Chapter: 9. Differential Equations

Exercise: Miscellaneous

1. For each of the differential equations given below, indicate its order and degree (if defined)

i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

Solution:

The given differential equation is $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

The highest order derivative in the equation is of the term $\frac{d^2y}{dx^2}$, thus the order of the equation is 2 and its highest power is 1. Therefore its degree is 1

ii) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

Solution:

The given differential equation is $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y-\sin x=0$

The highest order derivative in the equation is of the term $\left(\frac{dy}{dx}\right)^3$, thus the order of the equation is 1 and its highest power is 3. Therefore its degree is 3

iii) $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Solution:

The given differential equation is $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

The highest order derivative in the equation is of the term $\frac{d^4y}{dx^4}$, thus the order of the equation is 4.

As the differential equation is not polynomial in its derivative, therefore its degree is not

defined.

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

i) $xy = ae^{x} + be^{-x} + x^{2} : x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx} - xy + x^{2} - 2 = 0$

Solution:

The given function is $xy = ae^{x} + be^{-x} + x^{2}$

Take derivative on both side:

$$\Rightarrow y + x\frac{dy}{dx} = ae^{x} - be^{-x} + 2x$$

Take derivative on both side:

$$\Rightarrow \frac{dy}{dx} + x\frac{d^{2}y}{dx^{2}} + \frac{dy}{dx} = ae^{x} + be^{-x} + 2$$

$$\Rightarrow x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx} = ae^{x} + be^{-x} + 2 \text{..........}(1)$$

The given differential equation is

$$x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx} - xy + x^{2} - 2 = 0$$

Solving LHS

Substitute $x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx}$ from the result (1) and xy

$$\Rightarrow \left(x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx}\right) - xy + x^{2} - 2$$

$$\Rightarrow \left(ae^{x}+be^{-x}+2\right)-\left(ae^{x}+be^{-x}+x^{2}\right)+x^{2}-2$$

$$\Rightarrow 2-x^{2}+x^{2}-2$$

$$\Rightarrow 0$$

Thus LHS = RHS the given function is the solution of the given differential equation

ii) $y=e^{x}\left(a\cos x+b\sin x\right):\dfrac{d^{2}y}{dx^{2}}-2\dfrac{dy}{dx}+2y=0$

Solution:

The given function is $y=e^{x}\left(a\cos x+b\sin x\right)$

Take derivative on both sides

$$\Rightarrow \frac{dy}{dx}=e^{x}\left(a\cos x+b\sin x\right)+e^{x}\left(-a\sin x+b\cos x\right)$$

$$\Rightarrow \frac{dy}{dx}=e^{x}\left(\left(a+b\right)\cos x+\left(b-a\right)\sin x\right)$$

Take derivative on both side

$$\Rightarrow \frac{d^{2}y}{dx^{2}}=e^{x}\left(\left(a+b\right)\cos x+\left(b-a\right)\sin x\right)+e^{x}\left(-\left(a+b\right)\sin x+\left(b-a\right)\cos x\right)$$

$$\Rightarrow \frac{d^{2}y}{dx^{2}}=e^{x}\left(\left(a+b+b-a\right)\cos x+\left(b-a-a-b\right)\sin x\right)$$

$$\Rightarrow \frac{d^{2}y}{dx^{2}}=e^{x}\left(2b\cos x-2a\sin x\right)$$

The given differential equation is

$$\frac{d^{2}y}{dx^{2}}-2\frac{dy}{dx}+2y=0$$

Solving LHS

$$\Rightarrow e^{x}\left(2b\cos x-2a\sin x\right)-2e^{x}\left(\left(a+b\right)\cos x+\left(b-a\right)\sin x\right)+2y$$

$\Rightarrow e^x\left(\left(2b-2a-2b\right)\cos x+\left(-2a-2b+2a\right)\sin x\right)-2y$

$\Rightarrow e^x\left(-2a\cos x-2b\sin x\right)-2y$

$\Rightarrow -2e^x\left(a\cos x+b\sin x\right)-2y$

$\Rightarrow 0$

Thus LHS = RHS, the given function is the solution of the given differential equation

iii) $y=x\sin 3x:\dfrac{d^2y}{dx^2}+9y-6\cos 3x=0$

Solution:

The given function is $y=x\sin 3x$

Take derivative on both side

$\Rightarrow \dfrac{dy}{dx}=\sin 3x+3x\cos 3x$

Take derivative on both side

$\Rightarrow \dfrac{d^2y}{dx^2}=3\cos 3x+3\left(\cos 3x+x\left(-3\sin 3x\right)\right)$

$\Rightarrow \dfrac{d^2y}{dx^2}=3\cos 3x+3\cos 3x-9x\sin 3x$

$\Rightarrow \dfrac{d^2y}{dx^2}=6\cos 3x-9x\sin 3x$

The given differential equation is

$\dfrac{d^2y}{dx^2}+9y-6\cos 3x=0$

Solving LHS

$\Rightarrow \dfrac{d^2y}{dx^2}+9y-6\cos 3x$

$$\Rightarrow (6\cos 3x - 9x\sin 3x) + 9(x\sin 3x) - 6\cos 3x$$

$$\Rightarrow 6\cos 3x - 9x\sin 3x + 9x\sin 3x - 6\cos 3x$$

Thus LHS = RHS, the given function is the solution of the given differential equation.

iv) $x^2 = 2y^2 \log y : (x^2 + y^2)\dfrac{dy}{dx} - xy = 0$

Solution:

The given function is $x^2 = 2y^2 \log y$

Take derivative on both side

$$\Rightarrow 2x = 2\left(2y\log y + y^2\left(\frac{1}{y}\right)\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{(2y\log y + y)}$$

Multiply numerator and denominator by y

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{(2y^2\log y + y^2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{(x^2 + y^2)}$$

The given differential equation is

$$(x^2 + y^2)\frac{dy}{dx} - xy = 0$$

Solving LHS

$$\Rightarrow (x^2 + y^2)\frac{dy}{dx} - xy$$

$$\Rightarrow (x^2 + y^2)\left(\frac{xy}{x^2 + y^2}\right) - xy$$

$$\Rightarrow xy - xy$$

$\Rightarrow 0$

Thus LHS = RHS, the given function is the solution of the given differential equation

3. Form the differential equation representing the family of curves given by $(x-a)^2+2y^2=a^2$, where a is an arbitrary constant

Solution:

Given family of curve $(x-a)^2+2y^2=a^2$

$$\Rightarrow x^2-2ax+a^2+2y^2=a^2$$

$$\Rightarrow x^2+2y^2=2ax..........(1)$$

Differentiate both side with respect to x

$$\Rightarrow 2(x-a)+4y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2(x-a)}{4y}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2a-2x}{4y}$$

Multiply numerator and denominator by x

$$\Rightarrow \frac{dy}{dx}=\frac{2ax-2x^2}{4xy}$$

Using expression (1) back substitute 2ax

$$\Rightarrow \frac{dy}{dx}=\frac{x^2+2y^2-2x^2}{4xy}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$$

Thus the differential equation for given family of curve is $\frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$

4. Prove that $x^2-y^2=c\left(x^2+y^2\right)^2$ is the general solution of differential equation $\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$, where c is a parameter

Solution:

Given differential equation

$$\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y}$$

As it can be seen that this is an homogenous equation. Substitute $y=vx$

$$\Rightarrow \frac{d(vx)}{dx}=\frac{x^3-3x(vx)^2}{(vx)^3-3x^2(vx)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{x^3\left(1-3v^2\right)}{x^3\left(v^3-3v\right)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2-v^4+3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v^4}{v^3-3v}$$

Separate the differential

$$\frac{v^3-3v}{1-v^4}dv=\frac{dx}{x}$$

Integrate both side

$$\int\frac{v^3-3v}{1-v^4}dv=\int\frac{dx}{x}$$

$$\Rightarrow\int\frac{v^3-3v}{1-v^4}dv=\log x+\log C$$

$$\Rightarrow I=\log x+\log C\left(I=\int\frac{v^3-3v}{1-v^4}dv\right)............(1)$$

Solving integral I

$$\Rightarrow I=\int\frac{v^3-3v}{1-v^4}dv$$

$$\Rightarrow\frac{v^3-3v}{1-v^4}=\frac{v^3-3v}{(1-v^2)(1+v^2)}$$

$$\Rightarrow\frac{v^3-3v}{1-v^4}=\frac{v^3-3v}{(1-v)(1+v)(1+v^2)}$$

Using partial fraction

$$\Rightarrow\frac{v^3-3v}{1-v^4}=\frac{A}{1-v}+\frac{B}{1+v}+\frac{Cv+D}{1+v^2}$$

Solving for A, B, C and D

$$A=-\frac{1}{2}$$

$$B=\frac{1}{2}$$

$$C=-2$$

$$D=0$$

$$\Rightarrow\frac{v^3-3v}{1-v^4}=\frac{-\frac{1}{2}}{1-v}+\frac{\frac{1}{2}}{1+v}+\frac{-2v+0}{1+v^2}$$

$$I=-\frac{1}{2}\int\frac{1}{1-v}dv+\frac{1}{2}\int\frac{1}{1+v}dv-\int\frac{2v}{1+v^2}dv$$

$$I=-\frac{1}{2}\left(-\log\left(1-v\right)\right)+\frac{1}{2}\left(\log\left(1+v\right)\right)-\log\left(1+v^2\right)$$

$$I=\frac{1}{2}\left(\log\left(1-v^2\right)\right)-\frac{2}{2}\log\left(1+v^2\right)$$

$$I=\frac{1}{2}\left(\log\frac{\left(1-v^2\right)}{\left(1+v^2\right)^2}\right)$$

$$\Rightarrow I=\frac{1}{2}\left(\log\frac{\left(1-\frac{y^2}{x^2}\right)}{\left(1+\frac{y^2}{x^2}\right)^2}\right)$$

$$\Rightarrow I=\frac{1}{2}\left(\log\frac{x^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right)$$

$$\Rightarrow I=\frac{1}{2}\left(\log\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right)+\frac{1}{2}\log x^2$$

$$\Rightarrow I=\frac{1}{2}\left(\log\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right)+\log x$$

Back substitute I in expression (1)

$$\Rightarrow\frac{1}{2}\left(\log\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right)+\log x=\log x+\log C$$

$$\Rightarrow\log\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}=2\log C$$

$$\Rightarrow \frac{x^2-y^2}{\left(x^2+y^2\right)^2}=C^2$$

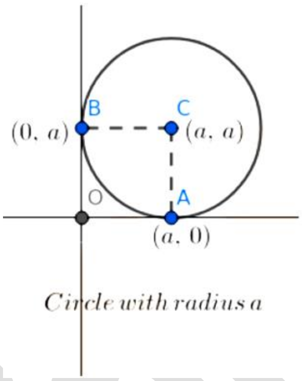
$$\Rightarrow x^2-y^2=c\left(x^2+y^2\right)^2=\left(c=C^2\right)$$

Thus for given differential equation, its general solution is $x^2-y^2=c\left(x^2+y^2\right)^2$

5. Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes

Solution:

Draw a circle according to question



The equation of the given circle will be

$$(x-a)^2+(y-a)^2=a^2$$

Differentiate both side with respect to x

$$\Rightarrow 2(x-a)+2(y-a)\frac{dy}{dx}=0$$

$$\Rightarrow x-a+yy'-ay'=0$$

$$\Rightarrow a=\frac{x+yy'}{1+y'}$$

Using equation of circle

$$\Rightarrow (x-a)^2+(y-a)^2=a^2$$

$$\Rightarrow \left(x-\left(\frac{x+yy'}{1+y'}\right)\right)^2+\left(y-\left(\frac{x+yy'}{1+y'}\right)\right)^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow \left(\frac{x+xy'-x-yy'}{1+y'}\right)^2+\left(\frac{y+yy'-x-yy'}{1+y'}\right)^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow \left(\frac{y'(x-y)}{1+y'}\right)^2+\left(\frac{y-x}{1+y'}\right)^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow (x-y)^2\left(1+y'^2\right)=(x+yy')^2$$

Thus the differential equation for given family of curves is

$$(x-y)^2\left(1+y'^2\right)=(x+yy')^2$$

6. Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

Solution:

The given differential equation is $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$

$$\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=-\frac{dx}{\sqrt{1-x^2}}$$

Integrate both side

$$\Rightarrow \int\frac{dy}{\sqrt{1-y^2}}=-\int\frac{dx}{\sqrt{1-x^2}}$$

$$\Rightarrow \sin^{-1} y = -\sin^{-1} x + C$$

$$\Rightarrow \sin^{-1} y + \sin^{-1} x = C$$

Thus the general solution of given differential equation is $\sin^{-1} y + \sin^{-1} x = C$

7. Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ is given by $(x+y+1) = A(1-x-y-2xy)$ where A is a parameter

Solution:

The given differential equation is $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+2\left(\frac{1}{2}\right)y+\frac{1}{4}-\frac{1}{4}+1}{x^2+2\left(\frac{1}{2}\right)x+\frac{1}{4}-\frac{1}{4}+1}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$\Rightarrow \frac{dy}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}} = -\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Integrate both side

$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{4}\right)^2} = -\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{4}\right)^2}$$

$$\Rightarrow \frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] = -\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+C$$

$$\Rightarrow \frac{2}{\sqrt{3}}\left(\tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]\right)=C$$

$$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]=\frac{\sqrt{3}}{2}C$$

Thus the general solution for given differential equation is

$$\tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]=\frac{\sqrt{3}}{2}C$$

8. Find the equation of the curve passing through the point $\left(0,\frac{\pi}{4}\right)$ whose differential equation is $\sin x\cos y dx+\cos x\sin y dy=0$

Solution:

Given differential equation is

$$\sin x\cos y dx+\cos x\sin y dy=0$$

$$\Rightarrow \sin x\cos y dx+\cos x\sin y dy=0$$

Divide both side by $\cos x\cos y$

$$\Rightarrow \frac{\sin x\cos y dx+\cos x\sin y dy}{\cos x\cos y}=0$$

$$\Rightarrow \tan x dx+\tan y dy=0$$

$$\Rightarrow \tan y\, dy-\tan x\, dx$$

Integrate both side

$$\Rightarrow \int \tan y\, dy-\int \tan x\, dx$$

$$\Rightarrow \log(\sec y) = -\log(\sec x) + C$$

$$\Rightarrow \log(\sec y) + \log(\sec x) = C$$

$$\Rightarrow \log(\sec x \sec y) = C$$

$$\Rightarrow \sec x \sec y = k \left(k = e^{C}\right)$$

As curve passes through $\left(0, \frac{\pi}{4}\right)$

$$\sec 0 \sec\left(\frac{\pi}{4}\right) = k$$

$$\Rightarrow k = \sqrt{2}$$

$$\Rightarrow \sec x \sec y = \sqrt{2}$$

Thus the equation of required curve is $\sec x \sec y = \sqrt{2}$

9. Find the particular solution of the differential equation $\left(1+e^{2x}\right)dy + \left(1+y^{2}\right)e^{x}dx = 0$ given that $y = 1$ when $x = 0$

Solution:

The given differential equation is $\left(1+e^{2x}\right)dy + \left(1+y^{2}\right)e^{x}dx = 0$

Divide both side $\left(1+e^{2x}\right)\left(1+y^{2}\right)$

$$\frac{dy}{\left(1+y^{2}\right)} + \frac{e^{x}}{\left(1+e^{2x}\right)}dx = 0$$

$$\int \frac{dy}{\left(1+y^{2}\right)} = -\int \frac{e^{x}}{\left(1+e^{2x}\right)}dx$$

$$\tan^{-1} y = -\int \frac{e^{x}}{\left(1+\left(e^{x}\right)^{2}\right)}dx$$

Substitute $t=e^{x}$

$dt=e^{x}dx$

$\Rightarrow \tan^{-1}y=-\int \frac{1}{\left(1+t^{2}\right)}dt$

$\Rightarrow \tan^{-1}y=-\tan^{-1}t+C$

$\Rightarrow \tan^{-1}y=-\tan^{-1}e^{x}+C$

$\Rightarrow \tan^{-1}y+\tan^{-1}e^{x}=C$

As $y=1$ when $x=0$

$\tan^{-1}(1)+\tan^{-1}\left(e^{0}\right)=C$

$\Rightarrow \frac{\pi}{4}+\frac{\pi}{4}=C$

$\Rightarrow C=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}y+\tan^{-1}e^{x}=\frac{\pi}{2}$

Thus the required particular solution is $\tan^{-1}y+\tan^{-1}e^{x}=\frac{\pi}{2}$

10. Solve the differential equation $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^{2}\right)dy\left(y\neq 0\right)$

Solution:

The given differential equation is $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^{2}\right)dy$

