1+\log x)^2}{x} dx$

Put $1+\log x = t$, so that $\frac{1}{x}dx = dt$

Hence,

$$\int \frac{(1+\log x)^2}{x} dx = \int t^2 dt$$
$$= \frac{t^3}{3} + C$$
$$= \frac{(1+\log x)^3}{3} + C$$

Therefore, $\int \frac{(1+\log x)^2}{x} dx = \frac{(1+\log x)^3}{3} + C$

36. Find the integral of the function $\frac{(x+1)(x+\log x)^2}{x}$

Solution:

The given integral is $\int \frac{(x+1)(x+\log x)}{x} dx$

Put $t = x + \log x$, so that $dt = \left(1+\frac{1}{x}\right)dx$

The integral can be rewrite it as

$$\int \frac{(x+1)(x+\log x)}{x} dx = \int (x+\log x)\left(1+\frac{1}{x}\right) dx$$
$$= \int t dt$$
$$= \frac{t^2}{2} + C$$
$$= \frac{(x+\log x)^2}{2} + C$$

Therefore, $\int \frac{(x+1)(x+\log x)}{x} dx = \frac{(x+\log x)^2}{2} + C$

37. Find the integral of the function $\frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8}$

Solution:

The given integral is $\int \frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8} dx$

Put $\tan^{-1} x^4 = t$, so that $\frac{1}{1+x^8}\left(4x^3\right) dx = dt$

The integral becomes

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8} dx = \frac{1}{4}\int \sin t dt$$
$$= \frac{1}{4}(-\cos t) + C$$
$$= \frac{1}{4}\left(-\cos\left(\tan^{-1} x^4\right)\right) + C$$

Therefore, $\int \frac{x^3 \sin\left(\tan^{-1} x^4\right)}{1+x^8} dx = \frac{1}{4}\left(-\cos\left(\tan^{-1} x^4\right)\right) + C$

38. $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx =$

1) $10^x - x^{10} + C$
2) $10^x + x^{10} + C$
3) $\left(10^x + x^{10}\right)^{-1} + C$
4) $\log\left(10^x + x^{10}\right) + C$

Solution:

The given integral is $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$

Put $t = x^{10} + 10^x$, so that $dt = \left(10x^9 + 10^x \log_e 10\right) dx$

The integral becomes

$$\begin{aligned}\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx &= \int \frac{1}{t} dt \\ &= \log(t) + C \\ &= \log\left(x^{10} + 10^x\right) + C\end{aligned}$$

Therefore, the option 4 is correct.

39. $\int \frac{dx}{\sin^2 x \cos^2 x} =$

1) $\tan x + \cot x + C$
2) $\tan x - \cot x + C$
3) $\tan x \cot x + C$
4) $\tan x - \cot 2x + C$

Solution:

The given integral becomes

$$\begin{aligned}\int \frac{dx}{\sin^2 x \cos^2 x} &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) dx \\ &= \int \left(\sec^2 x + \csc^2 x\right) dx \\ &= \tan x - \cot x + C\end{aligned}$$

Therefore, option 2 is correct.

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Exercise: 7.3

1. Find the integral of the function $\sin^2(2x+5)$

Solution:

From trigonometry we have $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\sin^2(2x+5) = \frac{1-\cos 2(2x+5)}{2}$$
$$= \frac{1-\cos(4x+10)}{2}$$

Use the integrals $\int \cos kx dx = \frac{1}{k}\sin kx + C$

Consider the integral

$$\int \sin^2(2x+5)dx = \int \frac{1-\cos(4x+10)}{2}dx$$
$$= \frac{1}{2}\int 1dx - \frac{1}{2}\int \cos(4x+10)dx$$
$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$$
$$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$$

Therefore, $\int \sin^2(2x+5)dx = \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$

2. Find the integral of the function $\sin 3x.\cos 4x$

Solution:

From trigonometry, we have $\sin A\cos B = \frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

Use the integrals $\int \sin kx dx = -\frac{1}{k}\cos kx + C$

Consider the integral

$$\begin{aligned}\int \sin 3x \cos 4x\, dx &= \frac{1}{2}\int \{\sin(3x+4x)+\sin(3x-4x)\}dx \\ &= \frac{1}{2}\int \{\sin 7x + \sin(-x)\}dx \\ &= \frac{1}{2}\int \{\sin 7x - \sin x\}dx \\ &= \frac{1}{2}\left(-\frac{\cos 7x}{7}+\cos x\right)+C \\ &= -\frac{\cos 7x}{14}+\frac{\cos x}{2}+C\end{aligned}$$

Therefore, $\int \sin 3x \cos 4x\, dx = -\frac{\cos 7x}{14}+\frac{\cos x}{2}+C$

3. Find the integral of the function $\cos 2x \cos 4x \cos 6x$

Solution:

From trigonometry we have $\cos A \cos B = \frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

Consider the product $\cos 4x \cos 6x$

$$\begin{aligned}\cos 4x \cos 6x &= \frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\} \\ &= \frac{1}{2}\cos(10x)+\cos(-2x) \\ &= \frac{1}{2}(\cos 10x + \cos 2x)\end{aligned}$$

Hence, the given product can be written as

$$\begin{aligned}\cos 2x\cos 4x\cos 6x &= \frac{1}{2}\left(\cos 2x\cos 10x+\cos 2x\cos 2x\right)\\ &= \frac{1}{4}\left(2\cos 2x\cos 10x+2\cos^2 2x\right)\\ &= \frac{1}{4}\left(\cos 12x+cos\left(-8x\right)+\left(1+\cos 4x\right)\right)\\ &= \frac{1}{4}\left(1+\cos 12x+cos8x+cos4x\right)\end{aligned}$$

Use the integrals $\int \cos kxdx=\frac{1}{k}\sin kx+C$

Consider the integral

$$\begin{aligned}\int \cos 2x\cos 4x\cos 6xdx &= \frac{1}{4}\int 1+\cos 12x+cos8x+cos4xdx\\ &= \frac{1}{4}\left[x+\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+\frac{\sin 4x}{4}\right]+C\\ &= \frac{x}{4}+\frac{\sin 12x}{48}+\frac{\sin 8x}{32}+\frac{\sin 4x}{16}+C\end{aligned}$$

Therefore, $\int \cos 2x\cos 4x\cos 6xdx=\frac{x}{4}+\frac{\sin 12x}{48}+\frac{\sin 8x}{32}+\frac{\sin 4x}{16}+C$

4. Find the integral of the function $\sin^3\left(2x+1\right)$

Solution:

Method 1:

From trigonometry, we have $\sin 3\theta=3\sin\theta-4\sin^3\theta$

$$\sin^3\left(2x+1\right)=\frac{1}{4}\left(3\sin\left(2x+1\right)-\sin\left(6x+3\right)\right)$$

Use the integrals $\int \sin kxdx=-\frac{1}{k}\cos kx+C$

Consider

$$\begin{aligned}\int \sin^3\left(2x+1\right)dx &= \frac{1}{4}\int\left(3\sin\left(2x+1\right)-\sin\left(6x+3\right)\right)dx\\ &= \frac{3}{4}\int \sin\left(2x+1\right)dx-\frac{1}{4}\int \sin\left(6x+3\right)dx\end{aligned}$$

$$=\frac{3}{4}\left(-\frac{\cos(2x+1)}{2}\right)-\frac{1}{4}\left(-\frac{\cos(6x+3)}{6}\right)+C$$

$$=-\frac{3}{8}\cos(2x+1)-\frac{1}{24}\cos(6x+3)+C$$

Therefore, $\int \sin^3(2x+1)\,dx=-\frac{3}{8}\cos(2x+1)-\frac{1}{24}\cos(6x+3)+C$

Method: 2

Consider the integral

$$\begin{aligned}I&=\int \sin^3(2x+1)\\&=\int \sin^2(2x+1).\sin(2x+1)\,dx\\&=\int\left(1-\cos^2(2x+1)\sin(2x+1)\right)dx\\&=\int \sin(2x+1)\,dx-\int \cos^2(2x+1)\cdot\sin(2x+1)\,dx\end{aligned}$$

Suppose that

$$\begin{aligned}\cos(2x+1)&=t\\-2\sin(2x+1)\,dx&=dt\end{aligned}$$

Use the integrals $\int \sin kx dx=-\frac{1}{k}\cos kx+C$

$$\begin{aligned}\int \sin^3(2x+1)\,dx&=\int \sin(2x+1)\,dx+\frac{1}{2}\int \cos^2(2x+1)\cdot\left(-2\sin(2x+1)\right)dx\\&=-\frac{\cos(2x+1)}{2}+c+\frac{1}{2}\int t^2 dt\\&=-\frac{\cos(2x+1)}{2}+\frac{t^3}{6}+C\\&=-\frac{\cos(2x+1)}{2}+\frac{\cos^3(2x+1)}{6}+C\end{aligned}$$

Therefore, $\int \sin^3(2x+1)\,dx=-\frac{1}{2}\cos(2x+1)+\frac{1}{6}\cos^3(2x+1)+C$

5. Find the integral of the function $\sin^3 x \cos^3 x$

Solution:

Consider the integral

$$\int \sin^3 x \cos^3 x dx = \int \sin^2 x \cos^3 x \sin x\, dx$$
$$= \int \cos^3 x\left(1-\cos^2 x\right)\sin x\, dx$$

Put $\cos x = t$, so that $-\sin x dx = dt$

The integral becomes

$$\int \sin^3 x \cos^3 x dx = \int t^3\left(1-t^2\right)\left(-dt\right)$$
$$= -\int \left(t^3 - t^5\right)dt$$
$$= -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$$
$$= -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + C$$

Therefore, $\int \sin^3 x \cos^3 x dx = -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + C$

6. Find the integral of the function $\sin x \sin 2x \sin 3x$

Solution:

From the trigonometry, use the formula $\sin A \sin B = \frac{1}{2}\left\{\cos\left(A-B\right) - \cos\left(A+B\right)\right\}$

The given function is $\sin x \sin 2x \sin 3x$

$$\sin x \sin 2x \sin 3x = \frac{1}{2}\left(\sin x\left(2\sin 2x \sin 3x\right)\right)$$

$$= \frac{1}{2}\left(\sin x\left(\cos x - \cos 5x\right)\right)$$

$$= \frac{1}{2}\left(\sin x \cos x - \sin x \cos 5x\right)$$

$$=\frac{1}{4}(2\sin x\cos x-2\sin x\cos 5x)$$

$$=\frac{1}{4}(\sin 2x-\sin 6x+\sin 4x)$$

Use the integral $\int \sin kx dx=-\frac{1}{k}\cos kx+C$

Consider the integral

$$\int \sin x\sin 2x\sin 3x dx=\frac{1}{4}\int(\sin 2x-\sin 6x+\sin 4x)dx$$

$$=\frac{1}{4}\left(-\frac{\cos 2x}{2}\right)-\frac{1}{4}\left(-\frac{\cos 6x}{6}\right)+\frac{1}{4}\left(-\frac{\cos 4x}{4}\right)+C$$

$$=-\frac{1}{8}\cos 2x+\frac{1}{24}\cos 6x-\frac{1}{16}\cos 4x+C$$

Therefore, $\int \sin x\sin 2x\sin 3x dx=-\frac{1}{8}\cos 2x+\frac{1}{24}\cos 6x-\frac{1}{16}\cos 4x+C$

7. Find the integral of the function $\sin 4x\sin 8x$

Solution:

From trigonometry $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Hence,

$$\sin 4x\sin 8x=\frac{1}{2}(\cos(4x-8x)-\cos(4x+8x))$$

$$=\frac{1}{2}(\cos(-4x)-\cos 12x)$$

$$=\frac{1}{2}(\cos 4x-\cos 8x)$$

Use the integral $\int \cos kx dx=\frac{1}{k}\sin kx+C$

Consider the integral

$$\int \sin 4x \sin 8x dx = \frac{1}{2}\int (\cos 4x - \cos 8x)\, dx$$
$$= \frac{1}{2}\left(\frac{\sin 4x}{4} - \frac{\sin 8x}{8}\right) + C$$
$$= \frac{1}{8}\sin 4x - \frac{1}{16}\sin 8x + C$$

Therefore, $\int \sin 4x \sin 8x dx = \frac{1}{8}\sin 4x - \frac{1}{16}\sin 8x + C$

8. Find the integral of the function $\frac{1-\cos x}{1+\cos x}$

Solution:

From the trigonometric formula $1-\cos x = 2\sin^2\frac{x}{2}, 1+\cos x = 2\cos^2\frac{x}{2}$

Hence,

$$\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}$$
$$= \tan^2\left(\frac{x}{2}\right)$$
$$= \sec^2\left(\frac{x}{2}\right) - 1$$

Consider the integral

$$\int \frac{1-\cos x}{1+\cos x} dx = \int \sec^2\left(\frac{x}{2}\right) - 1 dx \qquad \bullet \int \sec^2(kx)\, dx = \frac{1}{k}\tan(kx) + c$$
$$= 2\tan\left(\frac{x}{2}\right) - x + C$$

9. Find the integral of the function $\frac{\cos x}{1+\cos x}$

Solution:

From trigonometry,

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$$

Hence,

$$\frac{\cos x}{1+\cos x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$= \frac{1}{2} - \frac{1}{2}\tan^2\frac{x}{2}$$

$$= \frac{1}{2} - \frac{1}{2}\left(\sec^2\frac{x}{2} - 1\right)$$

$$= \frac{1}{2} - \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}$$

$$= 1 - \frac{1}{2}\sec^2\frac{x}{2}$$

Consider the integral

$$\int \frac{\cos x}{1+\cos x}dx = \int 1 - \frac{1}{2}\sec^2\frac{x}{2}dx \qquad \bullet \int \sec^2(kx)\,dx = \frac{1}{k}\tan kx + c$$

$$= x - \frac{1}{2}\left(2\tan\frac{x}{2}\right) + C$$

$$= x - \tan\frac{x}{2} + C$$

Therefore, $\int \frac{\cos x}{1+\cos x}dx = x - \tan\frac{x}{2} + C$

10. Find the integral of the function $\sin^4 x$

Solution:

From trigonometry

$$\sin^4 x = \left(\sin^2 x\right)^2$$

$$= \left(\frac{1-\cos 2x}{2}\right)^2$$

$$=\frac{1}{4}\left(1+\cos^2 2x-2\cos 2x\right)$$

$$=\frac{1}{4}\left(1+\frac{1+\cos 4x}{2}-2\cos 2x\right)$$

$$=\frac{1}{8}\left(3+\cos 4x-4\cos 2x\right)$$

Use the integrals $\int \cos kx dx=\frac{1}{k}\sin kx+C$

Consider the integral

$$\int \sin^4 x dx=\frac{1}{8}\int 3+\cos 4x-4\cos 2x dx$$

$$=\frac{3x}{8}+\frac{\sin 4x}{32}-\frac{1}{2}\frac{\sin 2x}{2}+C$$

$$=\frac{3x}{8}+\frac{\sin 4x}{32}-\frac{\sin 2x}{4}+C$$

Therefore, $\int \sin^4 x dx=\frac{3x}{8}+\frac{\sin 4x}{32}-\frac{\sin 2x}{4}+C$

11. Find the integral of the function $\cos^4 2x$

Solution:

From trigonometry

$$\cos^4 2x=\left(\cos^2 2x\right)^2$$

$$=\left(\frac{1+\cos 4x}{2}\right)^2$$

$$=\frac{1}{4}\left(1+\cos^2 4x+2\cos 4x\right)$$

$$=\frac{1}{4}\left(1+\frac{1+\cos 8x}{2}+2\cos 4x\right)$$

$$=\frac{1}{8}\left(3+\cos 8x+2\cos 4x\right)$$

Use the integrals $\int \cos kx dx=\frac{1}{k}\sin kx+C$

Consider the integral

$$\int \cos^4 2x dx = \frac{1}{8}\int 3 + \cos 8x + 2\cos 4x dx$$
$$= \frac{3x}{8} + \frac{\cos 8x}{64} + \frac{1}{4}\frac{\cos 4x}{4} + C$$
$$= \frac{3x}{8} + \frac{\cos 8x}{64} + \frac{\cos 4x}{16} + C$$

Therefore, $\int \cos^4 2x dx = \frac{3x}{8} + \frac{\cos 8x}{64} + \frac{\cos 4x}{16} + C$

12. Find the integral of the function $\frac{\sin^2 x}{1+\cos x}$

Solution:

Consider the integral $\int \frac{\sin^2 x}{1+\cos x} dx$

$$\int \frac{\sin^2 x}{1+\cos x} dx = \int \frac{1-\cos^2 x}{1+\cos x} dx$$
$$= \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x} dx$$
$$= \int (1-\cos x) dx$$
$$= x - \sin x + C$$

