Miscellaneous Exercise on Chapter 1

1. Decide, among the following sets, which sets are subsets of one and another: $A = \{x : x \in \mathbf{R}$ and x satisfy $x^2 - 8x + 12 = 0\}$

$B = \{2, 4, 6\}, \quad C = \{2, 4, 6, 8, \ldots\}, D = \{6\}$

Solution

Given

$A = \{x : x \in R \text{ and } (x-2)(x-6) = 0\} = \{2, 6\}$

$B = \{2, 4, 6\}, C = \{2, 4, 6, 8, \ldots\}, D = \{6\}$

Here,

So, all elements of A are in B, A is a subset of B,

$A \subset B$

Hence all elements of A are in C, A is a subset of C,

$A \subset C$

$A = \{2, 6\}, B = \{2, 4, 6\}, C = \{2, 4, 6, 8 \ldots\}, D = \{6\}$

C is not a subset of any set.

So, all elements of D is in A, D is a subset of A,

That is $D \subset A$

Since all elements of D are in B, D is a subset of B,

That is $D \subset B$

2. In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.

(i) If $x \in A$ and $A \in B$, then $x \in B$

(ii) If $A \subset B$ and $B \in C$, then $A \in C$

(iii) If $A \subset B$ and $B \subset C$, then $A \subset C$

(iv) If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$

(v) If $x \in A$ and $A \not\subset B$, then $x \in B$

(vi) If $\mathrm{A} \subset \mathrm{B}$ and $x \notin \mathrm{B}$, then $x \notin \mathrm{A}$

Solution

(i) False. Let $\mathrm{A}=\{1\}$ and $\mathrm{B}=\{\{1\},2\}$. Clearly, $1 \in \mathrm{A}$ and $\mathrm{A} \in \mathrm{B}$ but $1 \notin \mathrm{B}$.

Thus, $x \in \mathrm{A}$ and $\mathrm{A} \in \mathrm{B}$ need not imply $x \in \mathrm{B}$.

(ii) False. Let $\mathrm{A}=\{1\}, \mathrm{B}=\{1,2\}$ and $\mathrm{C}=\{\{1,2\},3\}$. Clearly, $\mathrm{A} \subset \mathrm{B}$ and $\mathrm{B} \in \mathrm{C}$ but $\mathrm{A} \notin \mathrm{C}$.

Thus, $A \subset B$ and $B \in C$ need not imply $A \in C$.

(iii) True. Let x be any element of A. Then

$x \in \mathrm{A} \Rightarrow x \in \mathrm{B}$

$\Rightarrow x \in \mathrm{C} \mathrm{z}$

Thus, $x \in \mathrm{A} \Rightarrow x \in \mathrm{C}$ for all $x \in \mathrm{A}$, therefore, $\mathrm{A} \subset \mathrm{C}$.

Hence, $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

(iv) False. Let $\mathrm{A}=\{1,2\}, \mathrm{B}=\{2,3\}$ and $\mathrm{C}=\{1,2,5\}$. Clearly, $A \not\subset B$ since $1 \in A$ and $1 \notin B$. Also $B \notin C$ since $3 \in B$ and $3 \notin$ C.

But $A \subset C$.

Thus, $\mathrm{A} \not\subset \mathrm{B}$ and $\mathrm{B} \not\subset \mathrm{C}$ need not imply $\mathrm{A} \not\subset \mathrm{C}$.

(v) False. Let $\mathrm{A}=\{1,2\}$ and $\mathrm{B}=\{2,3\}$. Clearly, $1 \in \mathrm{A}$ and AB but $1 \notin \mathrm{B}$.

Thus, $x \in \mathrm{A}$ and $\mathrm{A} \not\subset \mathrm{B}$ need not imply $x \in \mathrm{B}$.

(vi) True. Let $\mathrm{A} \subset \mathrm{B}$ and $x \notin \mathrm{B}$. If possible, suppose $x \in \mathrm{A}$.