$ye^{\frac{x}{y}}\frac{dx}{dy}=xe^{\frac{x}{y}}+y^{2}$

$$\Rightarrow ye^{\frac{x}{y}}\frac{dx}{dy}-xe^{\frac{x}{y}}=y^2$$

$$\Rightarrow e^{\frac{x}{y}}\frac{\left[y\frac{dx}{dy}-x\right]}{y^2}=1$$

Substitute $z=e^{\frac{x}{y}}$

$$z=e^{\frac{x}{y}}$$

$$\frac{d}{dz}z=\frac{d}{dy}e^{\frac{x}{y}}$$

$$\Rightarrow \frac{dz}{dy}=\frac{d}{dy}\left(e^{\frac{x}{y}}\right)$$

$$\Rightarrow \frac{dz}{dy}=e^{\frac{x}{y}}\frac{d}{dy}\left(\frac{x}{y}\right)$$

$$\Rightarrow \frac{dz}{dy}=e^{\frac{x}{y}}\left[\left(\frac{1}{y}\right)\frac{dx}{dy}-\frac{x}{y^2}\right]$$

$$\Rightarrow \frac{dz}{dy}=e^{\frac{x}{y}}\left[\frac{y\frac{dx}{dy}-x}{y^2}\right]$$

$$\Rightarrow \frac{dz}{dy}=1$$

$$\Rightarrow dz=dy$$

$$\Rightarrow \int dz=\int dy$$

$$\Rightarrow z=y+C$$

$$\Rightarrow e^{\frac{x}{y}}=y+C$$

Thus the required general solution is $e^{\frac{x}{y}} = y + C$

11. Find a particular solution of the differential equation $(x-y)(dx+dy) = dx - dy$ given that $y = -1$ when $x = 0$. Hint (put $x - y = t$)

Solution:

Given differential equation is $(x-y)(dx+dy) = dx - dy$

$$\Rightarrow (x-y)dx - dx = (y-x)dy - dy$$

$$\Rightarrow (x-y+1)dy = (1-x+y)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-x+y}{x-y+1}$$

Put $x - y = t$

$$\Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 1 - \frac{dt}{dx} = \frac{dy}{dx}$$

$$\Rightarrow 1 - \frac{dt}{dx} = \frac{1-t}{1+t}$$

$$\Rightarrow \frac{dt}{dx} = 1 - \frac{1-t}{1+t}$$

$$\Rightarrow \frac{dt}{dx} = \frac{1+t-1+t}{1+t}$$

$$\Rightarrow \frac{dt}{dx} = \frac{2t}{1+t}$$

$$\Rightarrow \frac{1+t}{t} dt = 2dx$$

Integrating both side

$$\Rightarrow \int \frac{1+t}{t} dt = 2\int dx$$

$$\Rightarrow \int \frac{1}{t} dt + \int dt = 2x + C$$

$$\Rightarrow \log|t| + t = 2x + C$$

$$\Rightarrow \log|x-y| + x - y = 2x + C$$

$$\Rightarrow \log|x-y| - y = x + C$$

As y = -1 when x = 0

$$\Rightarrow \log|0-(-1)| - (-1) = 0 + C$$

$$\Rightarrow \log 1 + 1 = C$$

$$\Rightarrow C = 1$$

Thus the required particular solution is $\log|x-y| - y = x + 1$

12. Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1 \left(x \neq 0\right)$

Solution:

Given differential equation is $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is linear differential equation of the form $\Rightarrow \frac{dy}{dx} + py = Q$

$$p=\frac{1}{\sqrt{x}}$$

$$Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Calculating integrating factor

$$I.F=e^{\int pdx}$$

$$I.F=e^{\int \frac{1}{\sqrt{x}}dx}$$

$$I.F=e^{2\sqrt{x}}$$

The general solution is given by

$$y\times I.F=\int(Q\times IF)dx+C$$

$$y\times\left(e^{2\sqrt{x}}\right)=\int\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\times e^{2\sqrt{x}}\right)dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=\int\left(\frac{e^{-2\sqrt{x}+2\sqrt{x}}}{\sqrt{x}}\right)dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=2\sqrt{x}+C$$

Thus the general solution for the given differential equation is

$$ye^{2\sqrt{x}}=2\sqrt{x}+C$$

13. Find a particular solution of the differential equation $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x\,(x\neq 0)$ given that $y=0$ when $x=\frac{\pi}{2}$

Solution:

The given differential equation is

$$\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x$$

It is linear differential equation of the form $\frac{dy}{dx} + py = Q$

$$p = \cot x$$
$$Q = 4x\operatorname{cosec} x$$

Calculating integrating factor

$$I.F = e^{\int p dx}$$

$$I.F = e^{\int \cot x dx}$$

$$I.F = e^{\log|\sin x|}$$

$$I.F = \sin x$$

The general solution is given by

$$y \times I.F = \int (Q \times I.F)\,dx + C$$

$$\Rightarrow y \times \sin x = \int (4x\operatorname{cosec} x)\sin x\,dx + C$$

$$\Rightarrow y\sin x = 4\int x\,dx + C$$

$$\Rightarrow y\sin x = 4\left(\frac{x^2}{2}\right) + C$$

$$\Rightarrow y\sin x = 2x^2 + C$$

As $y = 0$ when $x = \frac{\pi}{2}$

$$0 \times \sin\left(\frac{\pi}{2}\right) = 2\left(\frac{\pi}{2}\right)^2 + C$$

$$C=-2\left(\frac{\pi^2}{4}\right)$$

$$C=-\frac{\pi^2}{2}$$

Thus the required particular solution is

$$y\sin x=2x^2-\frac{\pi^2}{2}$$

14. Find a particular solution of the differential equation $(x+1)\frac{dy}{dx}=2e^{-y}-1$ given that $y=0$ when $x=0$. $(x+1)\frac{dy}{dx}=2e^{-y}-1$

Solution:

The given differential equation is $(x+1)\frac{dy}{dx}=2e^{-y}-1$

$$\Rightarrow\frac{dy}{2e^{-y}-1}=\frac{dx}{x+1}$$

Integrate both side

$$\Rightarrow\int\frac{dy}{2e^{-y}-1}=\int\frac{dx}{x+1}$$

$$\Rightarrow\int\frac{dy}{2e^{-y}-1}=\log(x+1)+\log C.........(1)$$

Evaluating LHS integral

$$\Rightarrow\int\frac{dy}{2e^{-y}-1}=\int\frac{e^y dy}{2-e^y}$$

Put $t=2-e^y$

$$t=2-e^y$$

$$dt = e^y dy$$

$$\Rightarrow \int \frac{dy}{2e^{-y}-1} = -\int \frac{dt}{t}$$

$$\Rightarrow \int \frac{dy}{2e^{-y}-1} = \log \frac{1}{t}$$

$$\Rightarrow \int \frac{dy}{2e^{-y}-1} = \log \frac{1}{2-e^y}$$

Back substituting in expression (1)

$$\Rightarrow \int \frac{dy}{2e^{-y}-1} = \log(x+1) + \log C$$

$$\Rightarrow \log\left(\frac{1}{2-e^y}\right) = \log C(x+1)$$

$$\Rightarrow 2-e^y = \frac{1}{C(x+1)}$$

As $y=0$ when $x=0$

$$\Rightarrow 2-e^0 = \frac{1}{C(0+1)}$$

$$\Rightarrow 2-1 = \frac{1}{C}$$

$$\Rightarrow C = 1$$

Thus the required particular solution is

$$\Rightarrow 2-e^y = \frac{1}{(x+1)}$$

$$\Rightarrow e^y = 2-\frac{1}{(x+1)}$$

$$\Rightarrow e^y = \frac{2x+2-1}{(x+1)}$$

$$\Rightarrow e^{y} = \frac{2x+1}{(x+1)}$$

$$\Rightarrow y = \log\left(\frac{2x+1}{x+1}\right)$$

Thus for given conditions the particular solution is $y = \log\left(\frac{2x+1}{x+1}\right)$

15. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Solution:

Let the population at time t be y. The according to question $\frac{dy}{dt} \propto y$

$$\Rightarrow \frac{dy}{dt} = ky$$

$$\Rightarrow \frac{dy}{y} = kdt$$

Integrate both side

$$\Rightarrow \int \frac{dy}{y} = k\int dt$$

$$\Rightarrow \log y = kt + C$$

In 1999 (taking reference year) t = 0 population was y = 20000

$$\Rightarrow \log(20000) = k(0) + C$$

$$\Rightarrow C = \log(20000)$$

In 2004, t =5 population was y = 25000

$$\Rightarrow \log(25000) = k(5) + \log(20000)$$

$$\Rightarrow 5k = \log(25000) - \log(20000)$$

$$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right)$$

$$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right)$$

Thus the population relation will be

$$\log y = \frac{1}{5}\log\left(\frac{5}{4}\right)t + \log(20000)$$

For 2009, t =10

$$\Rightarrow \log y = \frac{1}{5}\log\left(\frac{5}{4}\right)\times 10 + \log(20000)$$

$$\Rightarrow \log y = 2\log\left(\frac{5}{4}\right) + \log(20000)$$

$$\Rightarrow \log y = \log\left(\frac{5}{4}\right)^2 + \log(20000)$$

$$\Rightarrow \log y = \log\left(\frac{25}{16}\times 20000\right)$$

$$\Rightarrow y = 31250$$

Thus the population in the year 2009, is 31250

16. The general solution of the differential equation $\frac{ydx - xdy}{y} = 0$

Solution:

Given differential equation

$$\frac{ydx - xdy}{y} = 0$$

Divide both side by x

$$\Rightarrow \frac{ydx - xdy}{xy} = 0$$

$$\Rightarrow \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrate both side

$$\Rightarrow \int \frac{dx}{x} - \int \frac{dy}{y} = 0$$

$$\Rightarrow \log\left|\frac{x}{y}\right| = \log k$$

$$\Rightarrow \frac{x}{y} = k$$

$$\Rightarrow y = Cx\left(C = \frac{1}{k}\right)$$

Thus the correct option is (C)

17. Find the general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$

Solution:

The given differential equation is

$$\frac{dx}{dy} + P_1 x = Q_1$$

It is linear differential equation and its general solution is

$$xe^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dy + C$$

With integrating factor $I.F = e^{\int P_1 dy}$

Thus the correct option is (C)

18. Find the general solution of the differential equation $e^x dy + \left(ye^x + 2x\right)dx = 0$

Solution:

The given differential equation is

$$e^x dy + \left(ye^x + 2x\right)dx = 0$$

$$\Rightarrow e^x \frac{dy}{dx} + ye^x = -2x$$

$$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$$

The given differential equation is of the form

$$\frac{dy}{dx} + Py = Q$$

$$\Rightarrow P = 1$$
$$\Rightarrow Q = -2xe^{-x}$$

Calculating integrating factor

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int dx}$$

$$\Rightarrow I.F = e^x$$

It is a linear differential equation and its general solution is

$$y\left(I.F\right) = \int \left(Q \times I.F\right)dx + C$$

$$y\left(e^x\right) = \int \left(-2xe^{-x} \times e^x\right)dy + C$$

$$\Rightarrow ye^x = -2\int x\,dx + C$$

$$\Rightarrow ye^x = -2\left(\frac{x^2}{2}\right) + C$$

$$\Rightarrow ye^x + x^2 = C$$

Thus the correct answer is option (C)

Exercise: 9.1

1. Determine order and degree (if defined) of differential equation $\frac{d^4y}{dx^4}+\sin(y''')=0$

Solution:

Rewrite the equation $\frac{d^4y}{dx^4}+\sin(y''')=0$. As

$\Rightarrow y''''+\sin(y''')=0$

The highest order between the two terms is of y'''' which is four.

The differential equation contains a trigonometric derivative term and is not

completely polynomial in its derivative, thus degree is not defined.

2. Determine order and degree (if defined) of differential equation $y'+5y=0$.

Solution:

The given differential equation is $y'+5y=0$

The highest order term is y', thus the order is one.

As the derivative is of completely polynomial nature is and highest power of derivative is

of y' which is one. Thus degree is one

3. Determine order and degree (if defined) of differential equation $\left(\frac{ds}{dt}\right)^4+3s\frac{d^2s}{dt^2}=0$

Solution:

The given differential equation is $\left(\frac{ds}{dt}\right)^4+3s\frac{d^2s}{dt^2}=0$

The highest order term is $\frac{d^2s}{dt^2}$, thus the order is two.

As the derivative is of completely polynomial nature is and highest power of

derivative term $\left(\frac{ds}{dt}\right)^4$ which is four. Thus the degree is four.

4. Determine order and degree (if defined) of differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2+\cos\left(\frac{dy}{dx}\right)=0$$

Solution:

The given differential equation is $\left(\frac{d^2y}{dx^2}\right)^2+\cos\left(\frac{dy}{dx}\right)=0$

The highest order term is $\frac{d^2y}{dx^2}$, thus the order is two

The differential equation contains a trigonometric derivative term and is not completely polynomial in its derivative, thus degree is not defined.

5. Determine order and degree (if defined) of differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2-\cos 3x+\sin 3x=0$$

Solution:

The given differential equation is $\left(\frac{d^2y}{dx^2}\right)^2-\cos 3x+\sin 3x=0$

$\left(\frac{d^2y}{dx^2}\right)^2-\cos 3x+\sin 3x=0$ the highest order term is $\frac{d^2y}{dx^2}$, thus the order is two

As the derivative is of completely polynomial nature is and highest power of derivative term which is two. Thus degree is two $\left(\frac{d^2y}{dx^2}\right)^2$

6. Determine order and degree (if defined) of differential equation $(y''')^2+(y'')^3+(y')^4+y^5=0$

Solution:

The given differential equation is $(y''')^2+(y'')^3+(y')^4+y^5=0$.

The highest order term is $(y''')^2$, thus the order is three.

The differential equation is of the polynomial form and the power of highest order term y''' is two, thus the degree is two

7. Determine order and degree (if defined) of differential equation $y'''+2y''+y'=0$

Solution:

The given differential equation is . y'''+2y''+y'=0

The highest order derivative in the differential equation is y'''. Thus its order is three.

The differential equation is polynomial with the highest order term y''' having a degree one. Thus the degree is one.

8. Determine order and degree (if defined) of differential equation $y'+y=e'$

Solution:

The given differential equation is $y'+y=e'$. Therefore

$\Rightarrow y'+y-e'=0$

The highest order derivative in the differential equation is y'. Thus its order is one.

The given equation is of polynomial form with the highest order term y' with degree one. Thus the degree is one.

9. Determine order and degree (if defined) of differential equation $y'+\left(y'\right)^{2}+2y=0$

Solution:

The given differential equation is $y'+\left(y'\right)^{2}+2y=0$

The highest order derivative in the differential equation is y'. Thus its order is one.

The given equation is of polynomial form with the highest order term y' with highest degree two. Thus the degree is two.

10. Determine order and degree (if defined) of differential equation $y''+2y'+\sin y=0$

Solution:

The given differential equation is $y''+2y'+\sin y=0$.

The highest order derivative in the differential equation is y''. Thus its order is two.

The given equation is of polynomial form with the highest order term y'' with the highest degree one. Thus the degree is one.

11. Find the degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$

Solution:

The given differential equation is $\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$

The differential equation is not polynomial in its derivative because of the term $\sin\left(\frac{dy}{dx}\right)$

thus its order is not defined.

The correct answer is (D).

