Chapter: 1. Relations and functions

Miscellaneous Exercise:

1. Let $f: R \to R$ be defined as $f(x)=10x+7$. Find the function $g: R \to R$ such that $gof = fog = I_R$

Solution: Given that $f: R \to R$ is defined as $f(x)=10x+7$

Check the function is one to one or not:

Suppose that $f(x)=f(y)$, where $x, y \in R$

It implies that

$$\begin{aligned} 10x+7 &= 10y+7 \\ 10x &= 10y \\ x &= y \end{aligned}$$

Therefore, $f(x)$ is a one-one function

Check the function is onto or not.

Suppose that $y \in R$, and suppose that $y=10x+7$

It implies that $x=\frac{y-7}{10} \in R$

For any $y \in R$, there exists $x=\frac{y-7}{10} \in R$ such that

$$f(x)=f\left(\frac{y-7}{10}\right)=10\left(\frac{y-7}{10}\right)+7=y-7+7=y$$

Therefore, $f(x)$ is onto

Hence, f is an invertible function

The inverse of the function $f(x)$ is $\frac{x-7}{10}$

2. Let $f(x):W\to W$ be defined as $f(n)=n-1$, if is odd and $f(n)=n+1$, if n is even. Show that $f(x)$ is invertible. Find the inverse of $f(x)$. Here, W is the set of all whole numbers.

Solution: Given that $f:W\to W$ is defined as $f(n)=\begin{cases} n-1 & \text{if n is odd} \\ n+1 & \text{if n is even} \end{cases}$

Check whether the function is one to one or not.

Suppose that $f(n)=f(m)$, where n,m are whole numbers.

If n is odd and m is even, then we will have $n-1=m+1$, it implies that $n-m=2$

This is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored

Therefore, both n and m must be either odd or even.

Now, if both n and m are odd.

$$\begin{aligned} f(n) &= f(m) \\ n-1 &= m-1 \\ n &= m \end{aligned}$$

If both n and m are even,

$$\begin{aligned} f(n) &= f(m) \\ n+1 &= m+1 \\ n &= m \end{aligned}$$

Therefore f is one – one.

To check whether the function is onto or not.

For any odd number $2r+1$ in co domain N is the image of $2r$ in domain N and any even number

For any even number $2r$ in codomain N is the image of $2r+1$in domain N

Hence, $f(x)$ is onto.

Therefore, the function $f(x)$ is invertible

Define $g:W \to W$ as $g(m)=\begin{cases} m+1 \text{ if } m \text{ is even} \\ m-1 \text{ if } m \text{ is odd} \end{cases}$

Suppose that n is odd number

$$\begin{aligned} gof(n) &= g(f(n)) \\ &= g(n-1) \\ &= n-1+1 \\ &= n \end{aligned}$$

Suppose n is even

$$\begin{aligned} gof(n) &= g(f(n)) \\ &= g(n+1) \\ &= n+1-1 \\ &= n \end{aligned}$$

When m is odd

$$\begin{aligned} fog(m) &= f(g(m)) \\ &= f(m-1) \\ &= m-1+1 \\ &= m \end{aligned}$$

When m is even

$$\begin{aligned} fog(m) &= f(g(m)) \\ &= f(m+1) \\ &= m+1-1 \\ &= m \end{aligned}$$

Hence, $g \circ f = I$, and $fog = I$

Therefore, the function $f(x)$ is invertible and the inverse of $f(x)$ is given by $g(x)$, it is same as $f(x)$

3. If $f:R \to R$ is defined by $f(x)=x^2-3x+2$, find $f(f(x))$

Solution: It is given that $f:R\to R$ is defined as $f(x)=x^2-3x+2$

$$\begin{aligned}f(f(x)) &= f(x^2-3x+2)\\ &=(x^2-3x+2)^2-3(x^2-3x+2)+2\\ &=(x^4+9x^2+4-6x^3-12x+4x^2)+(-3x^2+9x-6)+2\\ &=x^4-6x^3+10x^2-3x\end{aligned}$$

Therefore, $f\circ f(x)=x^4-6x^3+10x^2-3x$

4. Show that function $f:R\to\{x\in R:-1<x<1\}$ defined by $f(x)=\dfrac{x}{1+|x|},x\in R$ is one-one and onto function

Solution: Given that $f:R\to\{x\in R:-1<x<1\}$ is defined as $f(x)=\dfrac{x}{1+|x|},x\in R$

Check the function $f(x)$ is one to one or not.