Now, $x \in \mathrm{A}$ and $\mathrm{A} \subset \mathrm{B} \Rightarrow x \in \mathrm{B}$ which is a contradiction to given.

Therefore, our supposition is wrong. Hence $x \notin \mathrm{A}$.

Thus, $\mathrm{A} \subset \mathrm{B}$ and $x \notin \mathrm{B} \Rightarrow x \notin \mathrm{A}$.

3. Let A, B, and C be the sets such that $\mathrm{A} \cup \mathrm{B}=\mathrm{A} \cup \mathrm{C}$ and $\mathrm{A} \cap \mathrm{B}=\mathrm{A} \cap \mathrm{C}$. Show that $\mathrm{B}=\mathrm{C}$.

Solution

Let $x \in \mathrm{B}$

The subsets is combination of $\mathrm{A} \cup \mathrm{B}$

$\Rightarrow x \in \mathrm{A} \cup \mathrm{B}$

$\Rightarrow x \in \mathrm{A} \cup \mathrm{C}$

$[\because \mathrm{A} \cup \mathrm{B}=\mathrm{A} \cup \mathrm{C}$ (given)$]$

$\Rightarrow x \in \mathrm{A}$ or $x \in \mathrm{C}$

For Case I

Let $x \in \mathrm{A}$

$\Rightarrow x \in \mathrm{A}$ and $x \in \mathrm{B}$

$\Rightarrow x \in \mathrm{A} \cap \mathrm{B}$

$[\because \mathrm{A} \cap \mathrm{B} = \mathrm{A} \cap \mathrm{C}$ (given)]

$\Rightarrow x \in \mathrm{A} \cap \mathrm{C}$

$\Rightarrow x \in \mathrm{A}$ and $x \in \mathrm{C}$

$\Rightarrow x \in \mathrm{C}$ also

For Case II

Let $x \in \mathrm{C}$

$\therefore$ In each case $x \in \mathrm{B} \Rightarrow x \in \mathrm{C}$

$\because B \subset C$

Similarly, $\mathrm{C} \subset \mathrm{B}$

So, $\mathrm{B} = \mathrm{C}$.

4. Show that the following four conditions are equivalent:

(i) $A \subset B$

(ii) $A - B = \phi$

(iii) $\mathrm{A} \cup \mathrm{B} = \mathrm{B}$

(iv) $A \cap B = A$

Solution

Showing Condition (i) is equals to Condition (ii).

Let $A \subset B$

It means all elements of A are in B,

hence, A has no element different from B

$\Rightarrow A - B = \Phi$

Showing Condition (ii) is equals to Condition (iii).

$A - B = \varnothing$

It means A has no elements different from B

Therefore, all elements of A are in B

Since, $A \cup B = B$

Showing Condition (iii) is equals to Condition (iv).

$A \cup B = B$

It means all elements of A are in B,

So, the same elements of A and B must be the elements of A

So, $A \cap B = A$

Thus, all the four conditions are equals.

5. Show that if $\mathrm{A} \subset \mathrm{B}$, then $\mathrm{C}-\mathrm{B} \subset \mathrm{C}-\mathrm{A}$.

Solution

Given $\mathrm{A} \subset \mathrm{B}$

Let $x \in (\mathrm{C}-\mathrm{B}) \Rightarrow x \in \mathrm{C}$ and $x \notin \mathrm{B}$

$x \in \mathrm{C}$ and $x \notin \mathrm{A}$

$x \in (\mathrm{C}-\mathrm{A})$

It is proven that $(\mathrm{C}-\mathrm{B}) \subset (\mathrm{C}-\mathrm{A})$

6. Assume that $\mathrm{P(A)} = \mathrm{P(B)}$. Show that $\mathrm{A} = \mathrm{B}$

Solution

Let assume $\mathrm{P(A)} = \mathrm{P(B)}$,

i.e., Power set of $\mathrm{A} =$ Power set of B

To prove $A = B$.