12. Find the order of the differential equation $2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$

Solution:

The given differential equation is $2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$

The highest order term of the equation is $\frac{d^2y}{dx^2}$, thus the order is two

The correct answer is (A)

Exercise: 9.2

1. Verify the function $y = e^x + 1$ is solution of differential equation $y'' - y = 0$

Solution:

The given function is $y = e^x + 1$

Take its derivative

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x + 1\right)$$

$$\Rightarrow y' = e^x ....(1)$$

Take the derivative of the above equation

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(e^x\right)$$

$$\Rightarrow y'' = e^x$$

Using result from equation (1)

$$y'' - y' = 0$$

Thus, the given function is solution of differential equation $y'' - y' = 0$

2. Verify the function $y = x^2 + 2x + C$ is solution of differential equation $y' - 2x - 2 = 0$

Solution:

The given function is $y = x^2 + 2x + C$

Take its derivative

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + 2x + C\right)$$

$\Rightarrow y' = 2x + 2$

Thus, the given function is solution of differential equation $y' - 2x - 2 = 0$

3. Verify the function $y = \cos x + C$ is solution of differential equation $y' + \sin x = 0$

Solution:

The given function is $y = \cos x + C$

Take its derivative

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x + c)$$

$$\Rightarrow y' = -\sin x$$

$$\Rightarrow y' + \sin x = 0$$

Thus the given function is solution of differential equation $y' + \sin x = 0$

4. Verify the function $y = \sqrt{1+x^2}$ is solution of differential equation $y' = \frac{xy}{1+x^2}$

Solution:

The given function is $y = \sqrt{1+x^2}$

Take its derivative

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$y' = \frac{1}{2\sqrt{1+x^2}} \times \frac{d}{dx}\left(1+x^2\right)$$

$$y' = \frac{2}{2\sqrt{1+x^2}}$$

$$\Rightarrow y' = \frac{1}{\sqrt{1+x^2}}$$

Multiply numerator and denominator by $\sqrt{1+x^2}$

$$y' = \frac{1}{\sqrt{1+x^2}} \times \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

Substitute $y = \sqrt{1+x^2}$

$$\Rightarrow y' = \frac{xy}{1+x^2}$$

Thus the given function is solution of differential equation $y' = \frac{xy}{1+x^2}$

5. Verify the function $y = Ax$ is solution of differential equation $xy' = y\left(x \neq 0\right)$

Solution:

The given function is $y = Ax$

Take its derivative

$$\frac{dy}{dx} = \frac{d}{dx}\left(Ax\right)$$

$$\Rightarrow y' = A$$

Multiply by x on both side

$$xy' = Ax$$

Substitute $y = Ax$

$$\Rightarrow xy' = y$$

Thus the given function is solution of differential equation $xy' = y\left(x \neq 0\right)$

6. Verify the function $y = x\sin x$ is solution of differential equation $xy' = y + x\sqrt{x^2 - y^2}$ ($x \neq 0$ and $x > y$ or $x < -y$)

Solution:

The given function is $y = x\sin x$

Take its derivative

$$\frac{dy}{dx} = \frac{d}{dx}(x\sin x)$$

$$\Rightarrow y' = \sin x\frac{d}{dx}(x) + x\frac{d}{dx}(\sin x)$$

$$\Rightarrow y' = \sin x + x\cos x$$

Multiply by x on both side

$$xy' = x(\sin x + x\cos x)$$

$$xy' = x\sin x + x^2\cos x$$

Substitute $y = x\sin x$

$$\Rightarrow xy' = y + x^2\cos x$$

Use $\sin x = \frac{y}{x}$ and substitute $\cos x$

$$xy' = y + x^2\sqrt{1 - \sin^2 x}$$

$$xy' = y + x^2\sqrt{1 - \left(\frac{y}{x}\right)^2}$$

$$\Rightarrow xy' = y + x\sqrt{y^2 - x^2}$$

Thus the given function is solution of differential equation $\Rightarrow xy' = y + x\sqrt{y^2 - x^2}$

7. Verify the function $xy = \log y + c$ is solution of differential equation

$$y' = \frac{y^2}{1 - xy}\left(xy \neq 1\right)$$

Solution:

The given function is $xy = \log y + c$

Take derivative on both sides

$$\frac{d}{dx}(xy) = \frac{d}{dx}(\log y)$$

$$\Rightarrow y\frac{d}{dx}(x) = x\frac{d}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow y + xy' = \frac{1}{y}y'$$

$$\Rightarrow y^2 + xyy' = y'$$

Shift the y' term on one side and take it common

$$\Rightarrow (xy - 1)y' = -y^2$$

$$\Rightarrow y' = \frac{y^2}{1 - xy}$$

Thus the given function is solution of differential equation $y' = \frac{y^2}{1 - xy}$

8. Verify the function $y - \cos y = x$ is solution of differential equation

$$\left(y \sin y + \cos y + x\right)y' = 1$$

Solution:

The given function is $y - \cos y = x$

Take derivative on both side

$$\frac{dy}{dx}-\frac{d}{dx}(\cos y)=\frac{d}{dx}(x)$$

$$\Rightarrow y'+y'\sin y=1$$

$$\Rightarrow y'(1+\sin y)=1$$

$$\Rightarrow y'=\frac{1}{1+\sin y}$$

Multiply by $(y\sin y+\cos y+x)$ on both side

$$(y\sin y+\cos y+x)y'=\frac{(y\sin y+\cos y+x)}{1+\sin y}$$

Substitute $y=\text{cosy}+\text{x}$ in the numerator

$$(y\sin y+\cos y+x)y'=\frac{(y\sin y+y)}{1+\sin y}$$

$$(y\sin y+\cos y+x)y'=\frac{y(\sin y+y)}{1+\sin y}$$

$$\Rightarrow(y\sin y+\cos y+x)y'=y$$

Thus the given function is solution of differential equation $(y\sin y+\cos y+x)y'=y$

9. Verify the function $x+y=\tan^{-1}y$ is solution of differential equation $y^2y'+y^2+1=0$

Solution:

The given function is $x+y=\tan^{-1}y$

Take derivative on both side

$$\frac{d}{dx}(x+y)=\frac{d}{dx}(\tan^{-1}y)$$

$$1+y'=\left(\frac{1}{1+y^2}\right)y'$$

$$\Rightarrow y'\left[\frac{1}{1+y^2}-1\right]=1$$

$$\Rightarrow y'\left[\frac{1-\left(1+y^2\right)}{1+y^2}\right]=1$$

$$\Rightarrow y'\left[\frac{-y^2}{1+y^2}\right]=1$$

$$\Rightarrow -y^2y'=1+y^2$$

$$\Rightarrow y^2y'+y^2+1=0$$

Thus the given function is solution of differential equation $y^2y'+y^2+1=0$

10. Verify the function $y=\sqrt{a^2-x^2}\ \ x\in(-a,a)$ is solution of differential equation

$$x+y\frac{dy}{dx}=0\,(y\neq 0)$$

Solution:

The given function is $y=\sqrt{a^2-x^2}\ \ x\in(-a,a)$

Take derivative on both side

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}\frac{d}{dx}\left(a^2-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2x}{2\sqrt{a^2-x^2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{\sqrt{a^2-x^2}}$$

Substitute $y=\sqrt{a^2-x^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow x + y\frac{dy}{dx} = 0$$

Thus the given function is solution of differential equation $x + y\frac{dy}{dx} = 0 \left(y \neq 0 \right)$

11. Find the numbers of arbitrary constants in the general solution of a differential equation of fourth order

Solution:

The number of arbitrary constants in the general solution of a differential equation is equal to its order. As the given differential equation is of fourth order, thus it has four arbitrary constants in its solution.

The correct answer is (D).

12. Find the numbers of arbitrary constants in the particular solution of a differential equation of third order

Solution:

The particular solution of any differential equation does not have any arbitrary constants.

Thus it has zero constants in its solution.

The correct answer is (D).

Exercise: 9.3

1. Form a differential equation representing the family of the curve $\frac{x}{a}+\frac{y}{b}=1$ by eliminating arbitrary constants

Solution:

The given differential equation is $\frac{x}{a}+\frac{y}{b}=1$

Take the derivative on both sides

$$\frac{d}{dx}\left(\frac{x}{a}+\frac{y}{b}\right)=0$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}y'=0$$

Take the derivative of the above equation to eliminate the constants

$$\Rightarrow 0+\frac{1}{b}y''=0$$

$$\Rightarrow y''=0$$

The curve for given differential equations is $y''=0$

2. Form a differential equation representing the family of the curve $y^2=a\left(b^2-x^2\right)$ by eliminating arbitrary constants

Solution:

The given differential equation is $y^2=a\left(b^2-x^2\right)$

Take the derivative on both sides

$$2y\frac{dy}{dx}=a(-2x)$$

$$\Rightarrow 2yy'\frac{dy}{dx} = -2ax$$

$$\Rightarrow yy'\frac{dy}{dx} = -ax..........(1)$$

Take the derivative of the above equation to eliminate the constants

$$\Rightarrow yy''+y'y' = -a$$

$$\Rightarrow (y')^2 + yy'' = -a$$

Substitute this in result (1)

$$yy' = \left((y')^2 + yy''\right)x$$

$$\Rightarrow xyy''+x(y')^2 - yy' = 0$$

The curve for given differential equation is $xyy''+x(y')^2 - yy' = 0$

3. Form a differential equation representing the family of the curve $y = ae^{3x} + be^{-2x}$ by eliminating arbitrary constants

Solution:

The given differential equation is $y = ae^{3x} + be^{-2x}$

Take the derivative on both sides

$$y' = 3ae^{3x} - 2be^{-2x}$$

Take derivative of above equation

$$y'' = 9ae^{3x} + 4be^{-2x}$$

Compute $y'+2y$

$$y'+2y = 3ae^{3x} - 2be^{-2x} + 2ae^{3x} + 2be^{-2x}$$

$\Rightarrow 5a^{3x} = y'+2y$

$\Rightarrow a^{3x} = \frac{y'+2y}{5}$

Compute $3y-y'$

$3y-y' = 3ae^{3x}+3be^{-2x}-\left(3ae^{3x}-2be^{-2x}\right)$

$\Rightarrow 5b^{-2x} = 3y-y'$

$\Rightarrow b^{-2x} = \frac{3y-y'}{5}$

Substitute the above results in y''

$\Rightarrow y'' = 9\left(\frac{y'+2y}{5}\right)+4\left(\frac{3y-y'}{5}\right)$

$\Rightarrow y'' = \frac{5y'+30y}{5}$

$\Rightarrow y'' = y'+6y$

$\Rightarrow y''-y'-6y=0$

The curve for given differential equation is $y''-y'-6y=0$

4. Form a differential equation representing the family of the curve $y=e^{2x}\left(a+bx\right)$ by eliminating arbitrary constants

Solution:

The given differential equation is $y=e^{2x}\left(a+bx\right)$

Take the derivative on both sides

$y' = 2e^{2x}\left(a+bx\right)+e^{2x}\left(b\right)$

$\Rightarrow y' = e^{2x}\left(2a+2bx+b\right)$

Compute $y'-2y$

$$y'-2y=e^{2x}\left(2a+2bx+b\right)-2e^{2x}\left(a+bx\right)$$

$$\Rightarrow y'-2y=be^{2x}..........(1)$$

Take derivative of the above equation

$$\Rightarrow y''-2y'=2be^{2x}$$

Substitute the above results using result from equation (1)

$$\Rightarrow y''-2y'=2\left(y'-2y\right)$$

$$\Rightarrow y''-4y'+4y=0$$

The curve for given differential equation is $y''-4y'+4y=0$

5. Form a differential equation representing the family of the curve $y=e^{x}\left(a\cos x+b\sin x\right)$ by eliminating arbitrary constants.

Solution:

The given differential equation is $y=e^{x}\left(a\cos x+b\sin x\right)$

Take the derivative on both sides

$$y'=e^{x}\left(a\cos x+b\sin x\right)+e^{x}\left(-a\sin x+b\cos x\right)$$

$$\Rightarrow y'=e^{x}\left[\left(a+b\right)\cos x-\left(a-b\right)\sin x\right]$$

Take the derivative of the above equation

$$y''=e^{x}\left[\left(a+b\right)\cos x-\left(a-b\right)\sin x\right]+e^{x}\left[-\left(a+b\right)\sin x-\left(a-b\right)\cos x\right]$$

$$y''=e^{x}\left[\left(a+b-a+b\right)\cos x-\left(a-b+a+b\right)\sin x\right]$$

$$\Rightarrow y''=e^{x}\left[2b\cos x-2a\sin x\right]$$

$$\Rightarrow y'' = 2e^x\left[b\cos x - a\sin x\right]$$

$$\Rightarrow \frac{y''}{2} = e^x\left[b\cos x - a\sin x\right]$$

Add y on both side

$$y + \frac{y''}{2} = e^x\left(a\cos x + b\sin x\right) + e^x\left(b\cos x - a\sin x\right)$$

$$\Rightarrow y + \frac{y''}{2} = e^x\left(\left(a+b\right)\cos x - \left(a-b\right)\sin x\right)$$

Back substitute y'

$$y + \frac{y''}{2} = y'$$

$$\Rightarrow 2y + y'' = 2y'$$

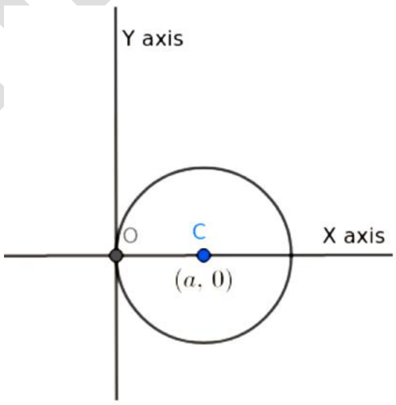
$$\Rightarrow y'' - 2y' + 2y = 0$$

The curve for given differential equation is $y'' - 2y' + 2y = 0$

6. Form the differential equation of the family of circles touching the y-axis at the origin

Solution:

Let a circle with radius a touches y-axis at origin



Thus the given circle will have the center at $(a, 0)$. So it equation will be

$$(x-a)^2+y^2=a^2$$

$$\Rightarrow x^2+y^2=2ax........(1)$$

Take the derivative of the above equation

$$2x+2yy'=2a$$

$$\Rightarrow x+yy'=a$$

Back substitute a in equation (1)

$$x^2+y^2=2(x+yy')x$$

$$\Rightarrow 2x^2+2yy'x=x^2+y^2$$

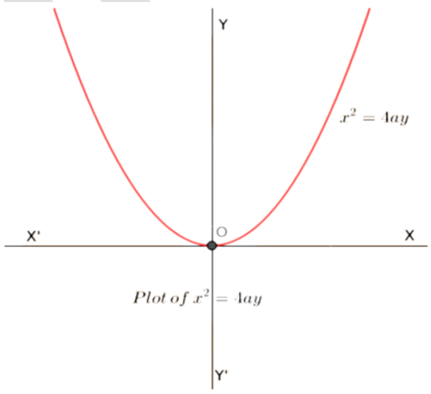
$$\Rightarrow x^2+2yy'x=y^2$$

Thus the differential equation for the given family of the circle is $x^2+2yy'x=y^2$

7. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis

Solution:

Draw a general parabola with given properties



$$x^2=4ay..........(1)$$

Take the derivative of the above equation

$2x = 4ay'$

$\Rightarrow a = \dfrac{x}{2y'}$

Back substitute a in equation (1)

$x^2 = 4\left(\dfrac{x}{2y'}\right)y$

$\Rightarrow x^2 = 2\left(\dfrac{x}{y'}\right)y$

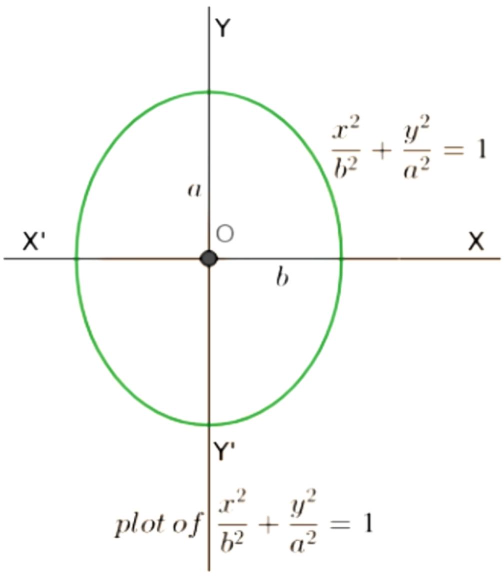
$\Rightarrow x^2y' - 2xy = 0$

Thus the differential equation for the given family of the parabolas is $x^2y' - 2xy = 0$

8. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin

Solution:

Draw a standard eclipse with given properties



$\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

Take the derivative of the above equation

$$\frac{2x}{b^2}+\frac{2yy^1}{a^2}=0$$

$$\Rightarrow \frac{x}{b^2}+\frac{yy^1}{a^2}=0......(1)$$

Take the derivative of the above equation

$$\frac{1}{b^2}+\frac{y'y'+yy''}{a^2}=0$$

$$\frac{1}{b^2}=-\frac{1}{a^2}\left(y'^2+yy''\right)$$

Back substitute in equation (1)

$$x\left[-\frac{1}{a^2}\left(y'^2+yy''\right)\right]+\frac{yy'}{a^2}=0$$

$$\Rightarrow -xy'^2-xyy''+yy'=0$$

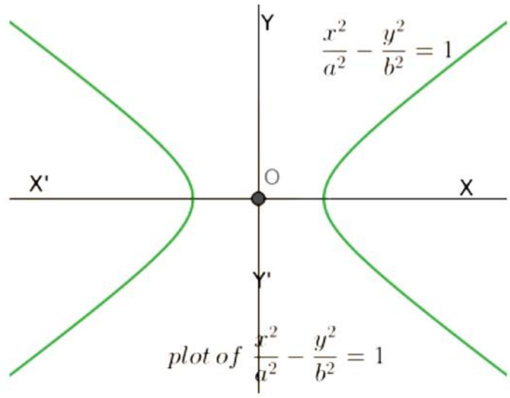
Thus the differential equation for the given family of the parabolas is

$$-xy'^2-xyy''+yy'=0$$

9. From the differential equation of the family of hyperbolas having foci on x-axis and centre at origin

Solution:

Draw a standard hyperbola with given properties



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Take the derivative of the above equation

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$$

$$\Rightarrow \frac{x}{b^2} + \frac{yy'}{a^2} = 0..........(1)$$

Take the derivative of the above equation

$$\frac{1}{b^2} + \frac{y'y' + yy''}{a^2} = 0$$

$$\frac{1}{b^2} = -\frac{1}{a^2}\left(y'^2 + yy''\right)$$

Back substitute in equation (1)

$$x\left[-\frac{1}{a^2}\left(y'^2 + yy''\right)\right] + \frac{yy'}{a^2} = 0$$

$$\Rightarrow -xy'^2 - xyy'' + yy' = 0$$

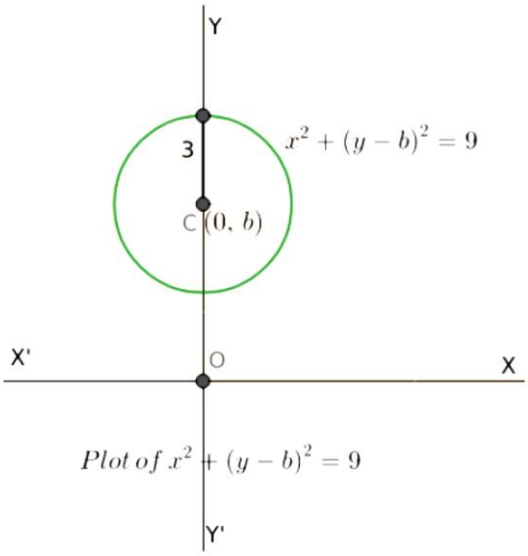
$$\Rightarrow -xyy'' - xy'^2 - yy' = 0$$