Suppose that $f(x)=f(y)$, where $x,y\in R$

It implies that $\dfrac{x}{1+|x|}=\dfrac{y}{1+|y|}$

If x is positive and y is negative,

$$\frac{x}{1+x}=\frac{y}{1-y}\Rightarrow x-xy=y+xy\Rightarrow 2xy=x-y$$

Since, x is positive and y is negative, $2xy$ is negative and $x-y$ is positive

Hence the condition $2xy=x-y$ is false

If x is positive, and y be negative can be ruled out.

Suppose that both x and y are positive

$$f(x)=f(y)$$
$$\frac{x}{1+x}=\frac{y}{1+y}$$
$$x+xy=y+xy$$
$$x=y$$

When x and y both are negative,

$$f(x)=f(y)$$
$$\frac{x}{1-x}=\frac{y}{1-y}$$
$$x-xy=y-xy$$
$$x=y$$

Therefore, $f(x)$ is one to one.

Check the function $f(x)$ is onto or not.

Suppose that $y\in R$ and y is negative real number

If y is negative, then, there exists $x=\frac{y}{1+y}$ such that

$$f(x)=f\left(\frac{y}{1+y}\right)=\frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|}=\frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)}=\frac{y}{1+y-y}=y$$

If y is positive, then, there exists $x=\frac{y}{1-y}\in R$ such that

$$f(x)=f\left(\frac{y}{1-y}\right)=\frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|}=\frac{\frac{y}{1-y}}{1+\left(\frac{-y}{1-y}\right)}=\frac{y}{1-y+y}=y$$

Therefore, the function $f(x)$ is onto.

5. Show that the function $f:R\to R$ given by $f(x)=x^3$ is injective

Solution: Given that the function $f: R \to R$ is defined as $f(x) = x^3$

Check the whether the function is one to one

Suppose that $f(x) = f(y)$, where $x, y \in R$

It implies that $x^3 = y^3 \Rightarrow x = y$

Hence, the function is one to one.

6. Given examples of two function $f: N \to Z$ and $g: Z \to Z$ such that gof is injective but $g(x)$ is not injective

Solution: Given that two functions $f: N \to Z$ and $g: Z \to Z$ such that $f(x) = x$ and $g(x) = |x|$

Since the function $f(x)$ is identity function it is one to one

But the function $g(x)$ is not one to one because for both $-1, 1$ has same image 1

Consider the compound function $gof(x)$ is defined as

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(x) \\ &= |x| \end{aligned}$$

Here the function $gof(x)$ is defined on the set of natural numbers

Hence the function $gof(x)$ is one to one

7. Given examples of two functions $f: N \to N$ and $g: N \to N$ such that $gof(x)$ is onto but $f(x)$ is not onto

Solution: Given $f: N \to N$ by $f(x) = x+1$ and $g: N \to N$ is defined as

$$g(x) = \begin{cases} x-1, \text{ if } x > 1 \\ 1, \quad \text{ if } x = 1 \end{cases}$$

We first show that $g(x)$ is not onto.

Consider element 1 in co-domain N . this element is not an image of any of the elements

in domain N .

Therefore, $f(x)$ is not onto.

Consider the function $g \circ f(x)$ is defined on the set of natural numbers

Such that

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(x+1) \\ &= x+1-1 \\ &= x \end{aligned}$$

For $y \in N$ there exists $x = y \in N$ such that $gof(x) = y$

Therefore, $gof(x)$ is onto.

8. Given a non-empty set X, consider $P(X)$ which the set of all subsets is of X .

Define the relation R in $P(X)$ is as follows:

For subsets A, B in $P(X)$, ARB if and only if A $\subset$ B. Is R an equivalence relation

on $P(X)$?

Justify you answer.

Solution: For every set is a subset it self

Hence the for any set A, it is subset itself, hence ARA for all $A \in P(X)$

Therefore, the relation R is reflexive.

