Chapter: 1. Relations and functions

Exercise 1.2

1. Show that the function $f: R_* \to R_*$ defined by $f(x)=\frac{1}{x}$ is one-one and onto, where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_*?

Solution: Consider the function $f: R_* \to R_*$ defined by $f(x)=\frac{1}{x}$

The function is one to one if and only if $x, y \in R_*$ such that $f(x)=f(y)$

It implies that $\frac{1}{x}=\frac{1}{y} \Rightarrow x=y$

Hence, $f(x)$ is one to one function.

For any $y \in R_*$ there exists $x \in R_*$ such that $x=\frac{1}{y} \in R_*$ and $f(x)=\frac{1}{\left(\frac{1}{y}\right)}=y$

Therefore, the function is both one to one and onto.

If the function $g(x): N \to R$ is defined as $g(x)=\frac{1}{x}$ then it is one to one but not onto

Because there is preimage for the value $2 \in R_*$ in the set of natural numbers

2. Check the infectivity and surjectivity of the following functions

a. $f: N \to N$ Given by $f(x)=x^2$

b. $f: Z \to Z$ Given by $f(x)=x^2$

c. $f: R \to R$ Given by $f(x)=x^2$

d. $f: N \to N$ Given by $f(x)=x^3$

e. $f: Z \to Z$ Given by $f(x)=x^3$

Solution: To check the function $f(x)$ is one to one, suppose that $f(x_1)=f(x_2)$ and show that $x_1 = x_2$ for all x_1, x_2 belongs to domain, To check the function $f(x)$ is onto, show that for every element in the codomain there exists preimage in the domain

a) The given function is $f(x) = x^2$ and domain and range both are equal to set of natural numbers.

The function is one to one because $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$

The function is not onto. Because there is no preimage for 2 in the set of natural numbers.

b) The given function is $f(x) = x^2$ and domain and range both are equal to set of all integers.

The function is not one to one because $f(2) = f(-2)$ but $2 \neq -2$

The function is not onto. Because there is no preimage for 2 in the set of natural numbers.

c) The given function is $f(x) = x^2$ and domain and range both are equal to set of all Real numbers.

The function is not one to one because $f(2) = f(-2)$ but $2 \neq -2$

The function is not onto. Because there is no square root for negative real numbers.

d) The given function is $f(x) = x^3$ and domain and range both are equal to set of natural numbers.

The function is one to one because $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

The function is not onto. Because there is no preimage for 2 in the set of natural numbers.

e) The given function is $f(x) = x^3$ and domain and range both are equal to set of all integers.

The function is one to one because $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

The function is not onto. Because there is no preimage for 2 in the set of integers

3. Prove that the greatest Integer function f: R → R given by f(x) = [x], is neither one – one nor onto, where [x] denotes the greatest integer less than or equal to x

Solution: Given that $f: R \to R$ and is defined by $f(x) = [x]$

This is not one to one function because $f(1.2) = 1, f(1.3) = 1$

This is not onto function because there is no pre image to any non - integer values

Therefore, the function is neither one to one nor onto.

4. Show that the Modulus Function $f: R \to R$ given by $f(x) = |x|$, is neither one – one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Solution: The function $f: R \to R$ is defined as $f(x) = |x|$

The function is not one to one because $f(1) = f(-1)$

There is no preimage to any negative real number in the real numbers

Therefore, the function $f(x) = |x|$ is neither one to one nor onto in the set of real numbers.

5. Show that the Signum function $f: R \to R$, given by $f(x) = \begin{cases} 1, & if \quad x > 0 \\ 0, & \text{if} \quad x = 0 \\ -1, & if \quad x < 0 \end{cases}$ is neither one-one nor onto

Solution: The given function $f(x) = \begin{cases} 1, & if \quad x > 0 \\ 0, & \text{if} \quad x = 0 \\ -1, & if \quad x < 0 \end{cases}$ is many to one function

For all positive real numbers, the functional values are equal

So that the function is not one to one

The function is not onto because there is no pre image to 2

Therefore, the signum function is neither one to one nor onto.