Thus the differential equation for the given family of the parabola is $xyy'' + xy'^2 - yy' = 0$

10. Form the differential equation of the family of circles having center on y-axis and radius 3 units

Solution:

Draw the circle with given properties



Plot of $x^2 + (y-b)^2 = 9$

$$x^2 + (y-b)^2 = 9$$

Take the derivative of the above equation

$$2x + 2(y-b)y' = 0$$

$$\Rightarrow x + (y-b)y' = 0$$

$$\Rightarrow (y-b) = -\frac{x}{y'}$$

Back substitute in the equation circle

$$x^2 + \left(\frac{-x}{y'}\right)^2 = 9$$

$$\Rightarrow x^2 + y'^2 + x^2 = 9y'^2$$

$$\Rightarrow (x^2 - 9)y'^2 + x^2 = 0$$

Thus the differential equation for the given family of the parabolas is $\left(x^2-9\right)y'^2+x^2=0$

11. Which of the following differential equation has $y=c_1e^x+c_2e^{-x}$ as the general solution?

A) $\frac{d^2y}{dx^2}+y=0$ B) $\frac{d^2y}{dx^2}-y=0$ C) $\frac{d^2y}{dx^2}+1=0$ D) $\frac{d^2y}{dx^2}-1=0$

Solution:

The given equation is $y=c_1e^x+c_2e^{-x}$

Differentiate the equation

$$\frac{dy}{dx}=c_1e^x-c_2e^{-x}$$

Differentiate the above equation

$$\frac{d^2y}{dx^2}=c_1e^x+c_2e^{-x}$$

Back substitute the y

$$\frac{d^2y}{dx^2}=y$$

$$\Rightarrow\frac{d^2y}{dx^2}-y=0$$

Thus the correct answer is option B

12. Which of the following differential equation has $y=x$ as one of its particular solutions?

A) $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=x$ B) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+xy=x$

C) $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=0$ D) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+xy=0$

Solution:

The given equation is $y=x$

Differentiate the equation

$$\frac{dy}{dx}=1$$

Differentiate the above equation

$$\frac{d^2y}{dx^2}=0$$

Deducing for the option

$$\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=0-x^2(1)+x(x)$$

$$\Rightarrow\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=-x^2+x^2$$

$$\Rightarrow\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy0$$

Thus the correct answer is option C

Exercise: 9.4

1. Find the general solution for $\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$

Solution:

The given differential equation is $\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$

Use trigonometric half-angle identities to simplify

$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\tan^2\frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx}=\sec^2\frac{x}{2}-1$$

Separate the differential and integrate

$$\int dy=\int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$\Rightarrow y=\int\sec^2\frac{x}{2}dx-\int dx$$

$$\Rightarrow y=2\tan\frac{x}{2}-x+c$$

Thus the general solution of given differential equation is $y=2\tan\frac{x}{2}-x+c$

2. Find the general solution for $\frac{dy}{dx}=\sqrt{4-y^2}\ (-2<y<2)$

Solution:

The given differential equation is $\frac{dy}{dx}=\sqrt{4-y^2}\left(-2<y<2\right)$

Simplify the expression

$$\frac{dy}{dx}=\sqrt{4-y^2}$$

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}}=dx$$

Use standard integration

$$\int \frac{dy}{\sqrt{4-y^2}}=\int dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2}=x+c$$

$$\Rightarrow \frac{y}{2}=\sin\left(x+c\right)$$

$$\Rightarrow y=2\sin\left(x+c\right)$$

Thus the general solution of given differentia equation is $y=2\sin\left(x+c\right)$

3. Find the general solutions for $\Rightarrow \frac{dy}{dx}+y=1\left(y\neq 1\right)$

Solution:

The given differential equation is $\Rightarrow \frac{dy}{dx}+y=1\left(y\neq 1\right)$

Simplify the expression

$$\frac{dy}{dx}+y=1$$

$$\Rightarrow \frac{dy}{dx} = 1 - y$$

$$\Rightarrow \frac{dy}{1-y} = dx$$

Use standard integration

$$\Rightarrow \int \frac{dy}{1-y} = \int dx$$

$$\Rightarrow -\log(1-y) = x + c$$

$$\Rightarrow \log(1-y) = -(x+c)$$

$$\Rightarrow 1 - y = e^{-(x+c)}$$

$$y = 1 - Ae^{-x} \left(A = e^{-c}\right)$$

Thus the general solution of given differential equation is $y = 1 - Ae^{-x}$

4. Find the general solution for $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

Solution:

The given differential equation is $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

Divide both side by $\tan x \tan y$

$$\frac{\sec^2 x \tan y dx + \sec^2 y \tan x dy}{\tan x \tan y} = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

Integrate both sides

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = 0$$

$$\int \frac{\sec^2 y}{\tan y} dy = -\int \frac{\sec^2 x}{\tan x} dx.........(1)$$

Use a substitution method for integration. Substitute $\tan x = u$

For integral on RHS

$$\Rightarrow \tan x = u$$

$$\Rightarrow \sec^2 x \, dx = du$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = \int \frac{du}{u}$$

$$\Rightarrow \int \frac{\sec^2 x}{\tan x} dx = \log u$$

$$\Rightarrow \int \frac{\sec^2 x}{\tan x} dx = \log(\tan x)$$

Thus evaluating result from (1)

$$\Rightarrow \log(\tan y) = -\log(\tan x) + \log(c)$$

$$\Rightarrow \log(\tan y) = \log\left(\frac{c}{\tan x}\right)$$

$$\Rightarrow \tan x \tan y = c$$

Thus the general solution of given differentia equation is $\tan x \tan y = c$

5. Find the general solution for $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$

Solution:

The given differential equation is $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$

Simplify the expression

$$dy=\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx$$

Integrate both sides

$$\int dy=\int\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx..........(1)$$

Use a substitution method for integration. Substitute $e^{x}+e^{-x}=t$

For integral on RHS

$$\Rightarrow e^{x}+e^{-x}=t$$

$$\Rightarrow\left(e^{x}+e^{-x}\right)dx=dt$$

$$\Rightarrow\int\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx=\int\frac{dt}{t}s$$

$$\Rightarrow\int\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx=\ln t+c$$

$$\Rightarrow\int\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx=\left(\log e^{x}+e^{-x}\right)+c$$

Thus evaluating result from (1)

$$y=\log\left(e^{x}+e^{-x}\right)+c$$

Thus the general solution of given differential equation is $y=\log\left(e^{x}+e^{-x}\right)+c$

6. Find the general solution for $\frac{dy}{dx}=\left(1+x^{2}\right)\left(1+y^{2}\right)$

Solution:

The given differential equation is $\frac{dy}{dx}=\left(1+x^{2}\right)\left(1+y^{2}\right)$

Simplify the expression

$$\frac{dy}{1+y^2}=\left(1+x^2\right)dx$$

Integrate both side

$$\int\frac{dy}{1+y^2}=\int\left(1+x^2\right)dx$$

Use standard integration

$$\tan^{-1}y=\int dx+\int x^2dx$$

$$\Rightarrow \tan^{-1}y=x+\frac{x^3}{3}+c$$

Thus the general solution of given differential equation is $\tan^{-1}y=x+\frac{x^3}{3}+c$

7. Find the general solution for $y\log y\,dx-xdy=0$

Solution:

The given differential equation is $y\log y\,dx-xdy=0$

Simplify the expression

$$y\log y\,dx=xdy$$

$$\Rightarrow \frac{dx}{x}=\frac{dy}{y\log y}$$

Integrate both sides

$$\int\frac{dx}{x}=\int\frac{dy}{y\log y}..........(1)$$

Use substitution method for integration on LHS. Substitute $\log y=t$

$$\log y=t$$

$$\Rightarrow \frac{1}{y} dy = dt$$

$$\Rightarrow \int \frac{dy}{y \log y} = \int \frac{dt}{t}$$

$$\Rightarrow \int \frac{dy}{y \log y} = \log t$$

$$\Rightarrow \int \frac{dy}{y \log y} = \log(\log y)$$

Evaluating expression (1)

$$\log(\log y) = \log x + \log c$$

$$\log(\log y) = \log c x$$

$$\log y = cx$$

$$\Rightarrow y = e^{cx}$$

Thus the general solution of given differential equation is $\Rightarrow y = e^{cx}$

8. Find the general solution for $x^5 \frac{dy}{dx} = -y^5$

Solution:

The given differential equation is $x^5 \frac{dy}{dx} = -y^5$

Simplify the expression

$$\frac{dy}{y^5} = -\frac{dx}{x^5}$$

Integrate both sides

$$\int \frac{dy}{y^5} = -\int \frac{dx}{x^{-5}}$$

$$\Rightarrow \int y^{-5} dy = -\int x^{-5} dx$$

$$\Rightarrow \frac{y^{-5+1}}{-5+1} = -\frac{x^{-5+1}}{-5+1} + c$$

$$\Rightarrow \frac{y^{-4}}{-4} = -\frac{x^{-4}}{-4} + c$$

$$\Rightarrow x^{-4} + y^{-4} = -4c$$

$$\Rightarrow x^{-4} + y^{-4} = A(A = -4c)$$

Thus the general solution of given differential equation is $x^{-4} + y^{-4} = A$

9. Find the general solution for $\frac{dy}{dx} = \sin^{-1} x$

Solution:

The given differential equation is $\frac{dy}{dx} = \sin^{-1} x$

Simplify the expression

$$dy = \sin^{-1} x \, dx$$

Integrate both side

$$\int dy = \int \sin^{-1} x \, dx$$

$$\Rightarrow y = \int 1 \times \sin^{-1} x \, dx$$

Use product rule of integration

$$\int \sin^{-1} x dx = \sin^{-1} x \int dx - \int \left(\frac{1}{\sqrt{1-x^2}} \int dx \right) dx$$

$$\Rightarrow \int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Substitute $1-x^2=t^2$

$1-x^2=t^2$

$\Rightarrow -2xdx=2tdt$

$\Rightarrow -xdx=tdt$

Evaluating the integral

$$\frac{2x^2+x}{(x+1)(x^2+1)}=\frac{A}{x+1}=\frac{Bx+C}{x^2+1}$$

$$\int \sin^{-1} xdx = x\sin^{-1} x+\int \frac{tdt}{\sqrt{t^2}}$$

$$\Rightarrow \int \sin^{-1} xdx = x\sin^{-1} x+t+c$$

$$\Rightarrow \int \sin^{-1} xdx = x\sin^{-1} x+\sqrt{1-x^2}+c$$

$$\Rightarrow y = x\sin^{-1} x+\sqrt{1-x^2}+c$$

Thus the general solution of given differential equation is $y=x\sin^{-1} x+\sqrt{1-x^2}+c$

10. Find the general solution for $e^x \tan ydx+(1-e^x)\sec^2 ydy=0$

Solution:

The given differential equation is $e^x \tan ydx+(1-e^x)\sec^2 ydy=0$

Simplify the expression

$$(1-e^x)\sec^2 y\,dy=-e^x \tan y\,dy$$

$$\Rightarrow \frac{\sec^2 y}{\tan y}dy=-\frac{e^x}{(1-e^x)}dx$$

Integrate both sides

$$\int \frac{\sec^2 y}{\tan y} dy = -\int \frac{e^x}{(1-e^x)} dx \text{...........}(1)$$

Substitute $\tan y = u$

$\tan y = u$

$\Rightarrow \sec^2 y = du$

Evaluating the LHS integral of (1)

$$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy = \int \frac{du}{u}$$

$$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy = \log u$$

$$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy = \log(\tan y)$$

Substitute $1 - e^x = v$

$1 - e^x = v$

$\Rightarrow -e^x dx = dv$

Evaluating the RHS integral of (1)

$$\Rightarrow -\int \frac{e^x}{(1-e^x)} dx = \int \frac{dv}{v}$$

$$\Rightarrow -\int \frac{e^x}{(1-e^x)} dx = \log v$$

$$\Rightarrow -\int \frac{e^x}{(1-e^x)} dx = \log(1-e^x)$$

Therefore the integral (1) will be

$$\log(\tan y) = \log(1-e^x) + \log c$$

$$\Rightarrow \log(\tan y) = \log c(1-e^x)$$

$$\Rightarrow \tan y = \log c(1-e^x)$$

Thus the general solution of given differential equation is $\tan y = \log c(1-e^x)$

11. Find the particular solution of $(x^3+x^2+x+1)\frac{dy}{dx} = 2x^2+x; y=1, x=0$ to satisfy the given condition

Solution:

The given differential equation is $(x^3+x^2+x+1)\frac{dy}{dx} = 2x^2+x; y=1, x=0$

Simplify the expression

$$\frac{dy}{dx} = \frac{2x^2+x}{(x^3+x^2+x+1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2+x}{(x+1)(x^2+1)}$$

$$\Rightarrow dy = \frac{2x^2+x}{(x+1)(x^2+1)}dx$$

Integrate both side

$$\int dy = \int \frac{2x^2+x}{(x+1)(x^2+1)}dx \text{..........}(1)$$

Use partial fraction method to simplify the RHS

$$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 2x^2+x = A(x^2+1)+(Bx+C)(x+1)$$

$$\Rightarrow 2x^2+x=(A+B)x^2+(B+C)x+(A+C)$$

By comparing coefficients

$$A+B=2$$
$$B+C=1$$
$$A+C=0$$

Solving this we get

$$\frac{2x^2+x}{(x+1)(x^2+1)}=\frac{\left(\frac{1}{2}\right)}{x+1}+\frac{\left(\frac{3}{2}\right)x+\left(-\frac{1}{2}\right)}{x^2+1}$$

$$\Rightarrow \frac{2x^2+x}{(x+1)(x^2+1)}=\frac{1}{2}\left(\frac{1}{x+1}+\frac{3x-1}{x^2+1}\right)$$

Rewriting the integral (1)

$$y=\frac{1}{2}\int\left(\frac{1}{x+1}+\frac{3x-1}{x^2+1}\right)dx$$

$$\Rightarrow y=\frac{1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{3x-1}{x^2+1}dx$$

$$\Rightarrow y=\frac{1}{2}\log(x+1)+\frac{3}{2}\int\frac{x}{x^2+1}dx-\frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$\Rightarrow y=\frac{1}{2}\log(x+1)+\frac{3}{4}\int\frac{2x}{x^2+1}dx-\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow y=\frac{1}{2}\log(x+1)+\frac{3}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+c$$

For $y=1$ when $x=0$

$$1=\frac{1}{2}\log(0+1)+\frac{3}{4}\log(0+1)-\frac{1}{2}\tan^{-1}0+c$$

$$\Rightarrow c=1$$

Thus the required particular solution is

$$\Rightarrow y=\frac{1}{2}\log(x+1)+\frac{3}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+1$$

12. Find the particular solution of $x(x^2-1)\frac{dy}{dx}=1; y=0$ when $x=2$ to satisfy the given condition

Solution:

The given differential equation is $x(x^2-1)\frac{dy}{dx}=1; y=0$ when x = 2

Simplify the expression

$$x(x^2-1)\frac{dy}{dx}=1$$

$$\Rightarrow dy=\frac{dx}{x(x^2-1)}$$

$$\Rightarrow dy=\frac{dx}{x(x-1)(x+1)}$$

Integrate both sides

$$\int dy=\int\frac{dx}{x(x-1)(x+1)}\text{............}(1)$$

Use partial fraction method to simplify the RHS

$$\frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

$$\Rightarrow 1=A(x^2-1)+Bx(x+1)+Cx(x-1)$$

$$\Rightarrow 1=(A+B+C)x^2+(B-C)x-A$$

By comparing coefficients

$$A+B+C=0$$

$B-C=0$

$-A=1$

Solving this we get

$$\frac{1}{x(x-1)(x+1)}=\frac{(-1)}{x}+\frac{\left(\frac{1}{2}\right)}{x-1}+\frac{\left(\frac{1}{2}\right)}{x+1}$$

$$\Rightarrow \frac{1}{x(x-1)(x+1)}=-\frac{1}{x}+\frac{1}{2}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)$$

Rewriting the integral (1)

$$y=\int\left(-\frac{1}{x}+\frac{1}{2}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)\right)dx$$

$$\Rightarrow y=\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{x-1}dx+\frac{1}{2}\int\frac{1}{x+1}dx$$

$$\Rightarrow y=-\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+\log c$$

$$\Rightarrow y=-\frac{2}{2}\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+\frac{2}{2}\log c$$

$$\Rightarrow y=\frac{1}{2}\left(-\log x^2+\log(x-1)+\log(x+1)+\log c^2\right)$$

$$\Rightarrow y=\frac{1}{2}\log\left[\frac{c^2\left(x^2-1\right)}{x^2}\right]$$

For y = 0 when x = 2

$$0=\frac{1}{2}\log\left[\frac{c^2\left(2^2-1\right)}{2^2}\right]$$

$$0=\log\left[\frac{3c^2}{4}\right]$$

$$\Rightarrow \frac{3c^2}{4} = 1$$

$$\Rightarrow c^2 = \frac{4}{3}$$

Thus the required particular solution is $y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$