Let $x \in \mathrm{A}$.

$\therefore |x| \subset \mathrm{A}$

Subsets of $\mathrm{P(A)}$

$\therefore \{x\} \in \mathrm{P(A)}$

$\{x| \in \mathrm{P(B)}$

$\{x\} \subset \mathrm{B}$

$x \in \mathrm{B}$

$A \subset B$

Similarly, $B \subset A$

Hence it $A = B$ is shown

7. Is it true that for any sets A and $B, P(A) \cup P(B) = P(A \cup B)$? Justify your answer.

Solution

Let

$A = \{1\}, B = \{2\}$, then

$P(A) = \{\phi, \{1\}\}$ and $P(B) = \{\phi, \{2\}\}$

$P(A) \cup P(B) = \{\phi, \{1\}\} \cup \{\phi, \{2\}|$

$= \{\phi, \{1\}, \{2\}\}$

Also $A \cup B = \{1, 2\}$, then

$P(A \cup B) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Now, $\{1, 2\} \in P(A \cup B)$ but $\{1, 2\} \notin P(A) \cup P(B)$

Therefore, $P(A) \cup P(B) \neq P(A \cup B)$

8. Show that for any sets A and B,

$A = (A \cap B) \cup (A - B)$ **and** $A \cup (B - A) = (A \cup B)$

Solution

Let consider $U = \{1, 2, 3, 4, 5\}$

$A = \{1, 2\}$

$B = \{2, 3, 4\}$

$A - B = A - (A \cap B) = \{1, 2\} - \{2\}$

$= \{1\}$

Also,

$B' = U - B = \{1, 2, 3, 4, 5\} - \{2, 3, 4\}$

$= \{1, 5\}$

$A - B = A \cap B' = \{1, 2\} \cap \{1, 5\}$

$=\{1\}$

$(A\cap B)\cup(A-B)\ =(A\cap B)\cup\left(A\cap B'\right)$

Applying the Distributive Law

$=A\cap\left(B\cup B'\right)$

$=A\cap U=A$

Also $A\cup(B-A)=A\cup\left(B\cap A'\right)$

Applying the Distributive Law

$=(A\cup B)\cap\left(A\cup A'\right)$

$=(A\cup B)\cap U$

$=A\cup B$

9. Using properties of sets, show that

(i) $A\cup(A\cap B)=A$

(ii) $A\cap(A\cup B)=A$.

Solution

In order to prove

(i) $A\cup(A\cap B)=A$

we should prove

$A\cup(A\cap B)$ is a subset of A

That is $A\cup(A\cap B)\subset A$

A is a subset of $A\cup(A\cap B)$

Then $A\subset A\cup(A\cap B)$

As set is a subset of itself, $A\subset A$

Also, A is a subset of $A\cap B$,

So, $A\subset A\cap B$

as all elements of set A are in $A\cap B$

Now,

$A\cup(A\cap B)$

Using distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$= (A) \cap (A \cup B)$

$= A$

Thus, $A \cup (A \cap B) = A$

Hence its proved

(ii) $A \cap (A \cup B) = A$

$A \cup (A \cap B)$

Using distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$= (A \cup A) \cap (A \cup B)$

$= (A) \cap (A \cup B)$

$= A$

Thus, $A \cup (A \cap B) = A$

Hence its proved

10. Show that $A \cap B = A \cap C$ need not imply $B = C$.

Solution

Let $A = \{1, 2, 3\}, B = \{2, 4\}, C = \{2, 5\}$

Then $A \cap B = \{2\} = A \cap C$

But $B \neq C$.