Suppose that $ARB \Rightarrow A \subset B$, it does not implies that $B \subset A$

Hence ARB does not implies that BRA

Therefore, the relation R is not symmetric

Suppose that ARB, BRC it implies that $A \subset B$ and $B \subset C$

It gives $A \subset C$, it means ARC

Therefore, the relation R is transitive

Hence, the relation R is not an equivalence relation as it is not symmetric

9. Find the number of all onto functions from the set {1, 2, 3, … , n) to itself.

Solution:

We know that Onto functions from the set $\{1,2,3,4,...,n\}$ to itself is simply a permutation on n symbols

Thus, the total number of onto maps from {1, 2, … , n} to itself is the same as the total number of permutations on n symbols 1, 2, …, n, which is n.

10. Let $S=\{a,b,c\}$ and $T=\{1,2,3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

i) $F=\{(a,3),(b,2),(c,1)\}$

ii) $F=\{(a,2),(b,1),(c,1)\}$

Solution: Let $S=\{a,b,c\}$ and $T=\{1,2,3\}$

i) Consider the function $F:S \to T$ defined as F: S → T is defined as $F=\{(a,3),(b,2),(c,1)\}$

$$F(a)=3, F(b)=2, F(c)=1$$

Therefore, $F^{-1}=\{(3,a),(2,b),(1,c)\}$

ii) Consider the function $F:S \to T$ is defined as $F=\{(a,2),(b,1),(c,1)\}$

Since $F(b)=F(c)=1$, the function F is not one – one.

Hence, F is not invertible

Therefore, F^{-1} does not exist.

11. Let $A=\{-1,0,1,2\}$ and $B=\{-4,-2,0,2\}$ are any two sets. Two functions $f(x), g(x)$ are defined from A to B as $f(x)=x^2-x, x\in A$, $g(x)=2\left|x-\frac{1}{2}\right|-1, x\in A$.

Are $f(x), g(x)$ will be equal. Justify your answer.

Solution: Given that $f(x)=x^2-x, x\in A$, $g(x)=2\left|x-\frac{1}{2}\right|-1, x\in A$

The roster form of the function $f(x)$ is $f(x)=\{(-1,2),(0,0),(1,0),(2,2)\}$

The roster form of the function $g(x)$ is $g(x)=\{(-1,2),(0,0),(1,0),(2,2)\}$

Hence the above two functions are equal

12. Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

A) 1 B) 2 C) 3 D) 4

Solution: Given that $A=\{1,2,3\}$

The smallest relation containing $(1,2)$ and $(1,3)$ which is reflexive and symmetric but not transitive relation is $R=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(3,1)\}$

Because relation R is reflexive as $(1,1),(2,2),(3,3)\in R$

Relation R is symmetric since ($(1,2),(2,1)\in R \Rightarrow (2,1),(3,1)\in R$

Relation R is not transitive as , but $(3,2)\notin R$.

If we add any two pairs $(3,2),(2,3)$ or both to relation R, then the relation becomes transitive also, so that the number of required relations is only one

This is matching with the option (A)

13. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

A) 1 B) 2 C) 3 D) 4

Solution: It is given that $A=\{1,2,3\}$.

The smallest equivalence relation containing $(1,2)$ is given by,

$$R_1 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$$

The remaining ordered pair are $(2,3),(3,2),(1,3),(3,1)$

Suppose that R_2 is another relation containing all the ordered pairs of R_1 and add $(2,3)$

To make R_2 is equivalence relation, for symmetry we must add $(3,2)$, for transitivity we have to add $(1,3)$ and $(3,1)$

Hence, there are two relations which are equivalence relations having $(1,2)$

This is matching with the option (B)

14. Let $f : R \to R$ be the signum function defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 1 \end{cases}$ and $g : R \to R$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then does $f \circ g$ and $g \circ f$ coincide in $(0,1]$?

Solution: Given that $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 1 \end{cases}$

And another function $g : R \to R$ is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.

Let $x \in (0,1]$, then $[x] = 1$, when $x = 1$, and $[x] = 0$, when $0 < x < 1$,

Consider the compound functions

$$
\begin{aligned}
fog(x) &= f(g(x)) \\
&= f([x]) \\
&= \begin{cases} f(1), & if \;\; x=1 \\ f(0), & if \;\; x \in (0,1) \end{cases} \\
&= \begin{cases} 1, & if \;\; x=1 \\ 0, & if \;\; x \in (0,1) \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
go(x) &= g(f(x)) \\
&= g(1) \qquad [as\, x > 0]
\end{aligned}
$$

When $x \in (0,1)$, we have $fog(x)=0$ and $gof(x)=1$

It implies $f \circ g(x) \neq g \circ f(x)$