13. Find the particular solution of $\cos\left(\frac{dy}{dx}\right) = a\,(a \in R)$; $y = 1$ when $x = 0$ to satisfy the given equation

Solution:

The given differential equation is $\cos\left(\frac{dy}{dx}\right) = a\,(a \in R)$; $y = 1$ when $x = 0$

Simplify the expression

$$\cos\left(\frac{dy}{dx}\right) = a$$

$$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$$

$$\Rightarrow dy = \cos^{-1} a\, dx$$

Integrate both sides

$$\int dy = \int \cos^{-1} a\, dx$$

$$\Rightarrow y = \cos^{-1} a \int dx$$

$$\Rightarrow y = x\cos^{-1} a + c$$

For y =1 when x = 0

$$1 = 0\cos^{-1} a + c$$

$\Rightarrow c = 1$

Thus the required particular solution is

$y = x\cos^{-1} a + 1$

$\Rightarrow \frac{y-1}{x} = \cos^{-1} a$

$\Rightarrow \cos\left(\frac{y-1}{x}\right) = a$

14. Find the particular solution of $\frac{dy}{dx} = y\tan x; y = 1$ when $x = 0$ to satisfy the given condition

Solution:

The given differential equation is $\frac{dy}{dx} = y\tan x; y = 1$ when $x = 0$

Simplify the expression

$\frac{dy}{dx} = y\tan x$

$\Rightarrow \frac{dy}{dx} = \tan x\, dx$

Integrate both sides

$\Rightarrow \int \frac{dy}{dx} = \int \tan x\, dx$

$\Rightarrow \int \log y = \log(\sec x) + \log c$

$\Rightarrow \log y = \log(c\sec x)$

$\Rightarrow y = c\sec x$

For y = 1 when x = 0

$1 = c\sec 0$

$\Rightarrow c = 1$

Thus the required particular solution is $y = \sec x$

15. Find the equation of a curve passing through the point $(0,0)$ and whose differential equation is $y' = e^x \sin x$

Solution:

The given differential equation is $y' = e^x \sin x$

The curve passes through $(0,0)$

Simplify the expression

$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$

$$\Rightarrow dy = e^x \sin x\, dx$$

Integrate both sides

$$\int dy = \int e^x \sin x\, dx$$

Use product rules for integration of RHS. Let

$$I = \int e^x \sin x\, dx$$

$$\Rightarrow I = \sin x \int e^x\, dx - \int \left(\cos x \int e^x dx\right) dx$$

$$\Rightarrow I = e^x \sin x - \left(\cos x \int e^x dx + \int \left(\sin x \int e^x dx\right) dx\right)$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - \int \left(e^x \sin x\right) dx$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$

$$\Rightarrow I = \frac{e^x}{2}(\sin x - \cos x)$$

Thus integral will be

$$y = \frac{e^x}{2}(\sin x - \cos x) + c$$

Thus as the curve passes through $(0,0)$

$$0 = \frac{e^0}{2}(\sin 0 - \cos 0) + c$$

$$0 = \frac{1}{2}(0-1) + c$$

$$\Rightarrow c = \frac{1}{2}$$

Thus the equation of the curve will be

$$y = \frac{e^x}{2}(\sin x - \cos x) + \frac{1}{2}$$

$$\Rightarrow y = \frac{e^x}{2}(\sin x - \cos x + 1)$$

16. For the differential equation $xy\frac{dy}{dx} = (x+2)(y+2)$ find the solution curve passing through the point $(1,-1)$

Solution:

The given differential equation is $xy\frac{dy}{dx} = (x+2)(y+2)$

The curve passes through $(1,-1)$

Simplify the expression

$$\Rightarrow\left(\frac{y}{y+2}\right)dy=\frac{(x+2)}{x}dx$$

$$\Rightarrow\left(1-\frac{2}{y+2}\right)dy=\frac{(x+2)}{x}dx$$

Integrate both side

$$\int\left(1-\frac{2}{y+2}\right)dy=\int\frac{(x+2)}{x}dx$$

$$\Rightarrow\int dy-2\int\frac{1}{y+2}dy=\int\frac{x}{x}dx+\int\frac{2}{x}dx$$

$$\Rightarrow y-2\log(y+2)=x+2\log x+c$$

$$\Rightarrow y-x=2\log(y+2)+2\log x+c$$

$$\Rightarrow y-x=2\log\left[x(y+2)\right]+c$$

$$\Rightarrow y-x=\log\left[x^2(y+2)^2\right]+c$$

Thus as the curve passes through $(1,-1)$

$$\Rightarrow -1-1=\log\left[(1)^2(-1+2)^2\right]+c$$

$$\Rightarrow -2=\log 1+c$$

$$\Rightarrow c=-2$$

Thus the equation of the curve will be

$$y-x=\log\left[x^2(y+2)^2\right]-2$$

$$\Rightarrow y-x+2=\log\left(x^2(y+2)^2\right)$$

17. Find the equation of a curve passing through the point $(0,-2)$ given that at any point (x,y) on the curve, the product of the slope of its tangent and -coordinate of the point is equal to the -coordinate of the point y x

Solution:

According to question, the equation is given by

$$y\frac{dy}{dx}=x$$

The curve passes through $(0,-2)$

Simplify the expression

$$\Rightarrow ydy=xdx$$

Integrate both sides

$$\int y\,dy=\int x\,dx$$

$$\Rightarrow \frac{y^2}{2}=\frac{x^2}{2}+c$$

$$\Rightarrow y^2-x^2=2c$$

Thus as the curve passes through $(0,-2)$

$$\Rightarrow (-2)^2-0^2=2C$$

$$\Rightarrow 4=2c$$

$$\Rightarrow c=2$$

Thus the equation of the curve will be

$$y^2-x^2=2(2)$$

$$\Rightarrow y^2-x^2=4$$

18. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve given that it passes through $(-2, 1)$.

Solution:

Let the point of contact of the tangent be (x, y). Then the slope of the segment joining point of contact and $(-4, -3)$

$$m = \frac{y+3}{x+4}$$

According to question the for the slope of tangent $\frac{dy}{dx}$ if follows

$$\frac{dy}{dx} = 2m$$

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right)$$

Simplify the expression

$$\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right)$$

$$\frac{dy}{y+3} = \frac{2}{x+4}dx$$

Integrate both sides

$$\int \frac{dy}{y+3} = \int \frac{2}{x+4}dx$$

$$\Rightarrow \log(y+3) = 2\log(x+4) + \log c$$

$$\Rightarrow \log(y+3) = \log(x+4)^2 + \log c$$

$$\Rightarrow \log(y+3) = \log c(x+4)^2$$

$$\Rightarrow y+3=c(x+4)^2$$

Thus as the curve passes through $(-2,1)$

$$1+3=c(-2+4)^2$$

$$\Rightarrow 4=4C$$

$$\Rightarrow c=1$$

Thus the equation of the curve will be $y+3=(x+4)^2$

19. The volume of spherical balloons being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds

Solution:

Let the volume of spherical balloon be V and its radius r. Let the rate of change of volume be k.

$$\frac{dV}{dt}=k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)=k$$

$$\Rightarrow \frac{4}{3}\pi\frac{d}{dt}\left(r^3\right)=k$$

$$\Rightarrow \frac{4}{3}\pi\left(3r^2\right)\frac{d}{dt}=k$$

$$\Rightarrow 4\pi r^2 dr=k\,dt$$

Integrate both sides

$$\int 4\pi r^2 dr=\int k\,dt$$

$$\Rightarrow 4\pi\int r^2 dr = kt + C$$

$$\Rightarrow \frac{4}{3}\pi r^3 = kt + C$$

At initial time, $t = 0$ and $r = 3$

$$\frac{4}{3}\pi 3^3 = k(0) + C$$

$$\Rightarrow C = 36\pi$$

At t = 3 the radius r = 6

$$\frac{4}{6}\pi\left(6^3\right) = k(3) + 36\pi$$

$$\Rightarrow 3k = 288\pi - 36\pi$$

$$\Rightarrow k = 84\pi$$

Thus the radius-time relation can be given by

$$\frac{4}{3}\pi r^3 = 84\pi t + 36\pi$$

$$\Rightarrow r^3 = 63t + 27$$

$$\Rightarrow r = (63t + 27)^{\frac{1}{3}}$$

The radius of balloon after t seconds given by $r = (63t + 27)^{\frac{1}{3}}$

20. In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs. 100 doubles itself in 10 years $(\log_e 2 = 0.6931)$

Solution:

Let the principal be p, according to question

$$\frac{dp}{dt} = \left(\frac{r}{100}\right)p$$

Simplify the expression

$$\frac{dp}{p}=\left(\frac{r}{100}\right)dt$$

Integrate both side

$$\int\frac{dp}{p}=\int\left(\frac{r}{100}\right)dt$$

$$\Rightarrow \log p=\frac{rt}{100}+c$$

$$\Rightarrow p=e^{\frac{rt}{100}+c}$$

$$\Rightarrow p=Ae^{\frac{rt}{100}}\left(A=e^{c}\right)$$

At t = 0, p = 100

$$100=Ae^{\frac{r(0)}{100}}$$

$$\Rightarrow A=100$$

Thus the principle and rate of interest relation

$$p=100e^{\frac{rt}{100}}$$

At $t=10, p=2\times 100=200$

$$200=100e^{\frac{r(10)}{100}}$$

$$\Rightarrow 2=e^{\frac{r}{10}}$$

Take logarithm on both side

$$\log\left(e^{\frac{r}{10}}\right)=\log(2)$$

$$\Rightarrow \frac{r}{10}=0.6931$$

$\Rightarrow r = 6.931$

Thus the rate of intersect $r = 6.931\%$

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years $\left(e^{0.5} = 1.648\right)$

Solution:

Let the principal be p, according to question principle increases at the rate of 5% per year

$$\frac{dp}{dt} = \left(\frac{5}{100}\right)p$$

Simplify the expression

$$\frac{dp}{dt} = \frac{p}{20}$$

$$\Rightarrow \frac{dp}{p} = \frac{1}{20}dt$$

Integrate both side

$$\int \frac{dp}{p} = \int \frac{1}{20}dt$$

$$\Rightarrow \log p = \frac{t}{20} + c$$

$$\Rightarrow p = e^{\frac{t}{20}+c}$$

$$\Rightarrow p = Ae^{\frac{t}{20}} \left(A = e^{c}\right)$$

At t = 0, p = 1000

$$1000 = Ae^{\frac{0}{20}}$$

$\Rightarrow A = 1000$

Thus the relation of principal and time relation

$\Rightarrow p = 1000e^{\frac{t}{20}}$

At t = 10

$p = 1000e^{\frac{10}{20}}$

$\Rightarrow p = 1000e^{0.5}$

$\Rightarrow p = 1000 \times 1.648$

$\Rightarrow p = 1648$

Thus after 10 this year the amount will become Rs. 1648

22. In a culture, the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present?

Solution:

Let the number of bacteria by y at time t. According to question

$\frac{dy}{dt} \propto y$

$\frac{dy}{dt} = cy$

Here c is constant

Simplify the expression

$\frac{dy}{y} = cdt$

Integrate both side

$$\int \frac{dy}{y} = \int c\,dt$$

$$\Rightarrow \log y = ct + D$$

$$\Rightarrow y = e^{ct+D}$$

$$\Rightarrow y = Ae^{ct}\left(A = e^{D}\right)$$

At t = 0, y = 100000

$$100000 = Ae^{c(0)}$$

$$\Rightarrow A = 100000$$

At $t = 2, y = \frac{11}{10}(100000) = 110000$

$$y = 100000\,e^{ct}$$

$$\Rightarrow 110000 = 100000e^{c(2)}$$

$$\Rightarrow e^{2c} = \frac{11}{10}$$

$$\Rightarrow 2c = \log\left(\frac{11}{10}\right)$$

$$\Rightarrow c = \frac{1}{2}\log\left(\frac{11}{10}\right)..........(1)$$

For $y = 200000$

$$200000 = 100000e^{ct}$$

$$\Rightarrow e^{ct} = 2$$

$$\Rightarrow ct = \log 2$$

$$\Rightarrow t = \frac{\log 2}{c}$$

Back substituting using expression (1)

$$t=\frac{\log 2}{\frac{1}{2}\log\left(\frac{11}{10}\right)}$$

$$\Rightarrow t=\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Thus time required for bacteria to reach 200000 is $\Rightarrow t=\frac{\log 2}{\frac{1}{2}\log\left(\frac{11}{10}\right)}$ hrs

23. Find the general solution of the differential equation $\frac{dy}{dx}=e^{x+y}$

Solution:

The given differential equation is $\frac{dy}{dx}=e^{x+y}$. Simplify the expression

$$\frac{dy}{dx}=e^{x}e^{y}$$

$$\Rightarrow \frac{dy}{e^{y}}=e^{x}dx$$

$$\Rightarrow e^{-y}dy=e^{x}dx$$

Integrate both side

$$\int e^{-y}dy=\int e^{x}dx$$

$$\Rightarrow -e^{-y}=e^{x}+D$$

$$\Rightarrow e^{x}+e^{-y}=-D$$

$$\Rightarrow e^{x}+e^{-y}=C\left(C=-D\right)$$

Thus the general solution of given differential equation is $e^{x}+e^{-y}=C$

Exercise: 9.5

1. Show that, differential equation $\left(x^2+xy\right)dy=\left(x^2+y^2\right)dx$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy}$$

Checking for homogeneity

$$F(x,y)=\frac{x^2+y^2}{x^2+xy}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)^2+(\lambda x)(\lambda y)}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda^2\left(x^2+y^2\right)}{\lambda^2+\left(x^2+xy\right)}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{x^2+y^2}{x^2+xy}$$

$$\Rightarrow F(\lambda x,\lambda y)=F(\lambda x,\lambda y)$$

Thus it is an homogeneous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{x^2+(vx)^2}{x^2+x(vx)}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x^2\left(1+v^2\right)}{x^2+(1+v)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1+v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{1+v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2-v-v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v}{1+v}$$

Separate the differentials

$$\frac{1+v}{1-v}dv=\frac{dx}{x}$$

Integrate both side

$$\int\frac{1+v}{1-v}dv=\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{2-(1-v)}{1-v}dv=\log x-\log k$$

$$\Rightarrow \int\frac{2}{1-v}dv-\int\frac{1-v}{1-v}dv=\log\frac{x}{k}$$

$$\Rightarrow -2\log(1-v)-\int dv=\log\frac{x}{k}$$

$$\Rightarrow -2\log(1-v)-v=\log\frac{x}{k}$$

$$\Rightarrow v=-\log\frac{x}{k}-2\log(1-v)$$

$$\Rightarrow v=\log\left(\frac{k}{x(1-v)^2}\right)$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \frac{y}{x} = \log\left(\frac{k}{x\left(1-\frac{y}{x}\right)^2}\right)$$

$$\Rightarrow \frac{y}{x} = \log\left(\frac{kx}{(x-y)^2}\right)$$

$$\Rightarrow e^{\frac{y}{x}} = \frac{kx}{(x-y)^2}$$

$$\Rightarrow (x-y)^2 = kxe^{-\frac{y}{x}}$$

The solution of the given differential equation $(x-y)^2 = kxe^{-\frac{y}{x}}$

2. Show that, differential equation $y' = \frac{x+y}{x}$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Checking for homogeneity

$$F(x,y) = \frac{x+y}{x}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda(x+y)}{\lambda(x)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{x+y}{x}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{x+(vx)}{x}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x(1+v)}{x}$$

$$\Rightarrow v+x\frac{dv}{dx}=1+v$$

$$\Rightarrow x\frac{dv}{dx}=1$$

Separate the differentials

$$dv=\frac{dx}{x}$$

Integrate both side

$$\int dv=\int\frac{dx}{x}$$

$$\Rightarrow \int dv=\log x+\log k$$

$$\Rightarrow v=\log kx$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \frac{y}{x}=\log kx$$

$$\Rightarrow y=x\log kx$$

The solution of the given differential equation $y=x\log kx$

3. Show that, differential equation $(x-y)dy-(x+y)dx=0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

Checking for homogeneity

$$F(x,y)=\frac{x+y}{x-y}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda x+\lambda y}{\lambda x-\lambda y}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda(x+y)}{\lambda(x-y)}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{x+y}{x-y}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogeneous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{x+(vx)}{x-vx}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x(1+v)}{x(1-v)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1+v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v}{1-v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v-v+v^2}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{1-v}$$