11. Let A and B be sets. If $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, show that $A = B$.

Solution

Given:

Let A and B be two sets such that $A \cap X = B \cap X = \varnothing$ and $A \cup X = B \cup X$ for some set X

To prove: $A = B$

Let $A = A \cap (A \cup X)$

Given $A \cup X = B \cup X$

$A = A \cap (B \cup X)$

Using distributive law:

$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

$=(A\cap B)\cup(A\cap X)$

As $A\cap X=\varnothing$ given

$=(A\cap B)\cup\varnothing$

So, $A=A\cap B$

Again,

Let $B=B\cap(B\cup X)$

Given $A\cup X=B\cup X$

$B=B\cap(A\cup X)$

Using distributive law:

$A\cup(B\cap C)=(A\cap B)\cup(A\cap C)$

$=(B\cap A)\cup(B\cap X)$

As $B\cap X=\Phi$

$=(B\cap A)\cup\Phi$

$B=B\cap A$

$B=A\cap B$

$A=A\cap B$ & $B=A\cap B$

$A=B$

Hence the both condition is proved.

12. Find sets A,B and C such that $\mathrm{A}\cap\mathrm{B},\mathrm{B}\cap\mathrm{C}$ and $\mathrm{A}\cap\mathrm{C}$ are non-empty sets and $\mathrm{A}\cap\mathrm{B}\cap\mathrm{C}=\phi$.

Solution

Let consider $\mathrm{A}=\{x,y\},\mathrm{B}=\{x,z\},\mathrm{C}=\{y,z\}$

So the sets will be $\mathrm{A}\cap\mathrm{B}=\{x\}\neq\phi,\mathrm{B}\cap\mathrm{C}=\{z|\neq\phi,\mathrm{A}\cap\mathrm{C}=\{y|\neq\phi$

and $A\cap B\cap C=\phi$.

13. In a survey of 600 students in a school, 150 students were found to be taking tea and 225 taking coffee, 100 were taking both tea and coffee. Find how many students were taking neither tea nor coffee?

Solution

Given,

Total students $=600$

$n(\mathrm{T})=$ Number of students drinking tea $=150$

$n(\mathrm{C})=$ Number of students drinking coffee $=225$

$n(\mathrm{T} \cap \mathrm{C})=n($ both tea and coffee $)=100$

Using the formula

$n(\mathrm{T} \cup \mathrm{C})=n(\mathrm{T})+n(\mathrm{C})-n(\mathrm{T} \cap \mathrm{C})$

Substitute the values

$=150+225-100=275$

i.e., number of students taking at least one of the two drinks

$=275$

$\therefore$ Number of students drinking tea or coffee

$=$Total number of students $-n(\mathrm{T} \cup \mathrm{C})$

$=600-275=325$

325 students were taking tea or coffee.

14. In a group of students, 100 students know Hindi, 50 know English and 25 know both. Each of the students knows either Hindi or English. How many students are there in the group?

Solution

Let H mean a set of students who knows Hindi and E mean a set of students who knows English.

Number of students knows Hindi $=\mathrm{n}(\mathrm{H})=100$

Number of students knows English $=n(E)=50$

Number of students knowing Hindi and English $=\mathrm{n}(\mathrm{H} \cap \mathrm{E})=25$

So, each of the students knows Hindi or English

Number of students in group

$=$ Number of students knows Hindi or English

$=n(H \cup E)$

Using the formula

$$n(H \cup E) = n(H) + n(E) - n(H \cap E)$$
$$= 100 + 50 - 25$$
$$= 125$$

Hence, 125 students in the group.

15. In a survey of 60 people, it was found that 25 people read newspaper $\mathrm{H}, 26$ read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and $\mathrm{I}, 3$ read all three newspapers. Find:

(i) the number of people who read at least one of the newspapers.

(ii) the number of people who read exactly one newspaper.