Therefore, $\int \frac{\sin^2 x}{1+\cos x} dx = x - \sin x + C$

13. Find the integral of the function $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

Solution:

Use the trigonometry

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}=\frac{2\cos^2 x-1-(2\cos^2\alpha-1)}{\cos x-\cos\alpha}$$
$$=\frac{2(\cos^2 x-\cos^2\alpha)}{\cos x-\cos\alpha}$$
$$=2(\cos x+\cos\alpha)$$

Consider the integral

$$\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx=2\int(\cos x+\cos\alpha)dx$$
$$=2\sin x+2x\cos\alpha+C$$

Therefore, $\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx=2\sin x+2x\cos\alpha+C$

14. Find the integral of the function $\frac{\cos x-\sin x}{1+\sin 2x}$

Solution:

From trigonometry, use the formula $\cos^2 x+\sin^2 x=1, \sin 2x=2\sin x\cos x$

$$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{\cos^2 x+\sin^2 x+2\sin x\cos x}$$
$$=\frac{\cos x-\sin x}{(\cos x+\sin x)^2}$$

Put $t=\cos x+\sin x,$ so that $dt=(\cos x-\sin x)dx$

Hence,

$$\int\frac{\cos x-\sin x}{1+\sin 2x}dx=\int\frac{\cos x-\sin x}{(\cos x+\sin x)^2}dx$$
$$=\int\frac{dt}{t^2}$$
$$=-\frac{1}{t}+C$$
$$=-\frac{1}{\cos x+\sin x}+C$$

Therefore, $\int \frac{\cos x-\sin x}{1+\sin 2x}dx=-\frac{1}{\cos x+\sin x}+C$

15. Find the integral of the function $\tan^3 2x\sec 2x$

Solution:

From trigonometry

$$\begin{aligned}\tan^3 2x\sec 2x &= \tan^2 2x\tan 2x\sec 2x\\ &=\left(\sec^2 2x-1\right)\tan 2x\sec 2x\\ &=\sec^2 2x.\tan 2x\sec 2x-\tan 2x\sec 2x\end{aligned}$$

Consider the integral

$$\int \tan^3 2x\sec 2xdx=\int \sec^2 2x.\tan 2x\sec 2x-\tan 2x\sec 2xdx$$

Put $t=\sec 2x,$ so that $dt=2\sec 2x\tan 2x$

The above integral becomes

$$\begin{aligned}\int \tan^3 2x\sec 2xdx &= \int \sec^2 2x.\tan 2x\sec 2x-\tan 2x\sec 2xdx\\ &=\frac{1}{2}\int \mathrm{t}^2\,dt-\frac{1}{2}\int dt\\ &=\frac{1}{2}\left(\frac{t^3}{3}\right)-\frac{t}{2}+C\\ &=\frac{\sec^3 2x}{6}-\frac{\sec 2x}{2}+C\end{aligned}$$

Therefore, $\int \tan^3 2x\sec 2xdx=\frac{\sec^3 2x}{6}-\frac{\sec 2x}{2}+C$

16. Find the integral of the function $\tan^4 x$

Solution:

$$\begin{aligned}\int \tan^4 xdx &= \int \left(\sec^2 x-1\right)^2 dx\\ &=\int \sec^4 x+1-2\sec^2 xdx\\ &=\int \left(\sec^2 x\right)\left(\sec^2 x\right)+1-2\sec^2 xdx\end{aligned}$$

$$= \int (1+\tan^2 x)(\sec^2 x) + 1 - 2\sec^2 x dx$$
$$= \int \sec^2 x + \tan^2 x \sec^2 x + 1 - 2\sec^2 x dx$$
$$= \int (\tan x)^2 \sec^2 x dx - \int \sec^2 x dx + \int 1 dx$$

Put $\tan x = t$, so that $\sec^2 x dx = dt$

Hence, the above integral becomes

$$\int \tan^4 x dx = \int t^2 dt - \tan x + x + C$$
$$= \frac{t^3}{3} - \tan x + x + C$$
$$= \frac{1}{3}\tan^3 x - \tan x + C$$

Therefore, $\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + C$

17. Find the integral of the function $\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Solution:

Consider the integral

$$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin x}{\cos^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx$$
$$= \int \tan x \sec x dx + \int \cot x \csc x dx$$
$$= \sec x - \csc x + C$$

Therefore, $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \sec x - \csc x + C$

18. Find the integral of the function $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

Solution:

Consider the integral

$$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx = \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x}dx$$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x}dx$$

$$= \int \frac{1}{\cos^2 x}dx$$

$$= \int \sec^2 x dx$$

$$= \tan x + C$$

Therefore, $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx = \tan x + C$

19. Find the integral of the function $\frac{1}{\sin x \cos^3 x}$

Solution:

Consider the integral is $\int \frac{1}{\sin x \cos^3 x}dx$

$$\int \frac{1}{\sin x \cos^3 x}dx = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}dx$$

$$= \int \frac{\sin x}{\cos^3 x}dx + \int \frac{1}{\sin x \cos x}dx$$

$$= \int \tan x \sec^2 x dx + \int \frac{1}{\frac{\sin x}{\cos x}\cos^2 x}dx$$

$$= \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x}dx$$

Put $\tan x = t$, $\sec^2 x dx = dt$

$$\int \frac{1}{\sin x \cos^3 x}dx = \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x}dx$$

$$= \int t dt + \int \frac{1}{t}dt$$

$$= \frac{t^2}{2} + \log|t| + C$$

$$= \frac{\tan^2 x}{2} + \log(\tan x) + C$$

Therefore, $\int \frac{1}{\sin x \cos^3 x} dx = \frac{\tan^2 x}{2} + \log(\tan x) + C$

20. Find the integral of the function $\frac{\cos 2x}{(\cos x + \sin x)^2}$

Solution:

Consider the integral

$$\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx$$
$$= \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2} dx$$
$$= \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)} dx$$

Put $\cos x + \sin x = t$, so that $(-\sin x + \cos x) dx = dt$

The above integral becomes

$$\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)} dx$$
$$= \int \frac{dt}{t}$$
$$= \log|t| + C$$
$$= \log|\cos x + \sin x| + C$$

Therefore, $\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \log|\cos x + \sin x| + C$

21. Find the integral of the function $\sin^{-1}(\cos x)$

Solution:

Consider the integral

$$\int \sin^{-1}(\cos x) dx = \int \sin^{-1}\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx$$ • with restricted domain

$$=\int\left(\frac{\pi}{2}-x\right)dx$$

$$=\frac{\pi x}{2}-\frac{x^2}{2}+C$$

Therefore, $\int \sin^{-1}(\cos x)\,dx=\frac{\pi x}{2}-\frac{x^2}{2}+C$

22. The functional expression for the integral $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$

Solution:

Rewrite the integrand as below.

$$\frac{1}{\cos(x-a)\cos(x-b)}=\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x-b)-(x-a)\right]}{\cos(x-a)\cos(x-b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)\right]}{\cos(x-a)\cos(x-b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x-b)-\tan(x-a)\right]$$

Use the integral $\int \tan x dx=-\log(\cos x)+c$

The given integral becomes

$$\int\frac{1}{\cos(x-a)\cos(x-b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x-b)-\tan(x-a)\right]dx$$

$$=\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x-b)\right|+\log\left|\cos(x-a)\right|\right]$$

$$=\frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Therefore, $\int\frac{1}{\cos(x-a)\cos(x-b)}dx=\frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$

23. The functional expression $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to

A) $\tan x + \cot x + C$ B) $\tan x + \operatorname{cosec} x + C$

C) $-\tan x + \cot x + C$ D) $\tan x + \sec x + C$

Solution:

The given integral is

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx$$

$$= \int \left(\sec^2 x - \cos ec^2 x \right) dx$$

$$= \tan x + \cot x + C$$

Thus, the correct answer is (A)

24. The functional expression for the $\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ equals

A) $-\cot\left(xe^x\right) + C$ B) $\tan\left(xe^x\right) + C$ C) $\tan\left(e^x\right) + C$ D) $\cot\left(e^x\right) + C$

Solution:

The given integral is $\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$

Put $xe^x = t$, so that $\left(xe^x + e^x\right) dx = dt \Rightarrow e^x (x+1) dx = dt$

Hence, the integral becomes

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx = \int \frac{dt}{\cos^2 t} dt$$

$$= \int \sec^2 t dt$$

$$= \tan t + C$$

$$= \tan\left(xe^x\right) + C$$

Therefore, the option (B) is correct.

Exercise: 7.4

1. Integrate the function $\frac{3x^2}{1+x^6}$

Solution: The given integral is $\int \frac{3x^2}{1+x^6}dx$

Put $t = x^3$ so that $dt = 3x^2 dx$

Hence,

$$\int \frac{3x^2}{1+x^6}dx = \int \frac{dt}{1+t^2}$$
$$= \tan^{-1}(t) + C$$
$$= \tan^{-1}(x^3) + C$$

Therefore, $\int \frac{3x^2}{1+x^6}dx = \tan^{-1}(x^3) + C$

2. Integrate the function $\frac{1}{\sqrt{1+4x^2}}$

Solution: The given integral is $\int \frac{1}{\sqrt{1+4x^2}}dx$

Put $t = 2x$ so that $dt = 2dx$

Hence,

$$\int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{2dx}{\sqrt{1+4x^2}}$$
$$= \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$
$$= \frac{1}{2}\tan^{-1}(t) + C$$
$$= \frac{1}{2}\tan^{-1}(2x) + C$$

Therefore, $\int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\tan^{-1}(2x) + C$

3. Integrate the function $\frac{1}{\sqrt{(2-x)^2+1}}$

Solution: The given integral is $\int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{-1}{\sqrt{(2-x)^2+1}}dx$

Put $t = 2-x$ so that $dt = -dx$

Hence,

$$\int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{-1}{\sqrt{(2-x)^2+1}}dx$$

$$= -\int \frac{dt}{\sqrt{t^2+1}} \qquad \bullet \int \frac{1}{\sqrt{1+x^2}}dx = \log\left(x+\sqrt{1+x^2}\right)+C$$

$$= -\log\left((2-x)+\sqrt{(2-x)^2+1}\right)+C$$

$$= \log\left(\frac{1}{2-x+\sqrt{x^2-4x+5}}\right)+C$$

Therefore, $\int \frac{1}{\sqrt{(2-x)^2+1}}dx = \log\left(\frac{1}{2-x+\sqrt{x^2-4x+5}}\right)$

4. Integrate the function $\frac{1}{\sqrt{9-25x^2}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{9-25x^2}}dx$

Put $t = \frac{5x}{3}$ so that $dt = \frac{5}{3}dx$

Hence,

$$\int \frac{1}{\sqrt{9-25x^2}}dx = \frac{3}{5}\int \frac{1}{3\sqrt{1-\left(\frac{5}{3}x\right)^2}}\frac{5}{3}dx$$

$$= \frac{1}{5}\int \frac{1}{\sqrt{1-t^2}}dt$$

$$= \frac{1}{5}\sin^{-1}t + C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Therefore, $\int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$

5. Integrate the function $\frac{3x}{1+2x^4}$

Solution:

The given integral is $\int \frac{3x}{1+2x^4}dx$

Put $t = \sqrt{2}x^2$ so that $dt = 2\sqrt{2}xdx$

Hence,

$$\int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{2\sqrt{2}dx}{1+\left(\sqrt{2}x^2\right)^2}$$

$$= \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}t + C$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Therefore, $\int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$

6. Integrate the function $\frac{x^2}{1-x^6}$

Solution:

The given integral is $\int \frac{x^2}{1-x^6} dx$

Put $t = x^3$ so that $dt = 3x^2 dx$

Hence,

$$\int \frac{x^2}{1-x^6} dx = \frac{1}{3}\int \frac{3x^2}{1-\left(x^3\right)^2} dx$$
$$= \frac{1}{3}\int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\log\left(\frac{1+t}{1-t}\right) + C$$
$$= \frac{1}{3}\log\left(\frac{1+x^3}{1-x^3}\right) + C$$

Therefore, $\int \frac{x^2}{1-x^6} dx = \frac{1}{3}\log\left(\frac{1+x^3}{1-x^3}\right) + C$

7. Integrate the function $\frac{x-1}{\sqrt{x^2-1}}$

Solution:

The given integral is $\int \frac{x-1}{\sqrt{x^2-1}} dx$

Put $t = x^2 - 1$ so that $dt = 2x dx$

Hence,

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{t}} dt - \int \frac{1}{\sqrt{x^2-1}} dx$$
$$= \frac{1}{2}\left(2\sqrt{t}\right) - \log\left(x + \sqrt{x^2-1}\right) + C$$
$$= \sqrt{1-x^2} - \log\left|x + \sqrt{x^2-1}\right| + C$$

Therefore, $\int \frac{x-1}{\sqrt{x^2-1}}dx = \sqrt{1-x^2} - \log\left|x+\sqrt{x^2-1}\right| + C$

8. Integrate the function $\frac{x^2}{\sqrt{x^6+a^6}}$

Solution:

The given integral is $\int \frac{x^2}{\sqrt{x^6+a^6}}dx$

Put $t = x^3$ so that $dt = 3x^2dx$

Hence,

$$\int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{\left(x^3\right)^2+\left(a^3\right)^2}}dx$$
$$= \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$
$$= \frac{1}{3}\log\left(t+\sqrt{t^2+a^6}\right)+C$$
$$= \frac{1}{3}\log\left(x^3+\sqrt{x^6+a^6}\right)+C$$

Therefore, $\int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\log\left(x^3+\sqrt{x^6+a^6}\right)+C$

9. Integrate the function $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Solution:

The given integral is $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx$

Put $t = \tan x$ so that $dt = \sec^2 x dx$

Hence,

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log\left(t + \sqrt{t^2 + 4}\right) + C$$

$$= \log\left(\tan x + \sqrt{\tan^2 x + 4}\right) + C$$

Therefore, $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \log\left(\tan x + \sqrt{\tan^2 x + 4}\right)$

10. Integrate the function $\frac{1}{\sqrt{x^2 + 2x + 2}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{x^2 + 2x + 2}} \cdot dx = \int \frac{1}{\sqrt{(x+1)^2 + 1}} \cdot dx$

Put $t = x + 1$ so that $dt = dx$

Hence,

$$\int \frac{1}{\sqrt{(x+1)^2 + 1}} \cdot dx = \int \frac{1}{\sqrt{t^2 + 1}} \cdot dx$$

$$= \log\left(t + \sqrt{t^2 + 1}\right) + C$$

$$= \log\left(x + 1 + \sqrt{x^2 + 2x + 2}\right) + C$$

Therefore, $\int \frac{1}{\sqrt{(x+1)^2 + 1}} \cdot dx = \log\left(x + 1 + \sqrt{x^2 + 2x + 1}\right)$

11. Integrate the function $\frac{1}{9x^2 + 6x + 5}$

Solution:

The given integral is $\int \frac{1}{9x^2 + 6x + 5} dx$

$$\int \frac{1}{9x^2 + 6x + 5} dx = \int \frac{1}{(3x+1)^2 + 2^2} dx$$

Put, $3x+1=t$, so that $3dx=dt$

Hence,

$$\int \frac{1}{9x^2+6x+5}dx = \frac{1}{3}\int \frac{3}{(3x+1)^2+2^2}dx$$
$$= \frac{1}{3}\int \frac{dt}{t^2+2^2}$$
$$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{t}{2}\right)+C$$
$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

Therefore, $\int \frac{1}{9x^2+6x+5}dx = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$

12. Integrate the function $\frac{1}{\sqrt{7-6x+x^2}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{7-6x+x^2}}dx$

Put $t=x^3-1$ so that $dt=3x^2dx$

Hence,

$$\int \frac{1}{\sqrt{7-6x+x^2}}dx = \int \frac{1}{\sqrt{(x-3)^2-2}}dx$$
$$= \int \frac{1}{\sqrt{(x-3)^2-\left(\sqrt{2}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left(x+\sqrt{x^2-a^2}\right)+c$

$$\int \frac{1}{\sqrt{(x-3)^2-\left(\sqrt{2}\right)^2}}dx = \log\left(x-3+\sqrt{x^2-6x+7}\right)+C$$

Therefore, $\int \frac{1}{\sqrt{7-6x+x^2}}dx = \log\left(x-3+\sqrt{x^2-6x+7}\right)+C$

13. Integrate the function $\frac{1}{\sqrt{(x-1)(x-2)}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \int \frac{1}{\sqrt{x^2-3x+2}}dx$

The quadratic expression x^2-3x+2 can be written as below.

$$\begin{aligned} x^2-3x+2 &= \left(x-\frac{3}{2}\right)^2+2-\frac{9}{4} \\ &= \left(x-\frac{3}{2}\right)^2-\frac{1}{4} \\ &= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2 \end{aligned}$$