Separate the differentials

$$\frac{1-v}{1+v^2}dv=\frac{dx}{x}$$

Integrate both side

$$\int\frac{1-v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow\int\frac{1-v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow\int\frac{1}{1+v^2}dv-\int\frac{v}{1+v^2}dv=\log x+C$$

$$\Rightarrow\tan^{-1}v-\frac{1}{2}\int\frac{2v}{1+v^2}dv=\log x+C$$

$$\Rightarrow\tan^{-1}v-\frac{1}{2}\log\left(1+v^2\right)=\log x+C$$

$$\Rightarrow\tan^{-1}v=\log x+\frac{1}{2}\log\left(1+v^2\right)+C$$

$$\Rightarrow\tan^{-1}v=\frac{1}{2}\log x^2+\frac{1}{2}\log\left(1+v^2\right)+C$$

$$\Rightarrow\tan^{-1}v=\frac{1}{2}\log\left[x^2\left(1+v^2\right)\right]+C$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log\left[x^2\left(1+\frac{y^2}{x^2}\right)\right]+C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log\left(x^2+y^2\right)+C$$

The solution of the given differential equation $\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log\left(x^2+y^2\right)+C$

4. Show that, differential equation $\left(x^2-y^2\right)dx+2xy\,dy=0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx}=\frac{x^2-y^2}{2xy}$$

Checking for homogeneity

$$F(x,y)=-\frac{x^2-y^2}{2xy}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{(\lambda x)^2-(\lambda y)^2}{2(\lambda x)(\lambda y)}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda^2x^2-\lambda^2y^2}{2\lambda^2xy}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda^2\left(x^2-y^2\right)}{\lambda^2(2xy)}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{x^2-y^2}{2xy}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=-\frac{x^2-(vx)^2}{2x(vx)}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x^2(v^2-1)}{x^2(2v)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v^2-1}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1}{2v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=-\frac{1+v^2}{2v}$$

Separate the differentials

$$\frac{2v}{1+v^2}dv=-\frac{dx}{x}$$

Integrate both sides

$$\int\frac{2v}{1+v^2}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \log(1+v^2)=-\log x+C$$

$$\Rightarrow \log(1+v^2)+\log x=C$$

$$\Rightarrow \log\left[x(1+v^2)\right]=C$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \log\left[x\left(1+\frac{y^2}{x^2}\right)\right]=C$$

$$\Rightarrow \left(\frac{x^2+y^2}{x}\right) = k\, k = e^c$$

$$\Rightarrow x^2 + y^2 = kx$$

The solution of the given differential equations $x^2 + y^2 = kx$

5. Show that, differential equation $x^2 \frac{dy}{dx} = x^2 - 2y^2 + xy$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$$

Checking for homogeneity

$$F(x,y) = \frac{x^2 - 2y^2 + xy}{x^2}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda^2 x^2 - 2\lambda^2 y^2 + (\lambda x)(\lambda y)}{\lambda^2 x^2}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda^2 \left(x^2 - 2y^2 + xy\right)}{\lambda^2 x^2}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{x^2 - 2y^2 + xy}{x^2}$$

$$\Rightarrow F(x,y) = F(\lambda x, \lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{x^2 - 2(vx)^2 + x(vx)}{x^2}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x^2\left(1-2v^2+v\right)}{x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=1-2V^2+V$$

$$\Rightarrow x\frac{dv}{dx}=1-2V^2$$

Separate the differentials

$$\frac{1}{1-2v^2}dv=\frac{dx}{x}$$

Integrate both side

$$\int\frac{1}{1-2v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int\frac{1}{\frac{1}{2}-v^2}dv=\log x+C$$

$$\Rightarrow \frac{1}{2}\int\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2-v^2}dv=\log x+C$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{2\times\frac{1}{\sqrt{2}}}\right)\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right|=\log x+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right|=\log x+C$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\left(\frac{y}{x}\right)}{1-\sqrt{2}\left(\frac{y}{x}\right)}\right|=\log x+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\frac{\left|x+\sqrt{2}y\right|}{\left|x-\sqrt{2}y\right|}=\log x+C$$

The solution of the given differential equation $\frac{1}{2\sqrt{2}}\log\frac{\left|x+\sqrt{2}y\right|}{\left|x-\sqrt{2}y\right|}=\log x+C$

6. Show that, differential equation $xdy-ydx=\sqrt{x^2+y^2}dx$ is homogenous and solves it

Solution:

Rewrite the equations in standard form

$$xdy=\sqrt{x^2+y^2}dx+ydx$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x^2+y^2}+y}{x}$$

Checking for homogeneity

$$F(x,y)=\frac{\sqrt{x^2+y^2}+y}{x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\sqrt{\lambda^2x^2+\lambda^2y^2}+\lambda y}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\sqrt{\lambda^2\left(x^2+y^2\right)}+\lambda y}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda\left(\sqrt{x^2+y^2}+y\right)}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\sqrt{x^2+y^2}+y}{x}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{\sqrt{x^2 + (vx)^2} + vx}{x}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{x\left(\sqrt{1+v^2} + v\right)}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$$

Separate the differentials

$$\frac{1}{\sqrt{1+v^2}}dv = \frac{dx}{x}$$

Integrate both sides

$$\int \frac{1}{\sqrt{1+v^2}}dv = \int \frac{dx}{x}$$

$$\Rightarrow \log\left|v + \sqrt{1+v^2}\right| = \log x + \log C$$

Backs substitute $v = \frac{y}{x} : s$

$$\Rightarrow \log\left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| = \log Cx$$

$$\Rightarrow \log\left|\frac{y + \sqrt{1+x^2}}{x}\right| = \log Cx$$

$$y + \sqrt{1+x^2} = Cx^2$$

The solution of the given differential equation $y + \sqrt{1+x^2} = Cx^2$

7. Show that, differential equation

$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$$

$$\frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

Checking for homogeneity

$$F(x,y)=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)+y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda y}{\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda^2\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\lambda^2\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{\left\{x\cos\left(\frac{xv}{x}\right) + vx\sin\left(\frac{vx}{x}\right)\right\}vx}{\left\{vx\sin\left(\frac{vx}{x}\right) - x\cos\left(\frac{vx}{x}\right)\right\}x}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{x^2\{\cos v + v\sin v\}v}{x^2\{v\sin v - \cos v\}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

Separate the differentials

$$\frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}$$

Integrate both side

$$\int\frac{v\sin v - \cos v}{v\cos v}dv = 2\int\frac{dx}{x}$$

$$\int\frac{v\sin v}{v\cos v}dv - \int\frac{\cos v}{v\cos v} = 2\int\frac{dx}{x}$$

$$\Rightarrow \int\tan v dv - \int\frac{1}{v}dv = 2\log|x| + \log C$$

$$\Rightarrow \log|\sec v| - \log|v| = \log C|x|^2$$

$$\Rightarrow \log\frac{|\sec v|}{|v|} = \log C|x|^2$$

$$\Rightarrow \sec v = Cvx^2$$

Back substitute $v = \frac{y}{x}$: s

$$\Rightarrow \sec\frac{y}{x} = C\left(\frac{y}{x}\right)x^2$$

$$\Rightarrow \cos\frac{y}{x} = \frac{k}{xy} k = \frac{1}{C}$$

The solution of the given differential equation is $\cos\frac{y}{x} = \frac{k}{xy}$

8. Show that, differential equation $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

$$\frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

Checking for homogeneity

$$F(x,y) = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

$$F(\lambda x, \lambda y) = \frac{\lambda y - \lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda\left(y - x\sin\left(\frac{y}{x}\right)\right)}{\lambda x}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

$$\Rightarrow F(x, y) = F(\lambda x, \lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx - x\sin\left(\frac{vx}{x}\right)}{x}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{x(v - \sin v)}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow x\frac{dv}{dx} = -\sin v$$

Separate the differentials

$$\frac{1}{\sin v}dv = -\frac{dx}{x}$$

Integrate both side

$$\int \operatorname{cosec} v dv = -\int \frac{dx}{x}$$

$$\Rightarrow \log\left|\cos ec\, v - \cot v\right| = -\log x + \log C$$

$$\Rightarrow \log\left|\cos ec\, v - \cot v\right| = \log\frac{C}{x}$$

$$\Rightarrow \operatorname{cosec} v - \cot v = \frac{C}{x}$$

$$\Rightarrow \frac{1}{\sin v} - \frac{\cos v}{\sin v} = \frac{C}{x}$$

$$\Rightarrow 1 - \cos v = \frac{C}{x} \sin v$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow 1 - \cos \frac{y}{x} = \frac{C}{x} \sin \frac{y}{x}$$

$$\Rightarrow x\left(1 - \cos \frac{y}{x}\right) = C \sin\left(\frac{y}{x}\right)$$

The solution of the given differentia equation $x\left(1 - \cos \frac{y}{x}\right) = C \sin\left(\frac{y}{x}\right)$

9. Show that, differential equation $ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$ydx = 2xdy - x\log\left(\frac{y}{x}\right)dy$$

$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

Checking for homogeneity

$$F(x, y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda y}{2\lambda x - \lambda x \log\left(\frac{\lambda y}{\lambda x}\right)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda\left(2x - x\log\left(\frac{y}{x}\right)\right)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

$$\Rightarrow F(x, y) = F(\lambda x, \lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{xv}{x(2 - \log(v))}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2 - \log(v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{\text{v}\log v - v}{2 - \log v}$$

Separate the differentials

$$\frac{2 - \log v}{v\log v - v}dv = \frac{dx}{x}$$

Integrate both side

$$\int \frac{2-\log v}{v\log v - v}dv = \int \frac{dx}{x}$$

$$\int \frac{1+1-\log v}{v(\log v - v)}dv = \log x + \log C$$

$$\Rightarrow \int \frac{1}{v(\log v-1)}dv + \int \frac{1-\log v}{v(\log v-1)}dv = \log x + \log C$$

$$\Rightarrow \int \frac{1}{v(\log v-1)}dv - \int \frac{1}{v}dv = \log C\,x..........(1)$$

Solving:

$$\int \frac{1}{v(\log v-1)}dv$$

Substituting $\log v - 1 = t$

$\log v - 1 = t$

$$\Rightarrow \frac{1}{v}dv = dt$$

Thus the integral will be

$$\Rightarrow \int \frac{1}{v(\log v-1)}dv = \int \frac{dt}{t}$$

$$\Rightarrow \int \frac{1}{v(\log v-1)}dv = \log t$$

$$\Rightarrow \int \frac{1}{v(\log v-1)}dv = \log(\log v-1)$$

Using above result for solving (1)

$$\Rightarrow \log(\log v-1) - \log v = \log Cx$$

$$\Rightarrow \log \frac{\log v - 1}{v} = \log Cx$$

$$\Rightarrow \log \frac{\log v - 1}{v} = Cx$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow \frac{\log \frac{y}{x} - 1}{\frac{y}{x}} = Cx$$

$$\Rightarrow \log \frac{y}{x} - 1 = Cx\left(\frac{y}{x}\right)$$

$$\Rightarrow \log \frac{y}{x} - 1 = Cy$$

The solution of the given differential equation $\log \frac{y}{x} - 1 = Cy$

10. Show that, differential equation $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\frac{dx}{dy} = \frac{e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{\left(1 + e^{\frac{x}{y}}\right)}$$

Checking for homogeneity

$$F(x,y)=\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

$$F(\lambda x,\lambda y)=\frac{e^{\frac{\lambda x}{\lambda y}}\left(1-\frac{\lambda x}{\lambda y}\right)}{\left(1+e^{\frac{\lambda x}{\lambda y}}\right)}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $x=vy$

$$\frac{d(vy)}{dy}=-\frac{e^{\frac{vy}{y}}\left(1-\frac{vy}{y}\right)}{1+e^{\frac{vy}{y}}}$$

$$\Rightarrow v\frac{dy}{dy}+y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{1+e^{v}}$$

$$\Rightarrow v+y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{1+e^{v}}-v$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{-e^{v}+ve^{v}-v-ve^{v}}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{-\left(e^{v}+v\right)}{1+e^{v}}$$

Separate the differentials

$$\frac{1+e^{v}}{e^{v}+v}dv=-\frac{dy}{y}$$

Integrate both side

$$\int\frac{1+e^{v}}{e^{v}+v}dv=-\int\frac{dy}{y}$$

$$\int\frac{e^{v}+1}{e^{v}+v}dv=-\log y+\log C..........(1)$$

Solving the LHS integral. Substitute $e^{v}+v=t$

$$e^{v}+v=t$$

$$\Rightarrow\left(e^{v}+1\right)dv=dt$$

Solving the expression (1)

$$\Rightarrow\int\frac{1}{t}dt=\log\frac{C}{y}$$

$$\Rightarrow\log(t)=\log\frac{C}{y}$$

$$\Rightarrow\log\left(e^{v}+v\right)=\log\frac{C}{y}$$

$$\Rightarrow e^{v}+v=\frac{C}{y}$$

Back substitute $v=\frac{x}{y}$

$$\Rightarrow e^{\frac{x}{y}}+\frac{x}{y}=\frac{C}{y}$$

$$\Rightarrow ye^{\frac{x}{y}}+x=C$$

The solution of the given differential equation $ye^{\frac{x}{y}}+x=C$

11. For the differential equation $(x+y)dy+(x-y)dx=0$. Find the particular solution for the condition $y=1$ when $x=1$

Solution:

Given differential equation is

$$(x+y)dy+(x-y)dx=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x-y}{x+y}$$

Checking for homogeneity

$$F(x,y)=-\frac{x-y}{x+y}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda(x-y)}{\lambda(x+y)}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{(x-y)}{(x+y)}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogeneous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=-\frac{x-(vx)}{x+(vx)}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x(v-1)}{x(v+1)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-1}{v+1}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v-1-v^2-v}{v+1}$$

$$\Rightarrow x\frac{dv}{dx}=-\frac{1+v^2}{v+1}$$

Separate the differentials

$$\frac{v+1}{1+v^2}dv=-\frac{dx}{x}$$

Integrate both side

$$\int\frac{v+1}{1+v^2}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{v}{1+v^2}dv+\int\frac{1}{1+v^2}dv=-\log x+k$$

$$\Rightarrow \frac{1}{2}\log\left(1+v^2\right)+\tan^{-1}v+\log x=k$$

$$\Rightarrow \frac{1}{2}\log\left[x\left(1+v^2\right)\right]+\tan^{-1}v=k$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \frac{1}{2}\log\left[x\left(1+\frac{y^2}{x^2}\right)\right]+\tan^{-1}\frac{y}{x}=k$$

$$\Rightarrow \frac{1}{2}\log\left[\frac{x^2+y^2}{x}\right]+\tan^{-1}\frac{y}{x}=k$$

Now $y=1$ and $x=1$

$$\Rightarrow \frac{1}{2}\log\left[\frac{1^2+1^2}{1}\right]+\tan^{-1}\frac{1}{1}=k$$

$$k=\frac{1}{2}\log 2+\frac{\pi}{4}$$

The required particular solution

$$\Rightarrow \frac{1}{2}\log\left[\frac{x^2+y^2}{x}\right]+\tan^{-1}\frac{y}{x}=\frac{1}{2}\log 2+\frac{\pi}{4}$$

12. For the differential equation $x^2dy+\left(xy+y^2\right)dx=0$. Find the particular solution for the condition $y=1$ when $x=1$

Solution:

Given differential equation is $x^2dy+\left(xy+y^2\right)dx=0$

$$\Rightarrow \frac{dy}{dx}=-\frac{xy+y^2}{x^2}$$

Checking for homogeneity

$$F(x,y)=-\frac{xy+y^2}{x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{(\lambda x)(\lambda y)+\lambda^2y^2}{\lambda^2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda^2\left(xy+y^2\right)}{\lambda^2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{xy+y^2}{x^2}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = -\frac{-x(vx)+(vx)^2}{x^2}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = -\frac{vx^2+v^2x^2}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{x^2\left(v+v^2\right)}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = -v - v^2$$

$$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v$$

Separate the differentials

$$\frac{1}{v^2+2v}dv = -\frac{dx}{x}$$

Integrate both side

$$\int\frac{1}{v^2+2v}dv = -\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int\frac{v+2-v}{v(v+2)}dv = -\log x + \log C$$

$$\Rightarrow \frac{1}{2}\int\frac{v+2}{v(v+2)}dv - \frac{1}{2}\int\frac{v}{v(v+2)}dv = -\log x + \log C$$

$$\Rightarrow \frac{1}{2}\int\frac{1}{v}dv - \frac{1}{2}\int\frac{1}{v+2}dv = -\log x + \log C$$

$$\Rightarrow \frac{1}{2}\log v - \frac{1}{2}\log(v+2) = \log\frac{C}{x}$$

$$\Rightarrow \frac{1}{2}\log\frac{v}{v+2} = \log\frac{C}{x}$$

$$\Rightarrow \frac{v}{v+2} = \left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{v}{v+2} = \frac{C^2}{x^2}$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2} = \frac{C^2}{x^2}$$

$$\Rightarrow \frac{x^2 y}{y+2x} = C^2$$

Now y = 1 and x = 1

$$\Rightarrow \frac{1^2(1)}{1+2(1)} = C^2$$

$$\Rightarrow C^2 = \frac{1}{3}$$

The required particular solution

$$\Rightarrow \frac{x^2 y}{y+2x} = \frac{1}{3}$$

$$\Rightarrow y+2x = 3x^2 y$$

13. For the differential equation $\left[x\sin^2\left(\frac{x}{y}\right) - y\right]dx + xdy = 0$. Find the particular solution for the condition $y = \frac{\pi}{4}$ when x = 1