Solution

Number of people reads newspaper $\mathrm{H} = \mathrm{n(H)} = 25$,

Number of people reads newspaper $T = n(T) = 26$,

Number of people reads newspaper $I = n(I) = 26$,

Number of people who reading both $\mathrm{H} \,\&\, \mathrm{I} = \mathrm{n(H \cap I)} = 9$,

Number of people who reading both $H \,\&\, T = n(H \cap T) = 11$

Number of people who reading both $T \,\&\, I = n(T \cap I) = 8$

Number of people who reading all $H, T \,\&\, I = n(H \cap T \cap I) = 3$

(i) Number of persons who reading one of the newspapers

$$= n(\mathrm{H} \cup \mathrm{T} \cup \mathrm{I}) = n(\mathrm{H}) + n(\mathrm{T}) + n(\mathrm{I}) - n(\mathrm{H} \cap \mathrm{T}) - n(\mathrm{H} \cap \mathrm{I}) - n(\mathrm{I} \cap \mathrm{T}) + n(\mathrm{H} \cap \mathrm{T} \cap \mathrm{I})$$

Substitute the values

$$= 25 + 26 + 26 - 9 - 11 - 8 + 3 = 52.$$

(ii) Number of people who reads exactly one newspaper.

$$n[\mathrm{only(H)}] = n(\mathrm{H}) - n(\mathrm{H} \cap \mathrm{T}) - n(\mathrm{H} \cap \mathrm{I}) + n(\mathrm{H} \cap \mathrm{T} \cap \mathrm{I})$$

Substitute the values

$$= 25 - 11 - 9 + 3 = 8$$

$$n[\mathrm{only(T)}] = n(\mathrm{T}) - n(\mathrm{T} \cap \mathrm{H}) - n(\mathrm{T} \cap \mathrm{I}) + n(\mathrm{T} \cap \mathrm{H} \cap \mathrm{I})$$

Substitute the values

$$= 26 - 11 - 8 + 3 = 10$$

$n[\text{only (I)}] = n(\text{I}) - n(\text{I} \cap \text{H}) - n(\text{I} \cap \text{T}) + n(\text{T} \cap \text{H} \cap \text{I})$

Substitute the values

$= 26 - 8 - 9 + 3 = 12$

Number of persons who read exactly one newspaper

$= n(\text{ only H}) + n(\text{ only T}) + n(\text{ only I})$

Substitute the values

$= 8 + 10 + 12 = 30$

Hence, 30 people reading one newspaper exactly.

16. In a survey it was found that 21 people liked product A, 26 liked product B and 29 liked product C. If 14 people liked products A and $\text{B}, 12$ people liked products C and $\text{A}, 14$ people liked products B and C and 8 liked all the three products. Find how many liked product C only.

Solution

Let A,B,C set of people who liked product A, product B & product C respectively

Number of people liked product $A = n(A) = 21$

Number of people liked product $B = n(B) = 26$,

Number of people liked product $C = n(C) = 29$,

Number of people liked product A and $B = n(A \cap B) = 14$,

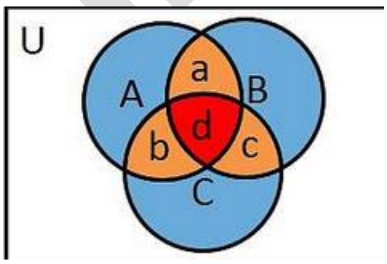
Number of people liked product C and $A = n(C \cap A) = 12$,

Number of people liked product B and $C = n(B \cap C) = 14$,

Number of people liked all three products A,B and C

$= n(A \cap B \cap C) = 8$

Let us draw a Venn diagram



Let a be the number of people who liked product A & B but not C.

Let **b** be the number of people who liked product A & C but not B.

Let **c** be the number of people who liked product **B** & C but not **A** .

Let **d** be the the number of people who liked all three products.

Number of people liked product C only

$=n(C)-b-d-c$

Now, $d=n(A\cap B\cap C)=8$

Given $n(A\cap C)=12$

$b+d=12$

Putting $d=8$

$b+8=12$

$b=12-8$

$b=4$

Similarly, $n(B\cap C)=14$

$c+d=14$

Putting $d=8$

$c+8=14$

$c=14-8$

$c=6$

Number of people who liked product C only $=n(C)-b-d-c$

$=29-4-8-6$

$=11$

Therefore, number of people liked product C only is 11.