Hence,

$$\begin{aligned} \int \frac{1}{\sqrt{(x-1)(x-2)}}dx &= \int \frac{1}{\sqrt{x^2-3x+2}}dx \\ &= \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx \\ &= \log\left(x-\frac{3}{2}+\sqrt{x^2-3x+2}\right)+C \end{aligned}$$

Therefore, $\int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \log\left(x-\frac{3}{2}+\sqrt{x^2-3x+2}\right)+C$

14. Find the integral of the function $\frac{1}{\sqrt{8+3x-x^2}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{8+3x-x^2}}dx$

The quadratic expression $8+3x-x^2$ can be written as

$$8+3x-x^2=\left(8+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2$$
$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

Hence, the integral becomes

$$\int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$

It gives

$$\int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}dx$$
$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$
$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Therefore, $\int \frac{1}{\sqrt{8+3x-x^2}}dx=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$

15. Integrate the function $\frac{1}{\sqrt{(x-a)(x-b)}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx$

The quadratic expression can be written as

$$(x-a)(x-b) = x^2 - (a+b) + ab$$
$$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 + \left(ab - \left(\frac{a+b}{2}\right)^2\right)$$

Hence, the integral can be rewrite as

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}dx = \int \frac{1}{\sqrt{\left(x-\left(\frac{a+b}{2}\right)\right)^2 + \left(ab-\left(\frac{a+b}{2}\right)^2\right)}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left(x+\sqrt{x^2+a^2}\right)+C$

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}dx = \int \frac{1}{\sqrt{\left(x-\left(\frac{a+b}{2}\right)\right)^2 + \left(ab-\left(\frac{a+b}{2}\right)^2\right)}}dx$$
$$= \log\left(x - \frac{a+b}{2} + \sqrt{(x-a)(x-b)}\right) + C$$

Therefore, $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx = \log\left(x - \frac{a+b}{2} + \sqrt{(x-a)(x-b)}\right) + C$

16. Find the integral of the function $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Solution:

The given integral is $\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx$

Put $t = 2x^2 + x - 3$, so that $dt = (4x+1)dx$

Hence, the given integral becomes

$$\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{dt}{\sqrt{t}}$$
$$= 2\sqrt{t} + C$$
$$= 2\sqrt{2x^2+x-3} + C$$

Therefore, $\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = 2\sqrt{2x^2+x-3} + C$

17. Find the integral of the function $\frac{x+2}{\sqrt{x^2-1}}$

Solution:

The given integral is $\int \frac{x+2}{\sqrt{x^2-1}}dx$

Put $t = x^2$, so that $dt = 2xdx$

Hence the integral can be rewrite as

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx$$
$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} + 2\int \frac{1}{\sqrt{x^2-1}}dx$$
$$= \sqrt{t} + 2\sin^{-1}x + C$$
$$= \sqrt{x^2-1} + 2\sin^{-1}x + C$$

Therefore, $\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left(x+\sqrt{x^2-1}\right) + C$

18. Find the integral of the function $\frac{5x-2}{1+2x+3x^2}$

Solution:

The given integral is $\int \frac{5x-2}{1+2x+3x^2}dx$

Suppose that $5x-2 = P\left(\frac{d}{dx}\left(1+2x+3x^2\right)\right) + Q$

It gives $5x-2=P(6x+2)+Q$

Comparing coefficient of x from both sides

$$6P=5\Rightarrow P=\frac{5}{6}$$

Comparing the constant terms from both sides

$$-2=2P+Q$$

$$2\left(\frac{5}{6}\right)+Q=-2$$

$$Q=-2-\frac{5}{3}$$

$$=-\frac{11}{3}$$

Hence, the given integral becomes

$$\int\frac{5x-2}{1+2x+3x^2}dx=\int\frac{\frac{5}{6}\left(\frac{d}{dx}\left(3x^2+2x+1\right)\right)-\frac{11}{3}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int\frac{6x+2}{1+2x+3x^2}dx-\frac{11}{3}\int\frac{1}{3\left(x^2+\frac{2}{3}x+\frac{1}{3}\right)}dx$$

Use the formula $\int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c$ to find the first integral

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\int\frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{1}{3}-\frac{1}{9}}dx$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\int\frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}dx$$

Use the formula $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\left(\frac{3}{\sqrt{2}}\right)\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)+C$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

Therefore, $\int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}\log\left|3x^2+2x+1\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$

19. Find the integral of the function $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

Solution:

The given integral is $\int\frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$

Suppose that $6x+7=P\left(\frac{d}{dx}\left(x^2-9x+20\right)\right)+Q$

It gives $6x+7=P(2x-9)+Q$

Comparing coefficient of x from both sides

$2P=6\Rightarrow P=3$

Comparing the constant terms from both sides

$$\begin{aligned}7&=-9P+Q\\-9(3)+Q&=7\\Q&=7+27\\&=34\end{aligned}$$

Hence, the given integral becomes

$$\int\frac{6x+7}{\sqrt{(x-5)(x-4)}}dx=\int\frac{3\left(\frac{d}{dx}\left(x^2-9x+20\right)\right)+34}{\sqrt{x^2-9x+20}}dx$$

$$=\int \frac{3(2x-9)}{\sqrt{x^2-9x+20}}dx+\int \frac{34}{\sqrt{\left(x-\frac{9}{2}\right)^2+20-\frac{81}{4}}}dx$$

Use the formula $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}+c$ to find the first integral

$$=6\sqrt{x^2-9x+20}+\int \frac{34}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2-a^2}}dx=\log\left(x+\sqrt{x^2-a^2}\right)+C$ to find the second integral

$$=6\sqrt{x^2-9x+20}+34\log\left(x-\frac{9}{2}+\sqrt{x^2-9x+20}\right)+C$$

Therefore,

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx=6\sqrt{x^2-9x+20}+34\log\left(x-\frac{9}{2}+\sqrt{x^2-9x+20}\right)+C$$

20. Find the integral of the function $\frac{x+2}{\sqrt{4x-x^2}}$

Solution:

The given integral is $\int \frac{x+2}{\sqrt{4x-x^2}}dx$

Suppose that $x+2=P\left(\frac{d}{dx}\left(4x-x^2\right)\right)+Q$

It gives $x+2=P(4-2x)+Q$

Comparing coefficient of x from both sides

$$-2P=1 \Rightarrow P=-\frac{1}{2}$$

Comparing the constant terms from both sides

$$2 = 4P + Q$$

$$4\left(-\frac{1}{2}\right) + Q = 2$$

$$Q = 2 + 2$$
$$= 4$$

Hence, the given integral becomes

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}\left(\frac{d}{dx}\left(4x-x^2\right)\right)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + \int \frac{4}{\sqrt{4x-x^2}}dx$$

Use the formula $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$ to find the first integral

$$= -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + \int \frac{4}{\sqrt{2^2-(x-2)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ to find the second integral

$$= -\sqrt{4x-x^2} + \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Therefore, $\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\sqrt{4x-x^2} + \sin^{-1}\left(\frac{x-2}{2}\right) + C$

21. Find the integral of the function $\frac{x+2}{\sqrt{x^2+2x+3}}$

Solution:

The given integral is $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$

Suppose that $x + 2 = P\left(\frac{d}{dx}\left(x^2+2x+3\right)\right) + Q$

It gives $x+2=P(2x+2)+Q$

Comparing coefficient of x from both sides

$$2P=1 \Rightarrow P=\frac{1}{2}$$

Comparing the constant terms from both sides

$$2=2P+Q$$
$$2=2\left(\frac{1}{2}\right)+Q$$
$$Q=2-1$$
$$=1$$

Hence, the given integral becomes

$$\frac{x+2}{\sqrt{x^2+2x+3}}=\int \frac{\frac{1}{2}\left(\frac{d}{dx}\left(x^2+2x+3\right)\right)+1}{\sqrt{x^2+2x+3}}dx$$

$$=\int \frac{\frac{1}{2}(2x+2)}{\sqrt{x^2+2x+3}}dx+\int \frac{1}{\sqrt{(x+1)^2+3-1}}dx$$

Use the formula $\int \frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c$ to find the first integral

$$=\frac{1}{2}\left(2\sqrt{x^2+2x+3}\right)+\int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2+a^2}}dx=\log\left(x+\sqrt{x^2+a^2}\right)+C$ to find the second integral

$$=\sqrt{x^2+2x+3}+\log\left(x+1+\sqrt{x^2+2x+3}\right)+C$$

Therefore, $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx=\sqrt{x^2+2x+3}+\log\left(x+1+\sqrt{x^2+2x+3}\right)+C$

22. Find the integral of the function $\frac{x+3}{x^2-2x-5}$

Solution:

The given integral is $\int \frac{x+3}{x^2-2x-5}dx$

Suppose that $x+3=P\left(\frac{d}{dx}\left(x^2-2x-5\right)\right)+Q$

It gives $x+3=P(2x-2)+Q$

Comparing coefficient of x from both sides

$$2P=1 \Rightarrow P=\frac{1}{2}$$

Comparing the constant terms from both sides

$$3=-2P+Q$$

$$-2\left(\frac{1}{2}\right)+Q=3$$

$$Q=3+1$$

$$=4$$

Hence, the given integral becomes

$$\int \frac{x+3}{x^2-2x-5}dx=\int \frac{\frac{1}{2}\left(\frac{d}{dx}\left(x^2-2x-5\right)\right)+4}{x^2-2x-5}dx$$

$$=\frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx+4\int \frac{1}{(x-1)^2-5-1}dx$$

Use the formula $\int \frac{f'(x)}{f(x)}dx=\log\left|f(x)\right|+c$ to find the first integral

$$\frac{1}{2}\log\left|x^2-2x-5\right|+4\int \frac{1}{(x-1)^2-\left(\sqrt{6}\right)^2}dx$$

Use the formula $\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+C$

$$=\frac{1}{2}\log\left|x^2-2x-5\right|+\frac{4}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right)+C$$

$$=\frac{1}{2}\log\left|x^2-2x-5\right|+\frac{2}{\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right)+C$$

Therefore, $\int\frac{x+3}{x^2-2x-5}dx=\frac{1}{2}\log\left|x^2-2x-5\right|+\frac{2}{\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right)+C$

23. Find the integral of the function $\frac{5x+3}{\sqrt{x^2+4x+10}}$

Solution:

The given integral is $\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx$

Suppose that $5x+3=P\left(\frac{d}{dx}\left(x^2+4x+10\right)\right)+Q$

It gives $5x+3=P(2x+4)+Q$

Comparing coefficient of x from both sides

$$2P=5\Rightarrow P=\frac{5}{2}$$

Comparing the constant terms from both sides

$$\begin{aligned}3&=4P+Q\\&=4\left(\frac{5}{2}\right)+Q\\&=10+Q\\Q&=-7\end{aligned}$$

Hence, the given integral becomes

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}\left(\frac{d}{dx}\left(x^2+4x+10\right)\right)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{(2x+4)}{\sqrt{x^2+4x+10}}dx - \int \frac{7}{\sqrt{(x+2)^2+6}}dx$$

Use the formula $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}+c$ to find the first integral

$$= \frac{5}{2}\left(2\sqrt{x^2+4x+10}\right) - 7\int \frac{1}{\sqrt{(x+2)^2+\left(\sqrt{6}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left(x+\sqrt{x^2+a^2}\right)+C$ to find the second integral

$$= 5\left(\sqrt{x^2+4x+10}\right) - 7\log\left|x+2+\sqrt{x^2+4x+10}\right|+C$$

Therefore, $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\left(\sqrt{x^2+4x+10}\right) - 7\log\left|x+2+\sqrt{x^2+4x+10}\right|+C$

24. $\int \frac{dx}{x^2+2x+2}$ equals

A) $x\tan^{-1}(x+1)+C$

B) $\tan^{-1}(x+1)+C$

C) $(x+1)\tan^{-1}x+C$

D) $\tan^{-1}x+C$

Solution:

The given integral is $\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1}$

Use the formula $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

Hence, $\int \frac{dx}{(x+1)^2+1} = \tan^{-1}(x+1)+C$

Therefore, the option (B), is correct.

25. $\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution:

The given integral is $\int \frac{dx}{\sqrt{9x-4x^2}} = \int \frac{dx}{2\sqrt{\frac{9}{4}x-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}$

Use the formula $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+C$

Hence,

$$\int \frac{dx}{\sqrt{9x-4x^2}} = \int \frac{dx}{2\sqrt{\frac{9}{4}x-x^2}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Therefore, option (B) is correct.

Exercise: 7.5

1. Find the integral of the function $\frac{x}{(x+1)(x+2)}$

Solution:

The integrand is $\frac{x}{(x+1)(x+2)}$,

- It has two linear factors in the denominator
- It is proper fraction

Suppose that $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

Simplifying, we get

$$x=A(x+2)+B(x+1) \qquad \ldots\ldots (1)$$

Plug in $x=-2$ in the equation (1),

$$-2=B(-1)$$
$$B=2$$

Plug in $x=-1$ in the equation (1),

$$-1=A(1)$$
$$A=-1$$

Hence, $\frac{x}{(x+1)(x+2)}=\frac{-1}{(x+1)}+\frac{2}{(x+2)}$

Consider the integral

$$\int \frac{x}{(x+1)(x+2)}dx = \int \frac{-1}{(x+1)}dx + \int \frac{2}{(x+2)}dx$$
$$= -\log(x+1) + 2\log(x+2) + C$$
$$= \log\frac{(x+2)^2}{|x+1|} + C$$

Therefore, $\int \frac{x}{(x+1)(x+2)}dx = \log\frac{(x+2)^2}{|x+1|} + C$

2. Find the integral of the function $\frac{1}{x^2-9}$

Solution:

The integrand is $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)}$,

- It has two linear factors in the denominator
- It is proper fraction

Suppose that $\frac{1}{(x-3)(x+3)} = \frac{A}{(x-3)} + \frac{B}{(x+3)}$

It implies

$$1 = A(x+3) + B(x-3) \qquad \text{….. (1)}$$

Plug in $x = -3$ in equation (1)

$$1 = A(0) + B(-6)$$
$$B = -\frac{1}{6}$$

Plug in $x = 3$ in equation (1)

$$1 = A(6) + B(0)$$
$$A = \frac{1}{6}$$

Consider the integral

$$\int \frac{1}{x^2-9}dx = \int \frac{1}{(x-3)(x+3)}dx$$
$$= \frac{1}{6}\int \frac{1}{x+3} - \frac{1}{x-3}dx$$
$$= \frac{1}{6}\left[\log|x+3| - \log|x-3|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{x+3}{x-3}\right| + C$$

Therefore, $\int \frac{1}{x^2-9}dx = \frac{1}{6}\log\left|\frac{x+3}{x-3}\right| + C$

3. Find the integral of the function $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Solution:

The integrand is $\frac{3x-1}{(x-1)(x-2)(x-3)}$

- It has three linear factors in the denominator
- It is proper fraction

Suppose that $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

It implies

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \ldots\ldots (1)$$

Plug in $x=1$ in equation (1)

$$2 = A(-1)(-2)$$
$$A = \frac{2}{2}$$
$$= 1$$

Plug in $x=2$ in equation (1)

$$5 = A(0) + B(1)(-1)$$
$$B = -5$$

Plug in $x = 3$ in equation (1)

$$8 = A(0) + B(0) + C(2)(1)$$
$$C = \frac{8}{2}$$
$$= 4$$

Consider the integral

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \frac{2}{x-1} - \frac{5}{x-2} + \frac{4}{x-3} dx$$
$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

Therefore, $\int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

4. Find the integral of the function $\frac{x}{(x-1)(x-2)(x-3)}$

Solution:

The integrand is $\frac{x}{(x-1)(x-2)(x-3)}$

- It has three linear factors in the denominator
- It is proper fraction

Suppose that $\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

It implies

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \text{..... (1)}$$

Plug in $x = 1$ in equation (1)

$$1 = A(-1)(-2)$$
$$A = \frac{1}{2}$$

Plug in $x = 2$ in equation (1)

$$2 = A(0) + B(1)(-1)$$
$$B = -2$$

Plug in $x = 3$ in equation (1)

$$3 = A(0) + B(0) + C(2)(1)$$
$$C = \frac{3}{2}$$

Consider the integral

$$\int \frac{x}{(x-1)(x-2)(x-3)} dx = \int \frac{\frac{1}{2}}{x-1} - \frac{2}{x-2} + \frac{\frac{3}{2}}{x-3} dx$$
$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$$

Therefore, $\int \frac{x}{(x-1)(x-2)(x-3)} dx = \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$

5. Find the integral of the function $\frac{2x}{x^2+3x+2}$

Solution:

The integrand is $\frac{2x}{x^2+3x+2} = \frac{2x}{(x+1)(x+2)}$

- It has two linear factors in the denominator
- It is proper fraction

Suppose that $\frac{2x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

It implies

$$2x = A(x+2) + B(x+1) \qquad \text{..... (1)}$$

Plug in $x = -1$ in equation (1)

$$-2 = A(1)$$
$$A = -2$$

Plug in $x=-2$ in equation (1)

$$-4=A(0)+B(-1)$$
$$B=4$$

Consider the integral

$$\int \frac{2x}{x^2+3x+2}dx=\int \frac{-2}{x+1}+\frac{4}{x+2}dx$$
$$=-2\log|x+1|+4\log|x+2|+C$$

Therefore, $\int \frac{2x}{x^2+3x+2}dx=-2\log|x+1|+4\log|x+2|+C$

6. Find the integral of the function $\frac{1-x^2}{x(1-2x)}$

Solution:

The integrand is $\frac{1-x^2}{x(1-2x)}=\frac{x^2-1}{(x)(2x-1)}$

- It has two linear factors in the denominator
- It is improper fraction

Suppose that $\frac{x^2-1}{(x)(2x-1)}=\frac{1}{2}+\frac{A}{x}+\frac{B}{2x-1}$

It implies

$$x^2-1=x(2x-1)+A(2x-1)+B(x) \quad \text{….. (1)}$$

Plug in $x=0$ in equation (1)

$$-1=A(-1)$$
$$A=1$$

Plug in $x=\frac{1}{2}$ in equation (1)

$$\frac{1}{4}-1=A(0)+B\left(\frac{1}{2}\right)$$

$$-\frac{3}{4}=B\left(\frac{1}{2}\right)$$

$$B=-\frac{3}{2}$$

Consider the integral

$$\int \frac{x^2-1}{(x)(2x-1)}dx=\int \frac{1}{2}+\frac{1}{x}+\frac{-\frac{3}{2}}{2x-1}dx$$

$$=\frac{x}{2}+\log|x|-\frac{3}{2}\log|2x-1|\left(\frac{1}{2}\right)+C$$

$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|2x-1|+C$$

Therefore, $\int \frac{x^2-1}{(x)(2x-1)}dx == \frac{x}{2}+\log|x|-\frac{3}{4}\log|2x-1|+C$

7. Find the integral of the function $\frac{x}{(x^2+1)(x-1)}$

Solution:

The integrand is $\frac{x}{(x^2+1)(x-1)}$

- It has one linear factor and one irreducible quadratic factor in the denominator
- It is proper fraction

Suppose that $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$

It implies

$$x=(Ax+B)(x-1)+C(x^2+1) \qquad \text{….. (1)}$$

Plug in $x=1$ in equation (1)

$$1 = C(2)$$
$$C = \frac{1}{2}$$

Compare the coefficient of x^2

$$0 = A + C$$
$$A = -C$$
$$= -\frac{1}{2}$$

Compare the constant terms

$$0 = -B + C$$
$$B = C$$
$$= \frac{1}{2}$$

Consider the integral

$$\int \frac{x}{(x^2+1)(x-1)}dx = \int \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x-1}dx$$
$$= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x-1}{x^2+1}dx$$
$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$
$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

Therefore, $\int \frac{x}{(x^2+1)(x-1)}dx = \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$

8. Find the integral of the function $\frac{x}{(x-1)^2(x+2)}$

Solution:

The integrand is $\frac{x}{(x-1)^2(x+2)}$

- It has one linear factors and one repeated linear factor in the denominator
- It is proper fraction

Suppose that $\frac{x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2}$

It implies

$$x=A(x-1)(x+2)+B(x+2)+C(x-1)^2 \qquad \ldots..(1)$$

Plug in $x=1$ in equation (1)

$$1=B(3)$$
$$B=\frac{1}{3}$$

Plug in $x=-2$ in equation (1)

$$-2=C(9)$$
$$C=-\frac{2}{9}$$

Compare the coefficient of x^2 in the equation (1)

$$0=A+C$$
$$A=-C$$
$$=\frac{2}{9}$$

Consider the integral

$$\int\frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\int\frac{1}{x-1}dx+\frac{1}{3}\int\frac{1}{(x-1)^2}dx-\frac{2}{9}\int\frac{1}{x+2}dx$$
$$=\frac{2}{9}\log|x-1|-\frac{1}{3}\frac{1}{(x-1)}-\frac{2}{9}\log|x+2|+C$$
$$=\frac{2}{9}\log\left(\frac{x-1}{x+2}\right)-\frac{1}{3(x-1)}+C$$

Therefore, $\int\frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\log\left(\frac{x-1}{x+2}\right)-\frac{1}{3(x-1)}+C$

9. Find the integral of the function $\frac{3x+5}{x^3-x^2-x+1}$

Solution:

The integrand is $\frac{3x+5}{x^2-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$

- It has one linear factors and one repeated linear factor in the denominator
- It is proper fraction

Suppose that $\frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$

It implies

$$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2 \qquad \text{….. (1)}$$

Plug in $x=1$ in equation (1)

$$8=B(2)$$
$$B=4$$

Plug in $x=-1$ in equation (1)

$$2=C(4)$$
$$C=\frac{1}{2}$$

Compare the coefficient of x^2 in the equation (1)

$$0=A+C$$
$$A=-C$$
$$=-\frac{1}{2}$$

Consider the integral

$$\int \frac{3x+5}{(x-1)^2(x+1)}dx = -\frac{1}{2}\int \frac{1}{x-1}dx + 4\int \frac{1}{(x-1)^2}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$

$$= -\frac{1}{2}\log|x-1| + 4\frac{1}{(x-1)} + \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) - \frac{4}{(x-1)} + C$$

Therefore, $\int \frac{3x+5}{(x-1)^2(x+1)}dx = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) - \frac{4}{(x-1)} + C$

10. Find the integral of the function $\frac{2x-3}{(x^2-1)(2x+3)}$

Solution:

The integrand is $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)}$

- It has three linear factors in the denominator
- It is proper fraction

Suppose that $\frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$

It implies

$$2x-3 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1) \qquad \text{..... (1)}$$

Plug in $x=1$ in equation (1)

$$-1 = A(10)$$
$$A = -\frac{1}{10}$$

Plug in $x=-1$ in equation (1)

$$-5 = B(-2)$$
$$B = \frac{5}{2}$$

Plug in $x=-\frac{3}{2}$ in equation (1)

$$-6=C\left(\frac{5}{4}\right)$$
$$C=-\frac{24}{5}$$

Consider the integral

$$\int\frac{2x-3}{(x-1)(x+1)(2x+3)}dx=-\frac{1}{10}\int\frac{1}{x-1}dx+\frac{5}{2}\int\frac{1}{(x+1)}dx-\frac{24}{5}\int\frac{1}{2x+3}dx$$
$$=-\frac{1}{10}\log|x-1|+\frac{5}{2}\log|x+1|-\frac{24}{5}\log|2x+3|\left(\frac{1}{2}\right)+C$$
$$=\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+C$$

Therefore, $\int\frac{2x-3}{(x-1)(x+1)(2x+3)}dx=\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+C$

11. Find the integral of the function $\frac{5x}{(x+1)(x^2-4)}$

Solution:

The integrand is $\frac{5x}{(x+1)(x^2-4)}=\frac{5x}{(x+1)(x-2)(x+2)}$

- It has three linear factors in the denominator
- It is proper fraction

Suppose that $\frac{5x}{(x+1)(x-2)(x+2)}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x+2}$

It implies

$$5x=A(x-2)(x+2)+B(x+1)(x+2)+C(x+1)(x-2) \quad \ldots.. (1)$$

Plug in $x=-1$ in equation (1)

$$-5 = A(-3)$$

$$A = \frac{5}{3}$$

Plug in $x = 2$ in equation (1)

$$10 = B(12)$$

$$B = \frac{5}{6}$$

Plug in $x = -2$ in equation (1)

$$-10 = C(-1)(-4)$$

$$C = -\frac{10}{4} = -\frac{5}{2}$$

Consider the integral

$$\int \frac{5x}{(x+1)(x^2-4)} dx = \int \frac{5x}{(x+1)(x-2)(x+2)} dx$$

$$= \frac{5}{3}\int \frac{1}{x+1} dx + \frac{5}{6}\int \frac{1}{(x-2)} dx - \frac{5}{2}\int \frac{1}{x+2} dx$$

$$= \frac{5}{3}\log|x+1| + \frac{5}{6}\log|x-2| - \frac{5}{2}\log|x+2| + C$$

Therefore, $\int \frac{5x}{(x+1)(x^2-4)} dx = \frac{5}{3}\log|x+1| + \frac{5}{6}\log|x-2| - \frac{5}{2}\log|x+2| + C$

12. Find the integrand of the function $\frac{x^3+x+1}{x^2-1}$

Solution:

The integrand is $\frac{x^3+x+1}{x^2-1}$

- It has reducible quadratic factor in the denominator
- It is improper fraction

$$\frac{x^3+x+1}{x^2-1}=\frac{x\left(x^2+1\right)+1}{x^2-1}$$
$$=\frac{x\left(x^2-1\right)+2x+1}{x^2-1}$$
$$=x+\frac{2x+1}{(x-1)(x+1)}$$

Suppose that $\frac{2x+1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$

It implies

$$2x+1=A(x+1)+B(x-1) \qquad \ldots\ldots (1)$$

Plug in $x=-1$ in equation (1)

$$-1=B(-2)$$
$$B=\frac{1}{2}$$

Plug in $x=1$ in equation (1)

$$3=A(2)$$
$$A=\frac{3}{2}$$

Consider the integral

$$\int\frac{x^3+x+1}{x^2-1}dx=\int x+\frac{2x+1}{(x-1)(x+1)}dx$$
$$=\frac{x^2}{2}+\int\frac{\frac{3}{2}}{x-1}dx+\frac{1}{2}\int\frac{1}{x+1}dx$$
$$=\frac{x^2}{2}+\frac{3}{2}\log|x-1|+\frac{1}{2}\log|x+1|+C$$

Therefore, $\int\frac{x^3+x+1}{x^2-1}dx=\frac{x^2}{2}+\frac{3}{2}\log|x-1|+\frac{1}{2}\log|x+1|+C$

13. Find the integral of the function $\frac{2}{(1-x)\left(1+x^2\right)}$

Solution:

The given integrand is $\frac{2}{(1-x)(1+x^2)}$

- It has irreducible quadratic factor and one linear factor in the denominator
- It is proper fraction

Suppose that $\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$

It implies that

$$2 = A(1+x^2) + (Bx+C)(1-x)$$

Plug in $x=1$

$$2 = A(2)$$
$$A = 1$$

Compare the coefficient of x^2

$$0 = A - B$$
$$B = A$$
$$= 1$$

Compare the constant term

$$2 = A + C$$
$$C = 2 - 1$$
$$= 1$$

Hence,

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{(1-x)} + \frac{x+1}{(1+x^2)}$$

Consider the integral

$$\int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{(1-x)}dx + \int \frac{x+1}{(1+x^2)}dx$$

$$= -\log|1-x| + \frac{1}{2}\int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$$= -\log|1-x| + \frac{1}{2}\log|x^2+1| + \tan^{-1}(x) + C$$

Therefore, $\int \frac{2}{(1-x)(1+x^2)}dx = -\log|1-x| + \frac{1}{2}\log|x^2+1| + \tan^{-1}(x) + C$

14. Find the integral of the function $\frac{3x-1}{(x+2)^2}$

Solution:

The integrand is $\frac{3x-1}{(x+2)^2}$

- It has repeated linear factors in the denominator
- It is proper fraction

Suppose that $\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$

It implies that

$$3x-1 = A(x+2) + B$$

Plug in $x = -2$

$$-7 = B$$

Compare the coefficient of x

$$3 = A$$

Hence,

$$\frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

Consider the integral

$$\int \frac{3x-1}{(x+2)^2}dx = \int \frac{3}{(x+2)}dx - \int \frac{7}{(x+2)^2}dx$$
$$= 3\log|x+2| + 7\frac{1}{x+2} + C$$
$$= 3\log|x+2| + \frac{7}{x+2} + C$$

Therefore, $\int \frac{3x-1}{(x+2)^2}dx = 3\log|x+2| + \frac{7}{x+2} + C$

15. Find the integral of the function $\frac{1}{x^4-1}$

Solution:

The integrand is $\frac{1}{x^4-1}$

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)}$$
$$= \frac{1}{(x+1)(x-1)(1+x^2)}$$

- The integrand has two linear factors and one irreducible quadratic factor in the denominator
- It is proper fraction.

Suppose that $\frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$

Simplifying

1 $$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1) \quad \ldots\ldots (1)$$

Plug in $x=-1$ in (1)

$$1 = A(2)(-2)$$
$$A = -\frac{1}{4}$$

Plug in $x = 1$ in (1)

$$1 = B(1+1)(1+1)$$
$$B = \frac{1}{4}$$

Comparing x^2 coefficient in the equation (1)

$$A + B + C = 0$$
$$-\frac{1}{4} + \frac{1}{4} + C = 0$$
$$C = 0$$

Plug in $x = 0$ in (1)

$$1 = A(-1) + B(1) + D(-1)$$
$$1 = -A + B - D$$
$$1 = \frac{1}{4} + \frac{1}{4} - D$$
$$D = -\frac{1}{2}$$

Hence, $\frac{1}{(x+1)(x-1)(1+x^2)} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} + \frac{-\frac{1}{2}}{x^2+1}$

Integrate both sides

$$\int \frac{1}{(x+1)(x-1)(1+x^2)} dx = -\frac{1}{4}\int\left(\frac{1}{x+1}\right)dx + \frac{1}{4}\int\left(\frac{1}{x-1}\right)dx + \int \frac{-1}{2(x^2+1)} dx$$
$$= -\frac{1}{4}\log(x+1) + \frac{1}{4}\log(x-1) - \frac{1}{2}\tan^{-1} x$$
$$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1} x + C$$

Therefore, $\int \frac{1}{(x+1)(x-1)(1+x^2)} dx = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1} x + C$

16. Find the integration of the function $\frac{1}{x(x^n+1)}$

Solution:

The given integrand can be rewrite as

$$\frac{1}{x\left(x^n+1\right)}=\frac{x^{n-1}}{x^n\left(x^n+1\right)}$$

Apply integration on both sides

$$\int \frac{1}{x\left(x^n+1\right)}dx=\int \frac{x^{n-1}}{x^n\left(x^n+1\right)}dx$$

Put $x^n=t$, so that $dt=nx^{n-1}dx$

$$\int \frac{1}{x\left(x^n+1\right)}dx=\frac{1}{n}\int \frac{dt}{t(t+1)}$$

$$=\frac{1}{n}\int \frac{1}{t}dt-\int \frac{1}{t+1}dt$$

$$=\frac{1}{n}\left|\log|t|-\log|t+1|\right|+C$$

$$=\frac{1}{n}\log\left|\frac{t}{t+1}\right|+C$$

$$=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

Therefore, $\int \frac{1}{x\left(x^n+1\right)}dx=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$

17. Find the integral of the function $\frac{\cos x}{(1-\sin x)(2-\sin x)}$

Solution:

The given integral is $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx$

Put $\sin x=t$ so that $\cos x dx=dt$

$$\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{1}{(1-t)(2-t)}dt$$

$$= \int \frac{1}{1-t} - \frac{1}{2-t}dt$$

$$= -\log(1-t) + \log(2-t) + C$$

$$= \log\left|\frac{2-t}{1-t}\right| + C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right|$$

Therefore, $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \log\left|\frac{2-\sin x}{1-\sin x}\right|$

18. Find the integral of the function $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Solution:

Consider the integrand $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Suppose that $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{A}{x^2+3} + \frac{B}{x^2+4}$

It implies that

$$(x^2+1)(x^2+2) = (x^2+3)(x^2+4) + A(x^2+4) + B(x^2+3)$$

Substitute $x^2 = -4$

$$(-3)(-2) = 0 + A(0) + B(-1)$$
$$B = -6$$

Substitute $x^2 = -3$

$$(-2)(-1) = A(1)$$
$$A = 2$$

Hence,

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=1+\frac{2}{x^2+3}-\frac{6}{x^2+4}$$

Integrate both sides

$$\int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=\int 1dx+\int\frac{2}{x^2+3}dx-\int\frac{6}{x^2+4}dx$$

$$=x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+C$$

Therefore, $\int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+C$

19. Find the integral of the function $\frac{2x}{(x^2+1)(x^2+3)}$

Solution:

Consider the integral $\int\frac{2x}{(x^2+1)(x^2+3)}dx$

We know that $\frac{1}{x^2+1}-\frac{1}{x^2+3}=\frac{2}{(x^2+1)(x^2+3)}$

$$\int\frac{2x}{(x^2+1)(x^2+3)}dx=\frac{1}{2}\int\frac{2x}{(x^2+1)}-\frac{2x}{x^2+3}dx$$

$$=\frac{1}{2}\left(\log(x^2+1)-\log(x^2+3)\right)+C$$

$$=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$$

Therefore, $\int\frac{2x}{(x^2+1)(x^2+3)}dx=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$

20. Find the integral of the function $\frac{1}{x(x^4-1)}$

Solution:

The given integral is $\int \frac{1}{x\left(x^4-1\right)}dx$

$$\int \frac{1}{x\left(x^4-1\right)}dx = \int \frac{x^3}{x^4\left(x^4-1\right)}dx$$
$$= \frac{1}{4}\int \frac{4x^3}{x^4\left(x^4-1\right)}dx$$

Put $x^4 = t$, so that $4x^3dx = dt$

Hence,

$$\int \frac{1}{x\left(x^4-1\right)}dx = \frac{1}{4}\int \frac{4x^3}{x^4\left(x^4-1\right)}dx$$
$$= \frac{1}{4}\int \frac{dt}{t\left(t-1\right)}$$
$$= \frac{1}{4}\int \frac{1}{t} - \frac{1}{t-1}dt$$
$$= \frac{1}{4}\left[\log|t| - \log|t-1|\right] + C$$
$$= \frac{1}{4}\log\left|\frac{t}{t-1}\right| + C$$
$$= \frac{1}{4}\log\left|\frac{x^4}{x^4-1}\right| + C$$

Therefore, $\int \frac{1}{x\left(x^4-1\right)}dx = \frac{1}{4}\log\left|\frac{x^4}{x^4-1}\right| + C$

21. Find the integral of the function $\frac{1}{\left(e^x-1\right)}$

Solution:

Substitute $e^x = t$, so that $e^x dx = dt$

Hence,

$$\int \frac{1}{e^x-1}dx = \int\left(\frac{1}{t-1}\right)\frac{dt}{t}$$
$$= \int \frac{1}{t(t-1)}dt$$

Use the partial fractions

$$\int \frac{1}{t(t-1)}dt = \int \frac{1}{t-1} - \frac{1}{t}dt$$
$$= \log|t-1| - \log|t| + C$$
$$= \log\left|\frac{t-1}{t}\right| + C$$

Substitute $t = e^x$

$$\int \frac{1}{e^x-1}dx = \log\left|\frac{e^x-1}{e^x}\right| + C$$

Therefore, $\int \frac{1}{e^x-1}dx = \log\left|\frac{e^x-1}{e^x}\right| + C$

22. $\int \frac{xdx}{(x-1)(x-2)} =$

A) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

B) $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

D) $\log|(x-1)(x-2)| + C$

Solution:

The given integral is $\int \frac{xdx}{(x-1)(x-2)}$

Suppose that $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

This can be written as

$$x = A(x-2) + B(x-1) \quad \ldots\ldots (1)$$

Plug in $x = 2$ in the equation (1)

$$2 = B$$

Plug in $x = 1$ in the equation (1)

$$1 = -A \Rightarrow A = -1$$

Hence, $\dfrac{x}{(x-1)(x-2)} = -\dfrac{1}{x-1} + \dfrac{2}{x-2}$

Hence, the integral becomes

$$\int \frac{x}{(x-1)(x-2)} dx = -\int \frac{1}{x-1} dx + \int \frac{2}{x-2} dx$$
$$= -\log|x-1| + 2\log|x-2| + C$$
$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Thus, the correct answer is (B)

23. $\displaystyle\int \frac{dx}{x(x^2+1)} =$

A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

B) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

C) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

D) $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Solution:

The integrand is $\dfrac{1}{x(x^2+1)}$

- One linear and one irreducible quadratic factor in the denominator
- It is proper fraction

Suppose that $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

Simplifying

$$1 = A\left(x^2+1\right)+\left(Bx+C\right)x \quad \ldots\ldots (1)$$

Plug in $x=0$ in the equation (1)

$$1 = A$$

Compare the coefficient of x^2 in the equation (1)

$$A+B=0 \Rightarrow B=-A \Rightarrow B=-1$$

Comparing the coefficient of x in the equation (1)

$$C=0$$

The integral can be written as

$$\int \frac{1}{x\left(x^2+1\right)}dx = \int \frac{1}{x}dx + \int \frac{-x}{x^2+1}dx$$

$$= \log|x| - \frac{1}{2}\log\left|x^2+1\right| + C$$

Therefore, the option (A) is correct.

Exercise: 7.6

1. Find the integral of the function $x\sin x$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral

$$\begin{aligned} I &= \int x\sin xdx \\ &= x\int \sin xdx - \int 1\int \sin xdxdx \quad \bullet f(x) = x, g(x) = \sin x \\ &= x(-\cos x) - \int -\cos xdx \\ &= -x\cos x + \sin x + C \end{aligned}$$

Therefore, $\int x\sin xdx = -x\cos x + \sin x + C$

2. Find the integral of the function $x\sin 3x$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral

$$\begin{aligned} I &= \int x\sin 3xdx \\ &= x\int \sin 3xdx - \int 1\int \sin 3xdxdx \quad \bullet f(x) = x, g(x) = \sin 3x \\ &= \frac{x}{3}(-\cos 3x) - \frac{1}{3}\int -\cos 3xdx \\ &= -\frac{1}{3}x\cos x + \frac{1}{9}\sin 3x + C \end{aligned}$$

Therefore, $\int x\sin 3xdx = -\frac{1}{3}x\cos x + \frac{1}{9}\sin 3x + C$

3. Find the integral of the function x^2e^x

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral

$$\begin{aligned} I &= \int x^2e^x dx \\ &= x^2\int e^x dx - \int 2x\int e^x dxdx \quad \bullet f(x)=x^2, g(x)=e^x \\ &= x^2(e^x) - 2\int xe^x dx \\ &= x^2e^x - 2(xe^x) + 2\int e^x dx \\ &= x^2e^x - 2xe^x + 2e^x + C \end{aligned}$$

Therefore, $\int x^2e^x dx = x^2e^x - 2xe^x + 2e^x + C$

4. Find the integral of the function $x\log x$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral

$$\begin{aligned} I &= \int x\log x\,dx \\ &= \log x\int xdx - \int \frac{1}{x}\int xdxdx \quad \bullet f(x)=\log x, g(x)=x \\ &= \frac{x^2}{2}(\log x) - \frac{1}{2}\int xdx \\ &= \frac{x^2\log x}{2} - \frac{x^2}{4} + C \end{aligned}$$

Therefore, $\int x\log x\,dx = \frac{x^2\log x}{2} - \frac{x^2}{4} + C$

5. Find the integral of the function $x\log 2x$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral

$$I = \int x\log 2x\,dx$$
$$= \log 2x\int xdx - \int \frac{2}{2x}\int xdxdx \qquad \bullet f(x) = \log 2x, g(x) = x$$
$$= \frac{x^2}{2}(\log 2x) - \frac{1}{2}\int xdx$$
$$= \frac{x^2\log 2x}{2} - \frac{x^2}{4} + C$$

Therefore, $\int x\log 2x\,dx = \frac{x^2\log 2x}{2} - \frac{x^2}{4} + C$

6. Find the integral of the function $x^2\log x$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral

$$I = \int x^2\log x\,dx$$
$$= \log x\int x^2dx - \int \frac{1}{x}\int x^2dxdx \qquad \bullet f(x) = \log x, g(x) = x^2$$
$$= \frac{x^3}{3}(\log x) - \frac{1}{3}\int x^2dx$$
$$= \frac{x^3\log x}{3} - \frac{x^3}{9} + C$$

Therefore, $\int x^2\log x\,dx = \frac{x^3\log x}{3} - \frac{x^3}{9} + C$

7. Find the integral of the function $x\sin^{-1}x$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(x)=\sin^{-1}x, g(x)=x$

$$\int x\sin^{-1}xdx = \sin^{-1}x\int xdx - \int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int xdx\right\}dx$$

$$=\sin^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{1}{\sqrt{1-x^2}}\frac{x^2}{2}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\int\sqrt{1-x^2}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right\}$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x-\sin^{-1}x\right\}+C$$

$$=\frac{1}{4}\left(2x^2-1\right)\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+C$$

Therefore, $\int x\sin^{-1}xdx=\frac{1}{4}\left(2x^2-1\right)\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+C$

8. Find the integral of the function $x\tan^{-1}x$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(x)=\tan^{-1}x, g(x)=x$

$$\int x\tan^{-1}xdx = \tan^{-1}x\int xdx - \int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int xdx\right\}dx$$

$$= \tan^{-1}x\left(\frac{x^2}{2}\right) - \int\frac{1}{1+x^2}.\frac{x^2}{2}dx = \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\left(x - \tan^{-1}x\right) + C$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1} + C$$

Therefore, $\int x\tan^{-1}xdx = \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1} + C$

9. Find the integral of the function $x\cos^{-1}x$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(x)=\cos^{-1}x, g(x)=x$

$$\int x\cos^{-1}x dx=\cos^{-1}x\int xdx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int xdx\right\}dx$$

$$=\cos^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{-1}{\sqrt{1-x^2}}\frac{x^2}{2}dx$$

$$=\frac{x^2\cos^{-1}x}{2}+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\int\left\{\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\left\{\int\sqrt{1-x^2}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right\}$$

$$=\frac{x^2\cos^{-1}x}{2}-\frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x-\sin^{-1}x\right\}+C$$

$$=\frac{1}{4}\left(2x^2-1\right)\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C$$

Therefore, $\int x\cos^{-1}x dx=\frac{1}{4}\left(2x^2-1\right)\cos^{-1}x-\frac{x}{4}\sqrt{1-x^2}+C$

10. Find the integral of the function $\left(\sin^{-1}x\right)^2$

Solution:

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx=f(x)\int g(x)dx-\int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(x)=\left(\sin^{-1}x\right)^2, g(x)=1$

$$\int\left(\sin^{-1}x\right)^2dx=\left(\sin^{-1}x\right)^2\int 1dx-\int\left\{\left(\frac{d}{dx}\left(\sin^{-1}x\right)^2\right)\int 1dx\right\}dx$$

$$=\left(\sin^{-1}x\right)^2(x)-\int 2\sin^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right)xdx$$

$$=x\left(\sin^{-1}x\right)^2-2\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

Again use integration by parts. Use $f(x)=\sin^{-1}x, g(x)=\frac{2x}{\sqrt{1-x^2}}$

It implies that

$$x\left(\sin^{-1}x\right)^2-2\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx=x\left(\sin^{-1}x\right)^2-\int\sin^{-1}x\left(\frac{2x}{\sqrt{1-x^2}}\right)dx$$
$$=x\left(\sin^{-1}x\right)^2-\sin^{-1}x\left(-2\sqrt{1-x^2}\right)$$
$$+\int\frac{1}{\sqrt{1-x^2}}\left(-2\sqrt{1-x^2}\right)dx$$
$$=x\left(\sin^{-1}x\right)^2-2\sin^{-1}x\left(-\sqrt{1-x^2}\right)-2\int 1dx$$
$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$$

Therefore, $\int\left(\sin^{-1}x\right)^2dx=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$

11. Find the integral of the function $\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Solution:

Consider the integral $\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Suppose that $\cos^{-1}x=t$, so that $-\frac{1}{\sqrt{1-x^2}}dx=dt$ and $x=\cos t$

The integral becomes $\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx=-\int t\cdot cost dt$

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx=f(x)\int g(x)dx-\int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(t)=t, g(t)=\cos t$

$$-\int t\cdot\cos t dt=-t\int\cos t dt+\int\left\{\left(\frac{d}{dt}t\right)\int\cos t dt\right\}dt$$
$$=-t\sin t+\int\sin t dt$$
$$=-t\sin t-\cos t+C$$
$$=-\cos x\cdot\sqrt{1-x^2}-x+C$$

Therefore, $\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx=-\sqrt{1-x^2}\cos x-x+C$

12. Find the integral of the function $x\sec^2 x$

Solution:

Given integral is $\int x\sec^2 x dx$

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx=f(x)\int g(x)dx-\int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(x)=x, g(x)=\sec^2 x$

$$\int x\sec^2 x dx=x\int\sec^2 x dx-\int 1\cdot\int\sec^2 x dxdx$$
$$=x\tan x-\int\tan x dx$$
$$=x\tan x+\log|\cos x|+C$$

Therefore, $\int x\sec^2 x dx=x\tan x+\log|\cos x|+C$

13. Find the integral of the function $\tan^{-1}x$

Solution:

To find the integral of the function $\tan^{-1}x$, consider the integral $\int 1\cdot\tan^{-1}x dx$

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx=f(x)\int g(x)dx-\int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(x)=\tan^{-1}x, g(x)=1$

Hence, the required integral becomes

$$\begin{aligned}\int 1\cdot \tan^{-1} x dx &= \tan^{-1} x\cdot \int 1 dx-\int \frac{x}{1+x^2}dx \\ &= x\tan^{-1} x-\frac{1}{2}\int \frac{2x}{1+x^2}dx \\ &= x\tan^{-1} x-\frac{1}{2}\log\left|1+x^2\right|+C\end{aligned}$$

Therefore, $\int \tan^{-1} x dx = x\tan^{-1} x-\frac{1}{2}\log\left|1+x^2\right|+C$

14. Find the integral of the function $x(\log x)^2 dx$

Solution:

Consider the integral of the function $\int x(\log x)^2 dx$

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx-\int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(x)=(\log x)^2, g(x)=x$

The given integral becomes

$$\begin{aligned}\int x(\log x)^2 dx &= (\log x)^2\cdot\frac{x^2}{2}-\int 2\log x\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right)dx \\ &= \frac{x^2}{2}(\log x)^2-\int x\log x dx \\ &= \frac{x^2}{2}(\log x)^2-\log x\left(\frac{x^2}{2}\right)+\int \frac{1}{x}\frac{x^2}{2}dx \\ &= \frac{x^2}{2}(\log x)^2-\frac{x^2}{2}(\log x)+\frac{x^2}{4}+C\end{aligned}$$

Therefore, $\int x(\log x)^2 dx = \frac{x^2}{2}(\log x)^2-\frac{x^2}{2}(\log x)+\frac{x^2}{4}+C$

15. Find the integral of the function $(x^2+1)\log x$

Solution:

Consider the integral of the function $\int (x^2+1)\log x dx$

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(x)=(\log x), g(x)=x^2+1$

The given integral becomes

$$\begin{aligned}\int (x^2+1)(\log x)dx &= (\log x)\cdot\left(\frac{x^3}{3}+x\right)-\int \frac{1}{x}\left(\frac{x^3}{3}+x\right)dx\\ &= \log x\left(\frac{x^3+3x}{3}\right)-\int \frac{x^2}{3}+1dx\\ &= \log x\left(\frac{x^3+3x}{3}\right)-\frac{x^3}{9}-x+C\end{aligned}$$

Therefore, $\int (x^2+1)(\log x)dx = \log x\left(\frac{x^3+3x}{3}\right)-\frac{x^3}{9}-x+C$

16. Find the integral of the function $e^x(\sin x+\cos x)$

Solution:

Consider the integral $\int e^x(\sin x+\cos x)dx$

Use the formula $\int e^x(f(x)+f'(x))dx = e^x f(x)+C$

Let $f(x)=\sin x \Rightarrow f'(x)=\cos x$

Hence, the given integral becomes

$$\begin{aligned}\int e^x(\sin x+\cos x)dx &= \int e^x(f(x)+f'(x))dx\\ &= e^x f(x)+C\\ &= e^x\sin x+C\end{aligned}$$

Therefore, $\int e^x(\sin x+\cos x)dx = e^x\sin x+C$

17. Find the integral of the function $\frac{xe^x}{(1+x)^2}$

Solution:

Consider the integral and then rewrite it as

$$\int \frac{xe^x}{(1+x)^2}dx = \int e^x\left(\frac{x}{(1+x)^2}\right)dx$$
$$= \int e^x\left(\frac{x+1-1}{(1+x)^2}\right)dx$$
$$= \int e^x\left(\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right)dx$$

Use the formula $\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x)+C$

Let $f(x)=\frac{1}{1+x} \Rightarrow f'(x) = -\frac{1}{(1+x)^2}$

Hence, the given integral becomes

$$\int e^x\left(\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right)dx = \int e^x\left(f(x)+f'(x)\right)dx$$
$$= e^x f(x)+C$$
$$= e^x\left(\frac{1}{1+x}\right)+C$$

Therefore, $\int \frac{xe^x}{(1+x)^2}dx = \frac{e^x}{1+x}+C$

18. Find the integral of the function $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

Solution: Consider the integrand

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2}e^x\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$

$$=\frac{1}{2}e^x\left(1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right)$$

$$=\frac{1}{2}e^x\left(\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right)$$