Solution:

Given differential equation is $\left[x\sin^2\left(\frac{x}{y}\right)-y\right]dx+xdy=0$

$$\Rightarrow \frac{dy}{dx}=-\frac{\left[x\sin^2\left(\frac{x}{y}\right)-y\right]}{x}$$

Checking for homogeneity

$$F(x,y)=-\frac{x\sin^2\left(\frac{x}{y}\right)-y}{x}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda x\sin^2\left(\frac{\lambda x}{\lambda y}\right)-\lambda y}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda\left(x\sin^2\left(\frac{y}{x}\right)-y\right)}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{x\sin^2\left(\frac{y}{x}\right)-y}{x}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=-\frac{x\sin^2\left(\frac{vx}{x}\right)-vx}{x}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{-x\sin^2(v)+vx}{x}$$

$$\Rightarrow v+x\frac{dv}{dx}=-\sin^2 v+v$$

$$\Rightarrow x\frac{dv}{dx}=-\sin^2 v$$

Separate the differentials

$$\cos ec^2 v dv=-\frac{dx}{x}$$

Integrate both sides

$$\int \cos ec^2 v dv=-\int \frac{dx}{x}$$

$$\Rightarrow -\cot v=-\log x-\log C$$

$$\Rightarrow \cot v=\log C x$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \cot \frac{y}{x}=\log C x$$

Now $y=\frac{\pi}{4}$ and x = 1

$$\cot \frac{\frac{\pi}{4}}{1}=\log C(1)$$

$$\Rightarrow \log C=\cot \frac{\pi}{4}$$

$$\Rightarrow \log C=1$$

$$\Rightarrow C=e$$

The required particular solution

$$\Rightarrow \cot \frac{y}{x}=\log |ex|$$

14. For the differential equation $\frac{dy}{dx}-\frac{y}{x}+\cos ec\left(\frac{y}{x}\right)=0$. Find the particular solution for the condition y = 0 and x = 1

Solution:

Given differential equation is $\frac{dy}{dx}-\frac{y}{x}+\cos ec\left(\frac{y}{x}\right)=0$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)$$

Checking for homogeneity

$$F(x,y)=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=v-\cos ec(v)$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=v-\cos ec(v)$$

$$\Rightarrow v+x\frac{dv}{dx}=v-\cos ec(v)$$

$$\Rightarrow x\frac{dv}{dx}=-\cos ec(v)$$

Separate the differentials

$$\sin v dv=-\frac{dx}{x}$$

Integrate both sides

$$\int \sin v dv = -\int \frac{dx}{x}$$

$$\Rightarrow \cos v = \log|Cx|$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow \cos\frac{y}{x} = \log|Cx|$$

Now Y = 0 and x = 1

$$\Rightarrow \cos\frac{0}{1} = \log|C1|$$

$$\Rightarrow \log C = 1$$

$$\Rightarrow C = 1$$

The required particular solution

$$\Rightarrow \cos\frac{y}{x} = \log|ex|$$

15. For the differential equation $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$. Find the particular solution for the condition y = 2 when x = 1

Solution:

Given differential equation is $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

Checking for homogeneity

$$F(x,y)=\frac{2xy+y^2}{2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{2(\lambda x)(\lambda y)+\lambda^2y^2}{2\lambda^2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{2xy+y^2}{2x^2}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{2x(vx)+(vx)^2}{2x^2}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{2v+v^2}{2}$$

$$\Rightarrow v+x\frac{dv}{dx}=v+\frac{v^2}{2}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2}{2}$$

Separate the differentials

$$\frac{dv}{v^2}=\frac{1}{2}\left(\frac{dx}{x}\right)$$

Integrate both sides

$$2\int\frac{dv}{v^2}=\int\left(\frac{dx}{x}\right)$$

$$\Rightarrow\frac{v^{-2+1}}{-2+1}=\log|x|+C$$

$$\Rightarrow -\frac{2}{v}=\log|x|+C$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow -\frac{2x}{y}=\log|x|+C$$

Now y = 2 and x = 1

$$\Rightarrow -\frac{2(1)}{2}=\log|1|+C$$

$$\Rightarrow C=-1$$

The required particular solution

$$\Rightarrow -\frac{2x}{y}=\log|x|-1$$

$$\Rightarrow y=\frac{2x}{1-\log|x|}(x\neq 0,e)$$

16. What substitution should be used for solving homogeneous differential equation $\frac{dx}{dy}=h\left(\frac{x}{y}\right)$

Solution:

The required equation substitution will be

$$\frac{x}{y}=v$$

$$\Rightarrow x=vy$$

The correct answer is (C)

17. Which of the following equation is homogeneous

A) $(4x+6y+5)dy-(3y+2x+4)dx=0$

B) $(xy)dx-(x^3+y^3)dy=0$

C) $(x^3+2y^2)dx+2xy\,dy=0$

D) $y^2dx+(x^2-xy-y^2)dy=0$

Solution:

For option (A)

$$F(x,y)=\frac{3y+2x+4}{4x+6y+5}$$

$$F(\lambda x,\lambda y)=\frac{3\lambda y+2\lambda x+4}{4\lambda x+6\lambda y+5}$$

$$F(\lambda x,\lambda y)\neq F(x,y)$$

For option (B)

$$F(x,y)=\frac{xy}{x^3+y^3}$$

$$F(\lambda x,\lambda y)=\frac{(\lambda x)(\lambda y)}{(\lambda x)^3+(\lambda y)^3}$$

$$F(\lambda x,\lambda y)=\frac{xy}{\lambda(x+y)}$$

$$F(\lambda x,\lambda y)\neq F(x,y)$$

For Option (C)

$$F(x,y)=-\frac{x^3+2y^2}{2xy}$$

$$F(\lambda x,\lambda y)=-\frac{\lambda^3x^3+2\lambda^2y^2}{2(\lambda x)(\lambda y)}$$

$$F(\lambda x,\lambda y)=-\frac{\lambda x^3+2y^2}{2xy}$$

$$F(\lambda x, \lambda y) \neq F(x, y)$$

For option (D)

$$F(x, y) = -\frac{y^2}{x^2 - xy - y^2}$$

$$F(\lambda x, \lambda y) = -\frac{\lambda^2 y^2}{\lambda^2 x^2 - (\lambda x)(\lambda y) - \lambda^2 y^2}$$

$$F(\lambda x, \lambda y) = -\frac{y^2}{x^2 - xy - y^2}$$

$$F(\lambda x, \lambda y) = F(x, y)$$

Thus the correct answer is option (D)

Exercise: 9.6

1. Find the general solution for the differential equation $\frac{dy}{dx}+2y=\sin x$

Solution:

The given differential equation is $\frac{dy}{dx}+2y=\sin x$

It is a linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=2$$
$$Q=\sin x$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int 2dx}$$

$$\Rightarrow I.F=e^{2x}$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)dx+C$$

$$ye^{2x}=\int \sin x\left(e^{2x}\right)dx+C$$

$$\Rightarrow ye^{2x}=I+C\left(I=\int \sin x\left(e^{2x}\right)dx\right)..........(1)$$

$$I=\int \sin x\left(e^{2x}\right)dx$$

$$\Rightarrow I=(\sin x)\int e^{2x}\,dx-\int\left((\sin x)'\int e^{2x}dx\right)dx$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin x-\int\left(\cos x\left(\frac{e^{2x}}{2}\right)\right)dx$$

$$\Rightarrow I = \frac{e^{2x}}{2}\sin x - \frac{1}{2}\int\left[\cos x\int e^{2x}dx - \int\left((\cos x)'\left(\int e^{2x}dx\right)\right)\right]dx$$

$$\Rightarrow I = \frac{e^{2x}}{2}\sin x - \frac{1}{2}\left[\frac{e^{2x}}{2}\cos x + \frac{1}{2}\int e^{2x}(\sin x)dx\right]$$

$$\Rightarrow I = \frac{e^{2x}}{2}\sin x - \frac{1}{2}\left[\frac{e^{2x}}{2}\cos x + \frac{1}{2}I\right]$$

$$\Rightarrow I = \frac{e^{2x}}{2}\sin x - \frac{e^{2x}}{4}\cos x - \frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I = \frac{e^{2x}}{2}\sin x - \frac{e^{2x}}{4}\cos x$$

$$\Rightarrow I = \frac{2e^{2x}}{5}\sin x - \frac{e^{2x}}{5}\cos x$$

$$\Rightarrow I = \frac{e^{2x}}{5}[2\sin x - \cos x]$$

Back substituting I in expression (1)

$$\Rightarrow ye^{2x} = \frac{e^{2x}}{5}[2\sin x - \cos x] + C$$

$$\Rightarrow y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$$

The general solution for given differential equation is $y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$

2. Find the general solution for the differential equation $\frac{dy}{dx} + 3y = e^{-2x}$

Solution:

The given differential equation is $\frac{dy}{dx} + 3y = e^{-2x}$

It is a linear differential equation of the form $\frac{dy}{dx}+px=Q$, with

$$p=3$$
$$Q=e^{-2x}$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int 3dx}$$

$$\Rightarrow I.F=e^{3x}$$

General solution is of the form

$$\Rightarrow y(I.F)=\int (Q\times I.F)dx+C$$

$$\Rightarrow ye^{3x}=\int e^{-2x}\left(e^{3x}\right)dx+C$$

$$\Rightarrow ye^{3x}=\int e^{-2x+3x}\,dx+C$$

$$\Rightarrow ye^{3x}=\int e^{x}\,dx+C$$

$$\Rightarrow ye^{3x}=e^{x}+C$$

$$\Rightarrow y=e^{-2x}+Ce^{-3x}$$

The general solution for given differential equation is $y=e^{-2x}+Ce^{-3x}$

3. Find the general solution for the differential equation $\frac{dy}{dx}+\frac{y}{x}=x^2$

Solution:

The given differential equation is $\frac{dy}{dx}+\frac{y}{x}=x^2$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\frac{1}{x}$$
$$Q=x^2$$

Calculate the integrating factor

$$\Rightarrow I.F=e^{\int\frac{1}{x}dx}$$

$$\Rightarrow I.F=e^{\log x}$$

$$\Rightarrow I.F=x$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)dx+C$$

$$\Rightarrow yx=\int x^2(x)dx+C$$

$$\Rightarrow xy=\int x^3dx+C$$

$$\Rightarrow xy=\frac{x^{3+1}}{3+1}+C$$

$$\Rightarrow xy=\frac{x^4}{4}+C$$

The general solution for given differential equation is $xy=\frac{x^4}{4}+C$

4. Find the general solution for the differential equation

$$\frac{dy}{dx}+(\sec x)y=\tan x\left(0\le x\le\frac{\pi}{2}\right)$$

Solution:

The given differential equation is $\frac{dy}{dx}+(\sec x)y=\tan x$

It is a linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$p = \sec x$

$Q = \tan x$

Calculate the integrating factor

$$I.F = e^{\int p dx}$$

$$\Rightarrow I.F = e^{\int \sec x dx}$$

$$\Rightarrow I.F = e^{\log(\sec x + \tan x)}$$

$$\Rightarrow I.F = (\sec x + \tan x)$$

General solution is of form

$$y(I.F) = \int (Q \times I.F) dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x) dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int \tan x \sec x \, dx + \int \tan^2 x \, dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1) dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \int \sec^2 x \, dx - \int dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \tan x - x + C$$

The general solution for given differential equation is

$$y(\sec x + \tan x) = \sec x + \tan x - x + C$$

5. Find the general solution for the differential equation

$$\cos^2 x \frac{dy}{dx} + y = \tan x \left(0 \le x \le \frac{\pi}{2}\right)$$

Solution:

The given differential equation is $\cos^2 x\frac{dy}{dx}+y=\tan x$

$$\frac{dy}{dx}+\left(\sec^2 x\right)y=\sec^2 x\tan x$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\sec^2 x$$
$$Q=\sec^2 x\tan x$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int \sec^2 xdx}$$

$$\Rightarrow I.F=e^{\tan x}$$

General solution is of the form

$$y\left(I.F\right)=\int\left(Q\times I.F\right)dx+C$$

$$\Rightarrow ye^{\tan x}=\int e^{\tan x}\left(\sec^2 x\tan x\right)dx+C$$

$$\Rightarrow ye^{\tan x}=I+C\left(I=\int e^{\tan x}\left(\sec^2 x\tan x\right)dx\right)\text{.............}(1)$$

Solving the integral I

$$I=\int e^{\tan x}\left(\sec^2 x\tan x\right)dx$$

Substitute $\tan x=t$

$$\tan x=t$$

$$\Rightarrow \sec^2 xdx=dt$$

$$\Rightarrow I=\int e^t t\,dt$$

$$\Rightarrow I=t\int e^t\,dt-\int\left(\left(t\right)'\int e^t dt\right)dt$$

$\Rightarrow I = te^t - I!\left(e^t\right)dt$

$\Rightarrow I = te^t - e^t$

Back substitute t:

$I = \tan xe^{\tan x} - e^{\tan x}$

Back substitute I in expression (1)

$\Rightarrow ye^{\tan x} = I + C$

$\Rightarrow ye^{\tan x} = \tan xe^{\tan x} - e^{\tan x} + C$

The general solution for given differential equation is $ye^{\tan x} = \tan xe^{\tan x} - e^{\tan x} + C$

6. Find the general solution for the differential equation $x\frac{dy}{dx} + 2y = x^2 \log x$

Solution:

The given differential equation is $x\frac{dy}{dx} + 2y = x^2 \log x$

$\frac{dy}{dx} + \frac{2}{x}y = x\log x$

It is linear differential equation of the form $\frac{dy}{dx} + py = Q$, with

$p = \frac{2}{x}$

$Q = x\log x$

Calculate the integrating factor

$I.F = e^{\int pdx}$

$\Rightarrow I.F = e^{\int \frac{2}{x}dx}$

$\Rightarrow I.F = e^{2\log x}$

$$\Rightarrow I.F = e^{\log x^2}$$

$$\Rightarrow I.F = x^2$$

General solution is of the form

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

$$\Rightarrow yx^2 = \int x^2 (x \log x)\,dx + C$$

$$\Rightarrow yx^2 = I + C\left(I = \int x^3 \log x dx\right) \ldots\ldots..(1)$$

Solving the integral I

$$I = \int x^3 \log x dx$$

$$\Rightarrow I = \log x \int x^3 dx - \int \left((\log x)' \int x^3 dx\right) dx$$

$$\Rightarrow I = \frac{x^4}{4}\log x - \int \left(\frac{1}{x}\left(\frac{x^4}{4}\right)\right) dx$$

$$\Rightarrow I = \frac{x^4}{4}\log x - \frac{1}{4}\int x^3\, dx$$

$$\Rightarrow I = \frac{x^4}{4}\log x - \frac{1}{4}\int \left(\frac{x^4}{4}\right)$$

$$\Rightarrow I = \frac{x^4}{4}\log x - \frac{x^4}{16}$$

Back substitute I in expression (1)

$$\Rightarrow yx^2 = I + C$$

$$\Rightarrow yx^2 = \frac{x^4}{4}\log x - \frac{x^4}{16} + C$$

$$\Rightarrow y = \frac{x^4}{16}(4\log x - 1) + Cx^{-2}$$

The general solution for given differential equation is $y=\frac{x^4}{16}(4\log x-1)+Cx^{-2}$

7. Find the general solution for the differential equation $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$

Solution:

The given differential equation is $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$

$$\frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x^2}$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\frac{1}{x\log x}$$

$$Q=\frac{2}{x^2}$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int\frac{1}{x\log x}dx}$$

Substitute $\log x=t$

$$\log x=t$$

$$\Rightarrow \frac{1}{x}dx=dt$$

$$\Rightarrow I.F=e^{\int\frac{1}{t}dt}$$

$$\Rightarrow I.F=e^{\log t}$$

$$\Rightarrow I.F=t$$

$$\Rightarrow I.F = \log x$$

General solution is of the form

$$\Rightarrow y(I.F) = \int (Q \times I.F)\,dx + C$$

$$\Rightarrow y\log x = \int \frac{2}{x^2}(\log x)\,dx + C$$

$$\Rightarrow y\log x = I + C\left(I = \int \frac{2}{x^2}(\log x)\,dx\right)............(1)$$

Solving the integral I

$$I = \int \frac{2}{x^2}(\log x)\,dx$$

$$I = 2\left[\log x \int \frac{1}{x^2}\,dx - \int \left((\log x)' \int \frac{1}{x^2}\,dx\right) dx\right]$$

$$I = 2\left[\log x\left(\frac{-1}{x}\right) - \int \left(\frac{1}{x}\left(\frac{-1}{x}\right)\right) dx\right]$$

$$I = 2\left[-\frac{\log x}{x} + \int \left(\frac{1}{x^2}\right) dx\right]$$

$$I = 2\left[-\frac{\log x}{x} - \frac{1}{x}\right]$$

Back substitute I in expression (1)

$$y\log x = I + C$$

$$\Rightarrow y\log x = 2\left[-\frac{\log x}{x} - \frac{1}{x}\right] + C$$

$$\Rightarrow y\log x = -\frac{2}{x}(\log x + 1) + C$$

The general solution for given differential equation is $y\log x = -\frac{2}{x}(\log x + 1) + C$

8. Find the general solution for the differential equation

$$\left(1+x^2\right)dy+2xydx=\cot xdx\left(x\neq 0\right)$$

Solution:

The given differential equation is

$$\left(1+x^2\right)dy+2xydx=\cot xdx\left(x\neq 0\right)$$

$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\cot x}{1+x^2}$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\frac{2x}{1+x^2}$$

$$Q=\frac{\cot x}{1+x^2}$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int\frac{2x}{1+x^2}dx}$$

Substitute $\log x=t$

$$1+x^2=t$$

$$\Rightarrow 2xdx=dt$$

$$\Rightarrow I.F=e^{\int\frac{1}{t}dt}$$

$$\Rightarrow I.F=e^{\log t}$$

$$\Rightarrow I.F=t$$

$$\Rightarrow I.F=1+x^2$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)\,dx+C$$

$$\Rightarrow y\left(1+x^{2}\right)=\int\frac{\cot x}{1+x^{2}}\left(1+x^{2}\right)dx+C$$

$$\Rightarrow y\left(1+x^{2}\right)=\int\cot x\,dx+C$$

$$\Rightarrow y\left(1+x^{2}\right)=\log|\sin x|+C$$

The general solution for given differential equation is $y\left(1+x^{2}\right)=\log|\sin x|+C$

9. Find the general solution for the differential equation

$$x\frac{dy}{dx}+y-x+xy\cot x=0\,(x\neq 0)$$

Solution:

The given differential equation is $x\frac{dy}{dx}+y-x+xy\cot x=0$

$$\Rightarrow\frac{dy}{dx}+\frac{y}{x}-1+y\cot x=0$$

$$\Rightarrow\frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\left(\frac{1}{x}+\cot x\right)$$
$$Q=1$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int\left(\frac{1}{x}+\cot x\right)dx}$$

$$\Rightarrow I.F = e^{\int \frac{1}{x}dx + \int \cot x dx}$$

$$\Rightarrow I.F = e^{\log x + \log(\sin x)}$$

$$\Rightarrow I.F = e^{\log(x \sin x)}$$

$$\Rightarrow I.F = x \sin x$$

General solution is of the form

$$y(I.F) = \int (Q \times I.F)dx + C$$

$$\Rightarrow y(x \sin x) = \int (1)(x \sin x)dx + C$$

$$\Rightarrow xy \sin x = I + C\left(I = \int x \sin x\, dx\right) \text{..........}(1)$$

Solving the integral I

$$I = \int x \sin x\, dx$$

$$\Rightarrow I = x\int \sin x\, dx - \int\left((x)' \int \sin x dx\right)dx$$

$$\Rightarrow I = x(-\cos x) + \int(\cos x)dx$$

$$\Rightarrow I = x \cos x + \sin x$$

Back substitute I in expression (1)

$$xy \sin x = I + C$$

$$\Rightarrow xy \sin x = -x \cos x + \sin x + C$$

$$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$$

The general solution for given differential equation is $y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$

10. Find the general solution for the differential equation $(x+y)\frac{dy}{dx} = 1$

Solution:

The given differential equation is $(x+y)\frac{dy}{dx}=1$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+y}$$

$$\Rightarrow \frac{dx}{dy}=x+y$$

$$\Rightarrow \frac{dx}{dy}-x=y$$

It is differential equation of the form $\frac{dx}{dy}+px=Q$, with

$$p=-1$$
$$Q=y$$

Calculate the integrating factor

$$I.F=e^{\int pdy}$$

$$\Rightarrow I.F=e^{\int -1dy}$$

$$\Rightarrow I.F=e^{-y}$$

General solution is of the form

$$x(I.F)=\int (Q\times I.F)dy+C$$

$$\Rightarrow x(e^{-y})=\int (y)(e^{-y})dy+C$$

$$\Rightarrow xe^{-y}=I+C\left(I=\int ye^{-y}dy\right).............(1)$$

Solving the integral I

$$I=\int ye^{-y}dy$$

$$\Rightarrow I=y\int e^{-y}dy-\int\left((y)'\int e^{-y}dy\right)dy$$

$$\Rightarrow I = -ye^{-y} + \int \left((1) e^{-y} \right) dy$$

$$\Rightarrow I = -ye^{-y} + \int e^{-y}\, dy$$

$$\Rightarrow I = -ye^{-y} - e^{-y}$$

Back substitute I in expression (1)

$$xe^{-y} = I + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$$

$$\Rightarrow x = -y - 1 + Ce^{y}$$

$$\Rightarrow x + y + 1 = Ce^{y}$$

The general solution for given differential equation is $x + y + 1 = Ce^{y}$

11. Find the general solution for the differential equation $ydx + \left(x - y^2\right)dy = 0$

Solution:

The given differential equation is $ydx + \left(x - y^2\right)dy = 0$

$$\Rightarrow \frac{dx}{dy} = \frac{y^2 - x}{y}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{y}\right)x = y$$

It is differential equation of the form $\frac{dx}{dy} + px = Q$, with

$$p = \frac{1}{y}$$
$$Q = y$$

Calculate the integrating factor

$I.F = e^{\int p dy}$

$\Rightarrow I.F = e^{\int \frac{1}{y} dy}$

$\Rightarrow I.F = e^{\log y}$

$\Rightarrow I.F = y$

General solution is of the form

$x(I.F) = \int (Q \times I.F) dy + C$

$\Rightarrow x(y) = \int (y)(y) dy + C$

$\Rightarrow xy = \int y^2 dy + C$

$\Rightarrow xy = \frac{y^3}{3} + C$

$\Rightarrow x = \frac{y^2}{3} + \frac{C}{y}$

The general solution for given differential equation is $x = \frac{y^2}{3} + \frac{C}{y}$

12. Find the general solution for the differential equation $(x+3y^2)\frac{dy}{dx} = y \, (y>0)$

Solution:

The given differential equation is $(x+3y^2)\frac{dy}{dx} = y$

$\Rightarrow \frac{dx}{dy} = \frac{x+3y^2}{y}$

$\Rightarrow \frac{dx}{dy} - \left(\frac{1}{y}\right)x = 3y$

It is differential equation of the form $\frac{dx}{dy}+px=Q$, with

$$p=-\frac{1}{y}$$

$$Q=3y$$

Calculate the integrating factor

$$I.F=e^{\int pdy}$$

$$\Rightarrow I.F=e^{-\int\frac{1}{y}dy}$$

$$\Rightarrow I.F=e^{-\log y}$$

$$\Rightarrow I.F=e^{\log y^{-1}}$$

$$\Rightarrow I.F=\frac{1}{y}$$

General solution is of the form

$$x(I.F)=\int(Q\times I.F)dy+C$$

$$\Rightarrow x\left(\frac{1}{y}\right)=\int(3y)\left(\frac{1}{y}\right)dy+C$$

$$\Rightarrow \frac{x}{y}=3\int dy+C$$

$$\Rightarrow \frac{x}{y}=3y+C$$

$$\Rightarrow x=3y^2+Cy$$

The general solution for given differential equation is $x=3y^2+Cy$

13. Find particular solution for $\frac{dy}{dx}+2y\tan x=\sin x$ satisfying $y=0$ when $x=\frac{\pi}{3}$

Solution:

The given differential equation is $\frac{dy}{dx}+2y\tan x=\sin x$

$$\frac{dy}{dx}+(2\tan x)y=\sin x$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=2\tan x$$
$$Q=\sin x$$

Calculate the integration factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{2\int \tan x dx}$$

$$\Rightarrow I.F=e^{2\log|\sec x|}$$

$$\Rightarrow I.F=e^{\log(\sec x)^2}$$

$$\Rightarrow I.F=\sec^2 x$$

General solution is of the form

$$\Rightarrow y(I.F)=\int(Q\times I.F)dx+C$$

$$\Rightarrow y\sec^2 x=\int(\sin x)(\sec^2 x)dx+C$$

$$\Rightarrow y\sec^2 x=\int \tan x\sec x\,dx+C$$

$$\Rightarrow y\sec^2 x=\sec x+C$$

$$\Rightarrow y=\cos x+C\cos^2 x$$

Given $y=0$ when $x=\frac{\pi}{3}$

$$0=\cos\left(\frac{\pi}{3}\right)+C\cos^2\left(\frac{\pi}{3}\right)$$

$$\Rightarrow 0=\frac{1}{2}+C\left(\frac{1}{2}\right)^2$$

$$\Rightarrow C=-2$$

Therefore the particular solution will be

$$\Rightarrow y=\cos x-2\cos^2 x$$

The particular solution for given differential equation satisfying the given conditions is

$$y=\cos x-2\cos^2 x$$

14. Find particular solution for $\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$ satisfying $y=0$ when $x=1$

Solution:

The given differential equation is $\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$

$$\frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y=\frac{1}{\left(1+x^2\right)^2}$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\frac{2x}{1+x^2}$$

$$Q=\frac{1}{\left(1+x^2\right)^2}$$

Calculate the integrating factor

$$I.F = e^{\int pdx}$$

$$\Rightarrow I.F = e^{\int \frac{2x}{1+x^2}dx}$$

$$\Rightarrow I.F = e^{2\log|\sec x|}$$

$$\Rightarrow I.F = e^{\log\left(1+x^2\right)}$$

$$\Rightarrow I.F = 1+x^2$$

General solution is of the form

$$y\left(I.F\right) = \int \left(Q \times I.F\right)dx + C$$

$$\Rightarrow y\left(1+x^2\right) = \int\left(\frac{1}{\left(1+x^2\right)^2}\right)\left(1+x^2\right)dx + C$$

$$\Rightarrow y\left(1+x^2\right) = \int \frac{1}{1+x^2}dx + C$$

$$\Rightarrow y\left(1+x^2\right) = \tan^{-1}x + C$$

Given $y = 0$ when $x = 1$

$$0\left(1+1\right) = \tan^{-1}\left(1\right) + C$$

$$\Rightarrow C + \frac{\pi}{4} = 0$$

$$\Rightarrow C = -\frac{\pi}{4}$$

Therefore the particular solution will be

$$\Rightarrow y\left(1+x^2\right) = \tan^{-1}x - \frac{\pi}{4}$$

The particular solution for given differential equation satisfying the given conditions is

$$y\left(1+x^2\right) = \tan^{-1}x - \frac{\pi}{4}$$

15. Find particular solution for $\frac{dy}{dx}-3y\cot x=\sin 2x$ satisfying $y=2$ when $x=\frac{\pi}{2}$

Solution:

The given differential equation is $\frac{dy}{dx}-3y\cot x=\sin 2x$

$$\frac{dy}{dx}+(-3\cot x)y=\sin 2x$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=-3\cot x$$
$$Q=\sin 2x$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{-3\int \cot x dx}$$

$$\Rightarrow I.F=e^{-3\log|\sin x|}$$

$$\Rightarrow I.F=e^{\log\left(\frac{1}{\sin^3 x}\right)}$$

$$\Rightarrow I.F=\frac{1}{\sin^3 x}$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)dx+C$$

$$\Rightarrow y\left(\frac{1}{\sin^3 x}\right)=\int(\sin 2x)\left(\frac{1}{\sin^3 x}\right)dx+C$$

$$\Rightarrow y\left(\frac{1}{\sin^3 x}\right)=2\int(\sin x\cos x)\left(\frac{1}{\sin^3 x}\right)dx+C$$

$$\Rightarrow \frac{y}{\sin^3 x} = 2\int\left(\frac{\cos x}{\sin^2 x}\right)dx + C$$

$$\Rightarrow \frac{y}{\sin^3 x} = 2\int \cot x \operatorname{cosec} x \, dx + C$$

$$\Rightarrow \frac{y}{\sin^3 x} = -2\operatorname{cosec} x + C$$

$$\Rightarrow y = -2\sin^2 x + C\sin^3 x$$

Given $y = 2$ when $x = \frac{\pi}{2}$

$$2 = -2\sin^2\left(\frac{\pi}{2}\right) + C\sin^3\left(\frac{\pi}{2}\right)$$

$$\Rightarrow C - 2 = 2$$

$$\Rightarrow C = 4$$

Therefore the particular solution will be

$$\Rightarrow y = -2\sin^2 x + 4\sin^3 x$$

The particular solution for given differential equation satisfying the given conditions is

$$y = -2\sin^2 x + 4\sin^3 x$$

16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point

Solution:

According to question the slope of tangent $\frac{dy}{dx}$ is equal to sum of the coordinate

$$\frac{dy}{dx} = x + y$$

The given differential equation is

$$\frac{dy}{dx}=x+y$$

$$\frac{dy}{dx}+(-1)y=x$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=-1$$
$$Q=x$$

Calculate he integration factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{-1\int dx}$$

$$\Rightarrow I.F=e^{-x}$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)dx+C$$

$$\Rightarrow y\left(e^{-x}\right)=\int x\left(e^{-x}\right)dx+C$$

$$\Rightarrow ye^{-x}=-xe^{-x}+\int\left(e^{-x}\right)dx+C$$

$$\Rightarrow ye^{-x}=-xe^{-x}-e^{-x}+C$$

$$\Rightarrow y=-x-1+Ce^{x}$$

$$\Rightarrow y+x+1=Ce^{x}$$

Given $y=0$ when $x=0$ as it passes through origin

$$0+0+1=Ce^{0}$$

$$\Rightarrow C=1$$

Therefore the equation of the required curve is $y+x+1=e^{x}$

17. Find the equation of a curve passing through the point (0,2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5

Solution:

Let the slope of tangent be $\frac{dy}{dx}$

According to question

$$x+y=\frac{dy}{dx}+5$$

The given differential equation is

$$x+y=\frac{dy}{dx}+5$$

$$\frac{dy}{dx}+(-1)y=x-5$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=1$$
$$Q=x-5$$

Calculate the integrating factor

$$I.\text{F}=\text{e}^{\int pdx}$$

$$\Rightarrow I.\text{F}=\text{e}^{-1\int dx}$$

$$\Rightarrow I.\text{F}=\text{e}^{-x}$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)dx+C$$

$$y\left(e^{-x}\right)=\int(x-5)\left(e^{-x}\right)dx+C$$

$$\Rightarrow y\left(e^{-x}\right)=\int x\left(e^{-x}\right)dx-\int e^{-x}\,dx+C$$

$$\Rightarrow y\left(e^{-x}\right)=x\int e^{-x}dx-\int\left((x)'\int e^{-x}dx\right)dx-5\int e^{-x}dx+C$$

$$\Rightarrow ye^{-x}=-xe^{-x}+\int\left(e^{-x}\right)dx+5e^{-x}+C$$

$$\Rightarrow ye^{-x}=-xe^{-x}+4e^{-x}+C$$

$$\Rightarrow y=-x+4+Ce^{-x}$$

$$\Rightarrow y+x-4=Ce^{-x}$$

Given as it passes through (0,2)

$$2+0-4=Ce^{0}$$

$$\Rightarrow C=-2$$

Therefore the equation of the required curve is $y+x+4=-2e^{x}$

18. Find the integrating factor of the differential equation $x\frac{dy}{dx}-y=2x^2$

Solution:

Given differential equation is $x\frac{dy}{dx}-y=2x^2$

$$\Rightarrow \frac{dy}{dx}-\left(\frac{1}{x}\right)y=2x^2$$

Thus it is a linear differential equation of the form $\frac{dy}{dx}+py=Q$

$$p=-\frac{1}{x}$$

$$I.F=e^{\int pdx}$$

$\Rightarrow I.F = e^{-\int \frac{1}{x}dx}$

$\Rightarrow I.F = e^{-\log|x|}$

$\Rightarrow I.F = e^{\log x^{-1}}$

$\Rightarrow I.F = \frac{1}{x}$

Therefore integrating factor is $\frac{1}{x}$.

Thus the correct option is (C)

19. Find the integrating factor of the differential equation

$\left(1-y^2\right)\frac{dx}{dy} + yx = ay\left(-1 < y < 1\right)$

Solution:

Given differential equation is $\left(1-y^2\right)\frac{dx}{dy} + yx = ay$

$\Rightarrow \frac{dx}{dy} + \left(\frac{y}{1-y^2}\right)x = \frac{ay}{\left(1-y^2\right)}$

Thus it is a linear differential equation of the form $\frac{dx}{dy} + px = Q$

$p = \frac{y}{1-y^2}$

$I.F = e^{\int pdy}$

$\Rightarrow I.F = e^{\int \frac{y}{1-y^2}dy}$

$\Rightarrow I.F = e^{-\frac{1}{2}\int \frac{-2y}{1-y^2}dy}$

$$\Rightarrow I.F = e^{-\frac{1}{2}\log\left(1-y^2\right)}$$

$$\Rightarrow I.F = e^{\log\left(1-y^2\right)^{-\frac{1}{2}}}$$

$$\Rightarrow I.F = \frac{1}{\sqrt{1-y^2}}$$

Therefore integrating factor is $\frac{1}{\sqrt{1-y^2}}$

Thus the correct option is (D)