Use the formula $\int e^x\left(f(x)+f'(x)\right)dx=e^x f(x)+C$

Let $f(x)=2\tan\frac{x}{2}\Rightarrow f'(x)=\sec^2\frac{x}{2}$

Hence, the given integral becomes

$$\int\frac{1}{2}e^x\left(\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right)dx=\frac{1}{2}\int e^x\left(f(x)+f'(x)\right)dx$$

$$=\frac{1}{2}e^x f(x)+C$$

$$=e^x\tan\frac{x}{2}+C$$

Therefore, $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx=e^x\tan\frac{x}{2}+C$

19. Find the integral of the function $e^{x}\left(\frac{1}{x}-\frac{1}{x^{2}}\right)$

Solution:

The given integral is $\int e^{x}\left(\frac{1}{x}-\frac{1}{x^{2}}\right)dx$

Use the formula $\int e^{x}\left(f(x)+f'(x)\right)dx=e^{x}f(x)+C$

Let $f(x)=\frac{1}{x}\Rightarrow f'(x)=-\frac{1}{x^{2}}$

Hence, the given integral becomes

$$\begin{aligned}\int e^{x}\left(\frac{1}{x}-\frac{1}{x^{2}}\right)dx&=\int e^{x}\left(f(x)+f'(x)\right)dx\\&=e^{x}f(x)+C\\&=\frac{e^{x}}{x}+C\end{aligned}$$

Therefore, $\int e^{x}\left(\frac{1}{x}-\frac{1}{x^{2}}\right)dx=\frac{e^{x}}{x}+C$

20. Find the integral of the function $\frac{(x-3)e^{x}}{(x-1)^{3}}$

Solution:

Consider the integral and rewrite it as

$$\begin{aligned}\int e^{x}\left\{\frac{x-3}{(x-1)^{3}}\right\}dx&=e^{x}\left\{\frac{x-1-2}{(x-1)^{3}}\right\}dx\\&=\int e^{x}\left\{\frac{1}{(x-1)^{2}}-\frac{2}{(x-1)^{3}}\right\}dx\end{aligned}$$

Use the formula $\int e^{x}\left(f(x)+f'(x)\right)dx=e^{x}f(x)+C$

Let $f(x)=\frac{1}{(x-1)^{2}}\Rightarrow f'(x)=\frac{-2}{(x-1)^{3}}$

Hence, the given integral becomes

$$\int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx = \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx$$

$$= \int e^x \left(f(x) + f'(x) \right) dx$$

$$= e^x f(x) + C$$

$$= \frac{e^x}{(x-1)^2} + C$$

Therefore, $\int \frac{(x-3)e^x}{(x-1)^3} dx = \frac{e^x}{(x-1)^2} + C$

21. Find the integral of the function $e^{2x} \sin x$

Solution:

Consider the integral $\int e^{2x} \sin x dx$

To find the integral of product of two functions, use the integration by parts

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int f'(x) \int g(x) dx dx$$

Consider the integral and take $f(x) = \sin x, g(x) = e^{2x}$

The given integral becomes

$$\int e^{2x}\sin x dx=\sin x\cdot\frac{e^{2x}}{2}-\int\cos x\left(\frac{e^{2x}}{2}\right)dx$$
$$=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\int e^{2x}\cos x dx$$
$$=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left(\cos x\left(\frac{e^{2x}}{2}\right)-\frac{1}{2}\int e^{2x}(-\sin x)dx\right)$$
$$=\frac{e^{2x}\sin x}{2}-\frac{1}{4}e^{2x}\cos x-\frac{1}{4}\int e^{2x}\sin x dx$$
$$\left(1+\frac{1}{4}\right)\int e^{2x}\sin x dx=\frac{e^{2x}\sin x}{2}-\frac{1}{4}e^{2x}\cos x$$
$$\frac{5}{4}\int e^{2x}\sin x dx=\frac{e^{2x}\sin x}{2}-\frac{1}{4}e^{2x}\cos x$$
$$\int e^{2x}\sin x dx=\frac{2}{5}e^{2x}\sin x-\frac{1}{5}e^{2x}\cos x$$

Therefore, $\int e^{2x}\sin x dx=\frac{2}{5}e^{2x}\sin x-\frac{1}{5}e^{2x}\cos x$

22. Find the integral of the function $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution:

Consider the integral $\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Put $x=\tan\theta$, so that $dx=\sec^2\theta d\theta$

The given integral becomes

$$\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx=\int\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$$
$$=\int\sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$
$$=2\int\theta\sec^2\theta d\theta$$

To find the integral of product of two functions, use the integration by parts

$$\int f(x)g(x)dx=f(x)\int g(x)dx-\int f'(x)\int g(x)dxdx$$

Consider the integral and take $f(\theta)=\theta, g(\theta)=\sec^2\theta$

The integral becomes

$$2\int \theta \sec^2 \theta d\theta = 2\theta \tan\theta - 2\int \tan\theta d\theta$$
$$= 2\theta \tan\theta - 2\log\sec\theta + C$$
$$= 2x\tan^{-1} x - \log\left(1+x^2\right) + C$$

Therefore, $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2x\tan^{-1} x - \log\left(1+x^2\right) + C$

23. $\int x^2 e^{x^3} dx$

A) $\frac{1}{3}e^{x^3} + C$

B) $\frac{1}{3}e^{x^2} + C$

C) $\frac{1}{2}e^{x^3} + C$

D) $\frac{1}{3}e^{x^2} + C$

Solution:

Given $\int x^2 e^{x^3} dx$

Put $x^3 = t$, so that $3x^2 dx = dt$

The given integral becomes

$$\int x^2 e^{x^3} dx = \frac{1}{3}\int \left(3x^2\right) e^{x^3} dx$$
$$= \frac{1}{3}\int e^t dt$$
$$= \frac{1}{3}\left(e^t\right) + C$$
$$= \frac{1}{3}e^{x^3} + C$$

This is matching with the option (A)

24. $\int e^x \sec x(1+\tan x)\,dx =$

A) $e^x \cos x + C$

B) $e^x \sec x + C$

C) $e^x \sin x + C$

D) $e^x \tan x + C$

Solution:

The given integral is $\int e^x \sec x(1+\tan x)\,dx$

Use the formula $\int e^x \{f(x)+f'(x)\}dx = e^x f(x) + C$

Consider the integral and rewrite it as

$$\begin{aligned}\int e^x \sec x(1+\tan x)\,dx &= \int e^x(\sec x + \sec x \tan x)\,dx\\ &= \int e^x(f(x)+f'(x))\,dx\\ &= e^x f(x) + C\\ &= e^x \sec x + C\end{aligned}$$

This is matching with the option (B)

Exercise: 7.7

1. Find the integral value of the function $\sqrt{4-x^2}$

Solution:

Consider the integral $\int \sqrt{4-x^2}\,dx=\int \sqrt{(2)^2-(x)^2}\,dx$

Use the formula $\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

Hence,

$$\begin{aligned}\int \sqrt{4-x^2}\,dx&=\int \sqrt{(2)^2-(x)^2}\,dx\\&=\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)+C\end{aligned}$$

Therefore, $\int \sqrt{4-x^2}\,dx=\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)+C$

2. Find the integral of the function $\sqrt{1-4x^2}$

Solution: Consider the integral

$$\begin{aligned}\int \sqrt{1-4x^2}\,dx&=\int \sqrt{1-(2x)^2}\,dx\\&=\frac{1}{2}\int \sqrt{1-t^2}\,dt\end{aligned}$$

Use the formula $\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

Hence,

$$\begin{aligned}\frac{1}{2}\int \sqrt{1-t^2}\,dt&=\frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}(t)+C\\&=\frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}(2x)+C\\&=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}(2x)+C\end{aligned}$$

Therefore, $\int \sqrt{1-4x^2}\,dx = \frac{x}{2}\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}(2x) + C$

3. Find the integral of the function $\int \sqrt{x^2+4x+6}\,dx$

Solution:

The given integral is $\int \sqrt{x^2+4x+6}\,dx$

$$\int \sqrt{x^2+4x+6}\,dx = \int \sqrt{x^2+4x+4+2}\,dx$$
$$= \int \sqrt{(x+2)^2 + \left(\sqrt{2}\right)^2}\,dx$$

Use the formula $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + C$

Hence, the given integral can be written as

$$\int \sqrt{x^2+4x+6}\,dx = \int \sqrt{x^2+4x+4+2}\,dx$$
$$= \int \sqrt{(x+2)^2 + \left(\sqrt{2}\right)^2}\,dx$$
$$= \frac{x+2}{2}\sqrt{x^2+4x+6} + \frac{2}{2}\log\left(x+2+\sqrt{x^2+4x+6}\right) + C$$
$$= \frac{x+2}{2}\sqrt{x^2+4x+6} + \log\left|x+2+\sqrt{x^2+4x+6}\right| + C$$

Therefore, $\int \sqrt{x^2+4x+6}\,dx = \frac{x+2}{2}\sqrt{x^2+4x+6} + \log\left|x+2+\sqrt{x^2+4x+6}\right| + C$

4. Find the integral of the function $\sqrt{x^2+4x+1}$

Solution:

The given integral is $\int \sqrt{x^2+4x+1}\,dx$

$$\int \sqrt{x^2+4x+1}\,dx = \int \sqrt{x^2+4x+4-3}\,dx$$
$$= \int \sqrt{(x+2)^2 - \left(\sqrt{3}\right)^2}\,dx$$

Use the formula $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)+C$

Hence, the given integral can be written as

$$\int \sqrt{x^2+4x+1}\,dx = \int \sqrt{x^2+4x+4-3}\,dx$$
$$= \int \sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\,dx$$
$$= \frac{x+2}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left(x+2+\sqrt{x^2+4x+1}\right)+C$$

Therefore, $\int \sqrt{x^2+4x+1}\,dx = \frac{x+2}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|x+2+\sqrt{x^2+4x+1}\right|+C$

5. $\sqrt{1-4x-x^2}$

Solution:

Consider, $=\int \sqrt{1-4x-x^2}\,dx$

$$= \int \sqrt{1-\left(x^2+4x+4-4\right)}dx$$

$$= \int \sqrt{1+4-(x+2)^2}\,dx$$

$$= \int \sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}\,dx$$

Since, $\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}\,\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$$\therefore I = \frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$$

6. $\sqrt{x^2+4x-5}$

Solution:

Let $I = \int \sqrt{x^2+4x-5}dx$

$$= \int \sqrt{\left(x^2+4x+4\right)-9}dx = \int \sqrt{(x+2)^2-(3)^2}\,dx$$

Since, $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

$$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right| + C$$

7. $\sqrt{1+3x-x^2}$

Solution:

Put, $I = \int \sqrt{1+3x-x^2}\,dx$

$$= \int \sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}dx$$

$$= \int \sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}dx = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}dx$$

Since, $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$$\therefore I = \frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2} + \frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$$

$$= \frac{2x-3}{4}\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$$

8. $\sqrt{x^2+3x}$

Solution:

Let $I = \int \sqrt{x^2+3x}dx$

$$= \int \sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}dx$$

$$= \int \sqrt{\left(x+\frac{3}{4}\right)^2 - \left(\frac{3}{2}\right)^2}\, dx$$

Since, $\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

$$\therefore I = \frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x} - \frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right| + C$$

$$= \frac{(2x+3)}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right| + C$$

9. $\sqrt{1+\frac{x^2}{9}}$

Solution:

Let $I = \int \sqrt{1+\frac{x^2}{9}}\,dx = \frac{1}{3}\int \sqrt{9-x^2}\,dx = \frac{1}{3}\int \sqrt{(3)^2+x^2}\,dx$

Since, $\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

$$\therefore I = \frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9} + \frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right] + C$$

$$= \frac{x}{6}\sqrt{x^2+9} + \frac{3}{2}\log\left|x+\sqrt{x^2+9}\right| + C$$

10. $\int \sqrt{1+x^2}$ is equal to

A. $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right| + C$

B. $\frac{2}{3}(1+x^2)^{\frac{2}{3}} + C$

C. $\frac{2}{3}x(1+x^2)^{\frac{2}{3}} + C$

D. $\frac{x^3}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right| + C$

Solution:

Since, $\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}\,\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$\therefore \int\sqrt{1+x^2}dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

Thus, the correct answer is A.

11. $\int\sqrt{x^2-8x+7}dx$ is equal

A. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Solution:

Let $I=\int\sqrt{x^2-8x+7dx}$

$=\int\sqrt{\left(x^2-8x+16\right)-9dx}$

$=\int\sqrt{(x-4)^2-(3)^2}\,dx$

Since, $\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\int\sqrt{x^2-8x+7}\right|+C$

Thus, the correct answer is D.

Exercise: 7.8

1. Find the value of $\int_a^b x\,dx$

Solution:

The given integral is $\int_a^b x\,dx$

Use the limit as sum formula

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{b-a}{n}$

Plug in $f(x)=x$

Hence,

$$\begin{aligned}
\int_a^b x\,dx &= (b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+a+h+....+a+(n-1)h\right] \\
&= (b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h(1+2+3+...+n-1)\right] \\
&= (b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(\frac{(n-1)(n)}{2}\right)\right] \\
&= (b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{(b-a)}{n}\left(\frac{(n-1)(n)}{2}\right)\right] \\
&= (b-a)\lim_{n\to\infty}\left[a+\left(\frac{(n-1)(b-a)}{2n}\right)\right] \\
&= (b-a)\left(a+\left(\frac{b-a}{2}\right)\right) \\
&= \frac{1}{2}(b-a)(b+a) \\
&= \frac{b^2-a^2}{2}
\end{aligned}$$

Therefore, $\int_a^b x\,dx = \frac{b^2-a^2}{2}$

2. Find the value the definite integral $\int_0^b (x+1)\,dx$ using the limit as sum concept.

Solution:

Consider the integral $\int_0^5 (x+1)\,dx$

Use the limit as sum formula

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{b-a}{n}=\frac{5}{n}, a=0, b=5$

Plug in $f(x)=x+1$

$$\int_0^5 (x+1)\,dx = (5)\lim_{n\to\infty}\frac{1}{n}\left[1+1+h+....+1+(n-1)h\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+h(1+2+3+...+n-1)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{5}{n}\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$=5\lim_{n\to\infty}\left[1+\frac{5}{n^2}\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$=5\left(1+\frac{5}{2}\right)$$

$$=\frac{35}{2}$$

Therefore, $\int_0^5 (x+1)\,dx = \frac{35}{2}$

3. Find the value of $\int_2^3 x^2 dx$ using limit as sum concept

Solution:

Consider the integral $\int_2^3 x^2 dx$

Use the limit as sum formula

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{3-2}{n}=\frac{1}{n}, a=2, b=3$

Plug in $f(x)=x^2$

$$\int_2^3 x^2dx = \lim_{n\to\infty}\frac{1}{n}\left[2^2+(2+h)^2+....+(2+(n-1)h)^2\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n(2^2)+h^2\left(1^2+2^2...+(n-1)^2\right)+4h(1+2+...+(n-1))\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n2^2+h^2\left(\frac{(n-1)(n)(2n-1)}{6}\right)+4h\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$= \lim_{n\to\infty}\left[4+\frac{1}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right)+4\left(\frac{(n-1)(n)}{2n^2}\right)\right]$$

$$= 4+\frac{2}{6}+4\left(\frac{1}{2}\right)$$

$$= \frac{19}{3}$$

Therefore, $\int_2^3 x^2dx = \frac{19}{3}$

4. Find the value of $\int_1^4 (x^2-x)dx$ using limit as sum concept

Solution:

Consider the integral $\int_1^4 (x^2-x)dx$

Use the limit as sum formula

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{4-1}{n}=\frac{3}{n}, a=1, b=4$

Plug in $f(x)=x^2-x$

$$\int_1^4 \left(x^2 - x\right)dx = \lim_{n\to\infty}\frac{3}{n}\left[\begin{array}{r} 1^2 + (1+h)^2 + ... + (1+(n-1)h)^2 \\ -(1+1+h+1+2h+...+1+(n-1)h) \end{array}\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\begin{array}{r} n(1^2) + h^2\left(1^2 + ... + (n-1)^2\right) + 2h(1+2+...+(n-1)) \\ -n-h(1+2..+(n-1)) \end{array}\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[n + h^2\left(\frac{(n-1)(n)(2n-1)}{6}\right) + 2h\left(\frac{(n-1)(n)}{2}\right) - n - h\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[h^2\left(\frac{(n-1)(n)(2n-1)}{6}\right) + h\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$\lim_{n\to\infty}\left[\frac{27}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right) + \frac{9}{n}\left(\frac{(n-1)(n)}{2n^2}\right)\right]$$

$$= 27\left(\frac{1}{3}\right) + \frac{9}{2}$$

$$= \frac{27}{2}$$

Therefore, $\int_1^4 \left(x^2 - x\right)dx = \frac{27}{2}$

5. Find the value of $\int_{-1}^{1} e^x dx$ using limit as sum concept.

Solution:

Consider the integral $\int_{-1}^{1} e^x dx$

Use the limit as sum formula

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + ... + f(a+(n-1)h)\right]$$

Here $h = \frac{1+1}{n} = \frac{2}{n}, a = -1, b = 1$

Plug in $f(x) = e^x$

$$\int_{-1}^{1} e^x dx = 2\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+....+f\left(-1+\frac{(n-1)\cdot 2}{n}\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2.\frac{2}{n}\right)}+....e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}-1}}{e^{\frac{2}{n}-1}}\right]$$

$$=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}-1}}\right]$$

$$==\frac{e^{-1}\times 2\left(e^2-1\right)}{\lim_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}-1}}{\frac{2}{n}}\right)\times 2}$$

$$=e^{-1}\left[\frac{2\left(e^2-1\right)}{2}\right]$$

$$=\frac{e^2-1}{e}$$

$$=e-\frac{1}{e}$$

Therefore, $\int_{-1}^{1} e^x dx = e-\frac{1}{e}$

6. Find the value of $\int_0^4\left(x+e^{2x}\right)dx$ using limit as sum

Solution:

Consider the integral $\int_0^4\left(x+e^{2x}\right)dx$

Use the limit as sum formula

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{4-0}{n}=\frac{4}{n}, a=0, b=4$

Plug in $f(x)=x+e^{2x}$

$$\int_0^4 \left(x+e^{2x}\right)dx=(4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+....+f((n-1)h)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^0\right)+\left(h+e^{2h}+\left(2h+e^{2\cdot 2h}\right)+.....+\left\{(n-1)h+e^{2(n-1)h}\right\}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+.....+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+....+e^{2(n-1)h}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+....+(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=\Rightarrow 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$=4(2)+4\lim_{n\to\infty}\frac{\left(e^8-1\right)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$=8+\frac{e^8-1}{2}$$

$$=\frac{e^8+15}{2}$$

Therefore, $\int_0^4 \left(x+e^{2x}\right)dx=\frac{e^8+15}{2}$

Exercise: 7.9

1. Find the value of $\int_{-1}^{1}(x+1)dx$

Solution:

Consider the integral $\int_{-1}^{1}(x+1)dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_{a}^{b} f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_{-1}^{1}(x+1)dx = \left[\frac{x^2}{2}+x\right]_{-1}^{1}$$
$$=\frac{1}{2}+1-\frac{1}{2}+1$$
$$=2$$

Therefore, $\int_{-1}^{1}(x+1)dx = 2$

2. Find the value of $\int_{2}^{3}\frac{1}{x}dx$

Solution: Consider the integral $\int_{2}^{3}\frac{1}{x}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_{a}^{b} f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_{2}^{3}\frac{1}{x}dx = [\log x]_{2}^{3}$$
$$=\log 3-\log 2$$
$$=\log\frac{3}{2}$$

Therefore, $\int_2^3 \frac{1}{x} dx = \log \frac{3}{2}$

3. Find the value of $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

Solution: Consider the integral $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x) dx = F(x) + C$ then $\int_a^b f(x) dx = F(b) - F(a)$"

Hence the integral becomes

$$\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx = \left[x^4 - \frac{5x^3}{3} + 3x^2 + 9x\right]_1^2$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= \frac{64}{3}$$

Therefore, $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx = \frac{64}{3}$

4. Find the value of $\int_0^{\frac{x}{4}} \sin 2x dx$

Solution: Consider the integral $\int_0^{\frac{x}{4}} \sin 2x dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x) dx = F(x) + C$ then $\int_a^b f(x) dx = F(b) - F(a)$"

Hence the integral becomes

$$\int_0^{\frac{\pi}{4}} \sin 2x dx = \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{2}\left(\cos \frac{\pi}{2} - \cos 0\right)$$

$$= \frac{1}{2}$$

Therefore, $\int_0^{\frac{\pi}{4}} \sin 2x dx = \frac{1}{2}$

5. $\int_0^{\frac{\pi}{2}} \cos 2x\, dx$

Solution: Consider the integral $\int_0^{\frac{\pi}{2}} \cos 2x\, dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x) + C$ then $\int_a^b f(x)dx = F(b) - F(a)$"

Hence the integral becomes

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos 2x\, dx &= \left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) \\ &= \frac{1}{2}\end{aligned}$$

Therefore, $\int_0^{\frac{\pi}{4}} \cos 2x dx = \frac{1}{2}$

6. Find the value of $\int_4^5 e^x dx$

Solution: Consider the integral $\int_4^5 e^x dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x) + C$ then $\int_a^b f(x)dx = F(b) - F(a)$"

Hence the integral becomes

$$\begin{aligned}\int_4^5 e^x dx &= \left[e^x\right]_4^5 \\ &= e^5 - e^4 \\ &= e^4(e-1)\end{aligned}$$

Therefore, $\int_4^5 e^x dx = e^4(e-1)$

7. Find the value of $\int_0^{\frac{\pi}{4}} \tan x\, dx$

Solution: Consider the integral $\int_0^{\frac{\pi}{4}} \tan x\, dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^{\frac{\pi}{4}} \tan x\, dx = \left[\log(\sec x)\right]_0^{\frac{\pi}{4}}$$
$$= \log\left(\sec\frac{\pi}{4}\right) - \log(\sec 0)$$
$$= \log\left(\sqrt{2}\right) - \log(1)$$
$$= \frac{1}{2}\log 2$$

Therefore, $\int_0^{\frac{\pi}{4}} \tan x\, dx = \frac{1}{2}\log 2$

8. Find the value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx\, dx$

Solution: Consider the integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx\, dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx\, dx = \left[\log\left(\csc x - cotx\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \log\left(\csc\frac{\pi}{4} - \cot\frac{\pi}{4}\right) - \log\left(\csc\frac{\pi}{6} - \cot\frac{\pi}{6}\right)$$

$$= \log\left(\sqrt{2}-1\right) - \log\left(2-\sqrt{3}\right)$$

$$= \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$$

Therefore, $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx\, dx = \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$

9. Find the value of $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Solution: Consider the integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x) + C$ then $\int_a^b f(x)dx = F(b) - F(a)$"

Hence the integral becomes

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[sin^{-1}x\right]_0^1$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Therefore, $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$

10. Find the value of $\int_0^1 \frac{dx}{1+x^2}$

Solution: Consider the integral $\int_0^1 \frac{dx}{1+x^2}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1$$
$$= \frac{\pi}{4}$$

Therefore, $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

11. Find the value of $\int_2^3 \frac{dx}{x^2-1}$

Solution: Consider the integral $\int_2^3 \frac{dx}{x^2-1}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_2^3 \frac{dx}{x^2-1} = \frac{1}{2}\left[\log\frac{x-1}{x+1}\right]_2^3$$
$$= \frac{1}{2}\left(\log\left(\frac{2}{4}\right) - \log\left(\frac{1}{3}\right)\right)$$
$$= \frac{1}{2}\log\left(\frac{3}{2}\right)$$

Therefore, $\int_2^3 \frac{dx}{x^2-1} = \frac{1}{2}\log\left(\frac{3}{2}\right)$

12. Find the value of $\int_0^{\frac{\pi}{2}} \cos^2 x dx$

Solution: Consider the integral $\int_0^{\frac{\pi}{2}} \cos^2 x dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1+\cos 2x dx$$

$$= \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}+0\right)$$

$$= \frac{\pi}{4}$$

Therefore, $\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{\pi}{4}$

13. Find the value of $\int_2^3 \frac{xdx}{x^2+1}$

Solution: Consider the integral $\int_2^3 \frac{xdx}{x^2+1}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_2^3 \frac{xdx}{x^2+1} = \frac{1}{2}\int_2^3 \frac{2xdx}{x^2+1}$$

$$= \frac{1}{2}\left(\log\left(x^2+1\right)\right)_2^3$$

$$= \frac{1}{2}\log(10) - \frac{1}{2}\log(5)$$

$$= \frac{1}{2}\log 2$$

Therefore, $\int_2^3 \frac{xdx}{x^2+1} = \frac{1}{2}\log 2$

14. Find the value of $\int_0^1 \frac{2x+3}{5x^2+1}dx$

Solution: Consider the integral $\int_0^1 \frac{2x+3}{5x^2+1}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^1 \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}dx + \frac{3}{5}\int_0^1 \frac{1}{x^2+\left(\frac{1}{\sqrt{5}}\right)^2}dx$$

$$= \left[\frac{1}{5}\log\left(5x^2+1\right)+\frac{3}{5}\left(\frac{\sqrt{5}}{1}\right)\tan^{-1}\left(\frac{x}{\frac{1}{\sqrt{5}}}\right)\right]_0^1$$

$$= \frac{1}{5}\log(6)-\log(1)+\frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)$$

$$= \frac{1}{5}\log 6+\frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)$$

Therefore, $\int_0^1 \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\log 6+\frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)$

15. Find the value of $\int_0^1 xe^{x^2}dx$

Solution: Consider the integral $\int_0^1 xe^{x^2}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^1 xe^{x^2}dx = \frac{1}{2}\int_0^1 2xe^{x^2}dx$$
$$= \frac{1}{2}\left[e^{x^2}\right]_0^1$$
$$= \frac{1}{2}(e-1)$$

Therefore, $\int_0^1 xe^{x^2}dx = \frac{1}{2}(e-1)$

16. Find the value of $\int_1^2 \frac{5x^2}{x^2+4x+3}$

Solution: Consider the integral $\int_1^2 \frac{5x^2}{x^2+4x+3}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_1^2 \frac{5x^2}{x^2+4x+3}dx = \int_1^2 \frac{5x^2+20x+15-20x-15}{x^2+4x+3}dx$$
$$= 5\int_1^2 1dx - 10\int_1^2 \frac{2x+4}{x^2+4x+3}dx + 25\int_1^2 \frac{1}{x^2+4x+3}dx$$
$$= 5(x)_1^2 - 10\left(\log\left(x^2+4x+3\right)\right)_1^2 + 25\int_1^2 \frac{1}{(x+2)^2-1}dx$$
$$= 5 - 10\log\left(\frac{15}{8}\right) + 25\left(\frac{1}{2}\right)\left(\log\left(\frac{x+2-1}{x+2+1}\right)\right)_1^2$$
$$= 5 - 10\log\left(\frac{15}{8}\right) + \frac{25}{2}\log\left(\frac{3}{5}\right) - \frac{25}{2}\log\left(\frac{1}{2}\right)$$

Therefore, $\int_1^2 \frac{5x^2}{x^2+4x+3}dx = 5 - 10\log\left(\frac{15}{8}\right) + \frac{25}{2}\log\left(\frac{3}{5}\right) - \frac{25}{2}\log\left(\frac{1}{2}\right)$

17. Find the value of $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right)dx$

Solution: Consider the integral $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right)dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x) + C$ then $\int_a^b f(x)dx = F(b) - F(a)$"

Hence the integral becomes

$$\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right)dx = \left[2\tan x + \frac{x^4}{4} + 2x\right]_0^{\frac{\pi}{4}}$$

$$= 2 + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

Therefore, $\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right)dx = 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$

18. Find the value of $\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}\right)dx$

Solution: Consider the integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx\, dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x) + C$ then $\int_a^b f(x)dx = F(b) - F(a)$"

Hence the integral becomes

$$\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}\right)dx = -\int_0^{\pi} \cos x dx$$

$$= -(\sin x)_0^{\pi}$$

$$= 0$$

Therefore, $\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}\right)dx = 0$

19. Find the value of $\int_0^2 \frac{6x+3}{x^2+4}dx$

Solution: Consider the integral $\int_0^2 \frac{6x+3}{x^2+4}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^2 \frac{6x+3}{x^2+4}dx = 3\int_0^2 \frac{2x+1}{x^2+4}dx$$
$$= 3\int_0^2 \frac{2x}{x^2+4}dx + 3\int_0^2 \frac{1}{x^2+4}dx$$
$$= \left[3\log\left(x^2+4\right)+\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$= 3\log(8)+\frac{3}{2}\left(\frac{\pi}{4}\right)-3\log 4$$
$$= 3\log 2+\frac{3\pi}{8}$$

Therefore, $\int_0^2 \frac{6x+3}{x^2+4}dx = 3\log 2+\frac{3\pi}{8}$

20. Find the value of $\int_0^1 \left(xe^x+\sin\frac{\pi x}{4}\right)dx$

Solution: Consider the integral $\int_0^1 \left(xe^x+\sin\frac{\pi x}{4}\right)dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^1 \left(xe^x+\sin\frac{\pi x}{4}\right)dx = \left[e^x(x-1)-\frac{4}{\pi}cos\frac{\pi x}{4}\right]_0^1$$
$$= -1(0-1)-\frac{4}{\pi}cos\left(\frac{\pi}{4}\right)+\frac{4}{\pi}$$
$$= 1-\frac{2\sqrt{2}}{\pi}+\frac{4}{\pi}$$

Therefore, $\int_0^1 \left(xe^x+\sin\frac{\pi x}{4}\right)dx = 1-\frac{2\sqrt{2}}{\pi}+\frac{4}{\pi}$

21. The value of $\int_{1}^{\sqrt{3}} \frac{dx}{1+x^2}$

A) $\frac{\pi}{3}$ B) $\frac{2\pi}{3}$ C) $\frac{\pi}{6}$ D) $\frac{\pi}{12}$

Solution: Consider the integral $\int_{1}^{\sqrt{3}} \frac{dx}{1+x^2}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_{a}^{b} f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_{1}^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_{1}^{\sqrt{3}}$$
$$= \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)$$
$$= \frac{\pi}{3} - \frac{\pi}{4}$$
$$= \frac{\pi}{12}$$

Therefore, this is matching with the option (D)

22. The value of $\int_{0}^{\frac{2}{3}} \frac{dx}{4+9x^2}$

A) $\frac{\pi}{6}$ B) $\frac{\pi}{12}$ C) $\frac{\pi}{24}$ D) $\frac{\pi}{4}$

Solution: Consider the integral $\int_{0}^{\frac{2}{3}} \frac{dx}{4+9x^2}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_{a}^{b} f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = \frac{1}{9}\int_0^{\frac{2}{3}} \frac{dx}{\frac{4}{9}+x^2}$$

$$== \frac{1}{9}\int_0^{\frac{2}{3}} \frac{dx}{\left(\frac{2}{3}\right)^2 + x^2}$$

$$= \frac{1}{6}\left[\tan^{-1}\left(\frac{3x}{2}\right)\right]_0^{\frac{2}{3}} \qquad \bullet\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

$$= \frac{1}{6}\left(\tan^{-1}1 - \tan^{-1}(0)\right)$$

$$= \frac{1}{6}\left[\frac{\pi}{4}\right]$$

$$= \frac{\pi}{24}$$

Therefore, this is matching with the option (C)

Exercise: 7.10

1. Find the value of $\int_0^1 \frac{x}{x^2+1}dx$

Solution: Consider the integral $\int_0^1 \frac{x}{x^2+1}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}dx$$
$$= \frac{1}{2}\left[\log\left|x^2+1\right|\right]_0^1$$
$$= \frac{1}{2}\log 2$$

Therefore, $\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\log 2$

2. Find the value of $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Solution: Consider the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Suppose that $\sin\phi = t$, so that $\cos\phi d\phi = dt$

Hence the integral becomes

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi(\cos\phi\, d\phi)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\left(1-\sin^2\phi\right)(\cos\phi\, d\phi)$$

$$= \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$= \int_0^1 \sqrt{t}\left(1+t^4-2t^2\right)dt$$

$$= \int_0^1 t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}} dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - 2\frac{t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154+42-132}{231}$$

$$= \frac{64}{231}$$

Therefore, $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \frac{64}{231}$

3. Find the value of $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Solution: Consider the integral $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Suppose that $x = \tan\theta$, so that $dx = \sec^2\theta d\theta$

Hence the integral becomes

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta d\theta$$

$$= [2\theta\tan\theta]_0^{\frac{\pi}{4}} + [2\log(\cos\theta)]_0^{\frac{\pi}{4}}$$

$$= 2\left(\frac{\pi}{4}\right) - \log 2$$

$$= \frac{\pi}{2} - \log 2$$

Therefore, $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \frac{\pi}{2} - \log 2$

4. Find the value of $\int_0^2 x\sqrt{x+2}dx$

Solution: Consider the integral $\int_0^2 x\sqrt{x+2}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Suppose that $x+2=t$, so that $dx = dt$

When $x=0 \Rightarrow t=2$ and when $x=2 \Rightarrow t=4$

Hence the integral becomes

$$\int_0^2 x\sqrt{x+2}dx = \int_2^4 (t-2)\sqrt{t}dt$$
$$= \int_2^4 \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$$
$$= \left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$
$$= \frac{2}{5}\left(32-4\sqrt{2}\right) - \frac{4}{3}\left(8-2\sqrt{2}\right)$$
$$= \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Therefore, $\int_0^2 x\sqrt{x+2}dx = \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$

5. Find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Solution: Consider the integral $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Suppose that $t=\cos x$, so that $dt = -\sin xdx$

When $x=0 \Rightarrow t=1$ and when $x=\frac{\pi}{2} \Rightarrow t=0$

Hence the integral becomes

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{dt}{1+t^2}$$
$$= \left[-\tan^{-1} t\right]_1^0$$
$$= \frac{\pi}{4}$$

Therefore, $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = \frac{\pi}{4}$

6. Find the value of $\int_0^2 \frac{dx}{x+4-x^2}$

Solution: Consider the integral $\int_0^2 \frac{dx}{x+4-x^2}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Rewrite the quadratic expression $x+4-x^2$ as

$$\begin{aligned} x+4-x^2 &= -\left(x^2-x-4\right) \\ &= \frac{17}{4}-\left(x-\frac{1}{2}\right)^2 \\ &= \left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2 \end{aligned}$$

Hence the integral becomes

$$\begin{aligned} \int_0^2 \frac{dx}{x+4-x^2} &= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} \\ &= \frac{1}{\sqrt{17}}\log\left(\frac{\frac{\sqrt{17}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}-\left(x-\frac{1}{2}\right)}\right)_0^2 \\ &= \frac{1}{\sqrt{17}}\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\right)-\log\left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right) \\ &= \frac{1}{\sqrt{17}}\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}\right) \\ &= \frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right) \end{aligned}$$

Therefore, $\int_0^2 \frac{dx}{x+4-x^2} = \frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$

7. Find the value of $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

Solution: Consider the integral $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Rewrite the quadratic expression x^2+2x+5 as

$$\begin{aligned} x^2+2x+5 &= (x+1)^2+4 \\ &= (x+1)^2+2^2 \end{aligned}$$

Hence the integral becomes

$$\begin{aligned} \int_{-1}^{1}\frac{dx}{x^2+2x+5} &= \int_{-1}^{1}\frac{dx}{(x+1)^2+2^2} \\ &= \frac{1}{2}\left[\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1} \\ &= \frac{1}{2}\tan^{-1}(1) \\ &= \frac{\pi}{8} \end{aligned}$$

Therefore, $\int_{-1}^{1}\frac{dx}{x^2+2x+5} = \frac{\pi}{8}$

8. Find the value of $\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$

Solution: Consider the integral $\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Suppose that $2x = t$, so that $2dx = dt$

When $x=1 \Rightarrow t=2$ and $x=2 \Rightarrow t=4$

Hence the integral becomes

$$\int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx=\frac{1}{2}\int_2^4\left(\frac{2}{t}-\frac{2}{t^2}\right)e^t dx$$

$$=\int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dx$$

$$=\left[\frac{e^4}{4}-\frac{e^2}{2}\right]$$

$$=\frac{e^2\left(e^2-2\right)}{4}$$

Therefore, $\int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx=\frac{e^2\left(e^2-2\right)}{4}$

9. The value of the integral $\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ is

A) 6 B) 0 C) 3 D) 4

Solution:

Consider $I=\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$

Let $x=\sin\theta \Rightarrow dx=\cos\theta d\theta$

When $x=\frac{1}{3},\theta=\sin^{-1}\left(\frac{1}{3}\right)$ and when $x=1,\theta=\frac{\pi}{2}$

$$\Rightarrow I=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(1-\sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{1}{3}}}{\sin^2\theta\sin^2\theta}\cos\theta d\theta = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}}\operatorname{cosec}^2\theta d\theta$$

$$\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} (\cot\theta)^{\frac{5}{3}}\operatorname{cosec}^2\theta d\theta$$

Put $\cot\theta = t \Rightarrow -\operatorname{cosec}^2\theta d\theta = dt$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right), t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}, t = 0$

$$\therefore I = \int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$$

$$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0} = \frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$= \frac{3}{8}[16]$$

$= 3\times 2$

=6

Thus, the correct answer is A

10. If $f(x) = \int_0^x t\sin t\, dt$, then $f'(x)$ is

A) $\cos x + x\sin x$ B) $x\sin x$ C) $x\cos x$ D) $\sin x + x\cos x$

Solution:

$$f(x) = \int_0^x t\sin t dt$$

Using integration by parts, we get

$$f(x) = t\int_0^x \sin t dt - \int_0^x \left\{\left(\frac{d}{dt}t\right)\int \sin t dt\right\} dt$$

$=\left[t(-\cos t)\right]_0^x-\int_0^x(-\cot t)\,dt$

$=\left[-t\cos t+\sin t\right]_0^x$

$=-x\cos x+\sin x$

$\Rightarrow f'(x)=-\left[\{x(-\sin x)\}+\cos x\right]+\cos x$

$=x\sin x-\cos x+\cos x$

$=x\sin x$

Thus, the correct answer is B

Infinity Learn by Sri Chaitanya Educational Institutions

Exercise: 7.11

1. $\int_0^{\frac{\pi}{2}} \cos^2 x dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx........(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx \qquad \left(\int_0^{0} f(x)dx = \int_0^{0} f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x dx.........(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Consider, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx........(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cos}}{\sqrt{\cos}+\sqrt{\sin x}}dx.........(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

3. $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}xdx}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx$

Solution:

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}xdx}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx.........(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx..........(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx \Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 xdx}{\sin^5 x + \cos^5 x} dx$

Solution:

$$\text{Consider, } I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 xdx}{\sin^5 x + \cos^5 x} dx.........(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} dx........(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx \Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

5. $\int_{-5}^{5} |x+2| dx$

Solution:

Let $I=\int_{-5}^{5}|x+2|\,dx$

As, $(x+2)\leq 0$ on [-5,-2] and $(x+2)\geq 0$ on [-2,5]

$\therefore \int_{5}^{-2}-(x+2)\,dx+\int_{-2}^{5}(x+2)\,dx$ $\quad\left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$

$$I=-\left[\frac{x^2}{2}+2x\right]_{-5}^{-2}+\left[\frac{x^2}{2}+2x\right]_{-2}^{5}$$

$$=-\left[\frac{(-2)^2}{2}+2(-2)-\frac{(-5)^2}{2}-2(-5)\right]+\left[\frac{(5)^2}{2}+2(5)-\frac{(-2)^2}{2}-2(-2)\right]$$

$$=-\left[2-4-\frac{25}{2}+10\right]+\left[\frac{25}{2}+10-2+4\right]$$

$$=-2+4+\frac{25}{2}-10+\frac{25}{2}+10-2+4$$

$$=29$$

6. $\int_{2}^{8}|x-5|\,dx$

Solution:

Consider, $I=\int_{2}^{8}|x-5|\,dx$

As $(x-5)\leq 0$ on $[2,5]$ and $(x-5)\geq 0$ on $[5,8]$

$I=\int_{2}^{5}-(x-5)\,dx+\int_{2}^{8}(x-5)\,dx$ $\quad\left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$

$$\left[\frac{x^2}{2}-5x\right]_{2}^{5}+\left[\frac{x^2}{2}-5x\right]_{5}^{8}$$

$$-\left[\frac{25}{2}-25-2+10\right]+\left[32-40-\frac{25}{2}+25\right]=9$$

7. $\int_0^1 x(1-x)^n\,dx$

Solution:

Consider, $I=\int_0^1 x(1-x)^n\,dx$

$$\therefore I=\int_0^1 (1-x)(1-(1-x))^n\,dx$$

$$=\int_0^1 (1-x)(x)^n\,dx=\int_0^1\left(x^n-x^{n+1}\right)dx$$

$$=\left[\frac{1}{n+1}-\frac{1}{n+2}\right]=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1 \qquad \left(\int_1^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$

$$=\frac{(n+2)-(n+1)}{(n+1)(n+2)}=\frac{1}{(n+1)(n+2)}$$

8. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$

Solution:

Let $I=\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx........(1)$

$$\therefore I=\int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx \qquad \left(\int_1^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan x}{1+\tan x}\right\}dx \Rightarrow I=\int_0^{\frac{\pi}{4}} \log\frac{2}{(1+\tan x)}\,dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log 2\,dx-\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log 2 dx \quad \text{(from (1))}$$

$$\Rightarrow 2I = [x \log 2]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I = \frac{\pi}{4} \log 2$$

$$\Rightarrow I = \frac{\pi}{8} \log 2$$

9. $\int_0^2 x\sqrt{2-x}\, dx$

Solution:

Consider, $I = \int_0^2 x\sqrt{2-x}\, dx$

$$I = \int_0^2 (2-x)\sqrt{x}\, dx \qquad \left(\int_1^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$= \int_0^2 \left\{2x^{\frac{1}{2}} - x^{\frac{3}{2}}\right\} dx = \left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{5}}\right)\right]_0^2$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^2 = \frac{4}{3}(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}}$$

$$= \frac{4\times 2\sqrt{2}}{3} - \frac{2}{5}\times 4\sqrt{2} = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{40\sqrt{2} - 24\sqrt{2}}{15} = \frac{16\sqrt{2}}{15}$$

10. $\int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)\, dx$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\log(2\sin x\cos x))\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\text{logsinx}-\text{logcosx}-\log 2)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log\sin x-\log\text{cosx}-\log 2)\,dx.........(1)$

Since, $\left(\int_0^a f(d)\,dx=\int_0^a f(a-x)\,dx\right)$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{\log\cos x-\log\sin x-\log 2\}\,dx.........(2)$

Adding (1) and (2), we get

$2I=\int_0^{\frac{\pi}{2}}(-\log 2-\log 2)\,dx$

$\Rightarrow 2I=-2\log 2\int_0^{\frac{\pi}{2}}1.dx$

$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$

$\Rightarrow I=\frac{\pi}{2}(-\log 2)$

$\Rightarrow I=\frac{\pi}{2}\left(\log\frac{1}{2}\right)$

$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$

11. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$

Solution:

Let $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$

As $\sin^2(-x)=(\sin(-x))^2=(-\sin x)^2=\sin^2 x$, therefore, $\sin^2 x$ is an even function

If $f(x)$ is an even function, then $\int_{-a}^{a}f(x)\,dx=2\int_0^a f(x)\,dx$

$$I = 2\int_0^{\frac{\pi}{2}} \sin^2 x dx = 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$$

$$I = \int_0^{\frac{\pi}{2}} (1-\cos 2x) dx = \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

12. $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

Solution:

Let $I = \int_0^{\pi} \frac{x\,dx}{1+\sin x}$..........(1)

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x(\pi - x)} dx$ $\left(\int_1^a f(x) dx = \int_0^a f(a-x) dx \right)$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx$........(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx$

$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$

$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx \Rightarrow 2I = \pi \int_0^{\pi} \{\sec^2 x - \tan x \sec x\} dx$

$\Rightarrow 2I = \pi[2] \Rightarrow I = \pi$

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$

Solution:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$.........(1)

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, thus, $\sin^2 x$ is an odd function

$f(x)$ is an odd function, then $\int_{-a}^{a} f(x) dx = 0$

$$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7 x dx=0$$

14. $\int_0^{2\pi}\cos^5 x\,dx$

Solution:

Let $I=\int_0^{2\pi}\cos^5 x\,dx..........(1)$

$\cos^5(2\pi-x)=\cos^5 x$

We know that, $\int_0^{2a} f(x)dx=2\int_0^{a} f(x)dx$, if $f(2a-x)=f(x)$

$=0$ if $f(2a-x)=-f(x)$

$\therefore I=2\int_0^{2\pi}\cos^5 xdx$

$\Rightarrow I=2(0)=0$ $\quad\left[\cos^5(\pi-x)=-\cos^5 x\right]$

15. Find the value of $\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx..........(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_1^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos x-\sin x}{1+\sin x\cos x}dx.......(2)$$

Adding (1) and (2) we get

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\frac{0}{1+\sin x\cos x}dx\Rightarrow I=0$$

16. Find the value of $\int_0^{\pi}\log(1+\cos x)dx$

Solution: Use the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

The given integral can be written as

$$I = \int_0^{\pi} \log(1+\cos x)dx$$
$$= \int_0^{\pi} \log(1+\cos(\pi - x))dx$$
$$I = \int_0^{\pi} \log(1-\cos x)dx$$

Adding the above two integrals

$$2I = \int_0^{\pi} \log(1+\cos x)dx + \int_0^{\pi} \log(1-\cos x)dx$$
$$= \int_0^{\pi} \log(1-\cos^2 x)dx$$
$$= \int_0^{\pi} \log(\sin^2 x)dx$$
$$= 2\int_0^{\pi} \log(\sin x)dx$$

Hence, $I = \int_0^{\pi} \log(\sin x)dx$

Use the property $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{If } f(2a-x) = f(x) \\ 0 & \text{If } f(2a-x) = -f(x) \end{cases}$

The above integral becomes

$$I = 2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} \log(\cos x)\,dx$$

$$2I = 2\int_0^{\frac{\pi}{2}} \log(\sin x \cos x)\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \int_0^{\frac{\pi}{2}} \log 2dx$$

$$= 2I - 2\left(\frac{\pi}{2}\log 2\right)$$

$$I = \pi \log 2$$

Therefore, $\int_0^{\pi} \log(1+\cos x)\,dx = \pi \log 2$

17. Find the value of $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

Solution: Consider the integral $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

Use the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

The given integral can be rewrite as

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}\,dx$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx$$

The sum of the above two integrals

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

$$= \int_0^a 1dx$$

$$= [x]_0^a$$

$$= a$$

Divide both sides by 2

$$I = \frac{a}{2}$$

Therefore, $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx = \frac{a}{2}$

18. Find the value of $\int_0^4 |x-1|\,dx$

Solution: Consider the integrand

$$|x-1| = \begin{cases} 1-x & 0<x<1 \\ x-1 & 1<x<4 \end{cases}$$

Hence, the integral becomes

$$\int_0^4 |x-1|\,dx = \int_0^1 (1-x)\,dx + \int_1^4 (x-1)\,dx$$

$$= \left(x - \frac{x^2}{2}\right)_0^1 + \left(\frac{x^2}{2} - x\right)_1^4$$

$$= 1 - \frac{1}{2} + \frac{16}{2} - 4 - \frac{1}{2} + 1$$

$$= 5$$

Therefore, $\int_0^4 |x-1|\,dx = 5$

19. Show that $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$, if $f(x)$ and $g(x)$ are defined $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Solution:

Consider the given integral $\int_0^a f(x)g(x)dx$

Use the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, the above integral rewrite as

$$\begin{aligned} I &= \int_0^a f(x)g(x)dx \\ &= \int_0^a f(a-x)g(a-x)dx \\ &= \int_0^a f(x)g(a-x)dx \end{aligned}$$

Adding both integrals

$$\begin{aligned} 2I &= \int_0^a f(x)g(x)dx + \int_0^a f(x)g(a-x)dx \\ &= \int_0^a f(x)\big(g(x)+g(a-x)\big)dx \\ &= 4\int_0^a f(x)dx \end{aligned}$$

Divide both sides with 2

$$I = 2\int_0^a f(x)dx$$

Therefore, $I = 2\int_0^a f(x)dx$

20. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 + x\cos x + \tan^5 x + 1\right)dx$ is

A) 0 B) 2 C) π D) 1

Solution: Use the property that

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{If } f(x) \text{ is odd function} \\ 2\int_0^a f(x)dx & \text{If } f(x) \text{is even function} \end{cases}$$

In the given integrand $x^3, x\cos x, \tan^3 x$ are odd functions.

Hence, the given integral can be rewrite it as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5 x+1\right)dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x^3+x\cos x+\tan^5 x dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1dx$$

$$=0+0+0+2\int_0^{\frac{\pi}{2}}1dx$$

$$=2[x]_0^{\frac{\pi}{2}}$$

$$=2\left(\frac{\pi}{2}\right)$$

$$=\pi$$

Consider, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3+x\cos x+\tan^5 x+1\right)dx=\pi$

This is matching with the option (C)

21. The value of $\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

A) 2 B) $\frac{3}{4}$ C) 0 D) -2

Solution: Suppose that the integral $I=\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$

Use the property $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

The integral becomes

$$I=\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

Adding the above two integrals

$$2I=\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx+\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\cos x}{4+3\sin x}\times\frac{4+3\sin x}{4+3\cos x}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\log(1)\,dx$$

$$=0$$

This is matching with the option (C)