6. Let $A=\{1,2,3\}$, $B=\{4,5,6,7\}$ and let $f=\{(1,4),(2,5),(3,6)\}$ be a function from A to B. Show that f is one – one.

Solution:

The function is defined as $f=\{(1,4),(2,5),(3,6)\}$

The first elements of ordered pairs in the function are different. So that the function is one to one.

7. In each of the following cases, state whether the function is one – one, onto or bijective. Justify your answer.

(i) $f:R\to R$ Defined by $f(x)=3-4x$

(ii) $f:R\to R$ Defined by $f(x)=1+x^2$

Solution:

(i) The given function is $f:R\to R$ Defined by $f(x)=3-4x$

Suppose that $f(x_1)=f(x_2)$

It implies that

$$\begin{aligned}3-4x_1&=3-4x_2\\-4x_1&=-4x_2\\x_1&=x_2\end{aligned}$$

Therefore, the function is one to one.

For any real number y in the set of all real numbers, there exists $\frac{3-y}{4}$ in the real numbers such that $f\left(\frac{3-y}{4}\right)=3-4\left(\frac{3-y}{4}\right)=y$

Hence, the function is onto.

Therefore, the function is both one to one and onto, it means it is bijective.

ii) Given that the function $f:R\to R$ is defined as $f(x)=1+x^2$

Suppose that $f(x_1)=f(x_2)$

It implies that

$$\begin{aligned}1+x_1^{\;2}&=1+x_2^{\;2}\\ x_1^{\;2}&=x_2^{\;2}\\ x_1&=\pm x_2\end{aligned}$$

Hence, the function is not one to one.

There is no preimage for the real number 2 , so that the function is not onto.

Therefore the function is neither one to one nor onto.

8. Let A and B be sets. Show that $f:A\times B\to B\times A$ such that $(a,b)=(b,a)$ is bijective function.

Solution: Given A and B be sets and the function defined as $f:A\times B\to B\times A$,

$$f(a,b)=(b,a)$$

Suppose that $f(a_1,b_1)=f(a_2,b_2)$, it implies that $(b_1,a_1)=(b_2,a_2)$

Hence, $(a_1,b_1)=(a_2,b_2)$, the function is one to one.

For any ordered pair $(b,a)\in B\times A$, there exists $(a,b)\in A\times B$ such that $f(a,b)=(b,a)$

Therefore, the function f is both one to one and onto.

9. Let $f:N\to N$ be defined by $f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n\in N$. State whether the function f is bijective. Justify your answer

Solution: The given function $f:N\to N$ is defined as

$$f(n)=\begin{cases}\frac{n+1}{2}, & \text{if } n \text{ is odd}\\ \frac{n}{2}, & \text{if } n \text{ is even}\end{cases} \quad \text{For all } n\in N$$

The function is not one to one because $f(1)=f(2)$

$$f(1)=\frac{1+1}{2}=1 \text{ and } f(2)=\frac{2}{2}=1$$

Hence, the function is not one to one.

Consider a natural number n in co-domain N

Suppose that n is odd, the value of n is in the form of $n=2r+1$, there exists $4r+1\in N$ such that $f(4r+1)=\frac{4r+1+1}{2}=2r+1$

Suppose that n is even, the value of n is in the form of $n=2r$, for some $r\in N$, there exists $4r\in N$ such that $f(4r)=\frac{4r}{2}=2r$

Therefore, $f(x)$ is onto.

Hence, the function is not bijective function.

10. Let $A=R-\{3\}$ and $B=R-\{1\}$. Consider the function $f:A\to B$ defined by $f(x)=\left(\frac{x-2}{x-3}\right)$. Is f ono-one and onto? Justify your answer

Solution: The given function $f:A\to B$ is defined as $f(x)=\left(\frac{x-2}{x-3}\right)$

where $A=R-\{3\}, B=R-\{1\}$

Suppose that $x,y\in A$ such that $f(x)=f(y)$

Hence, $\frac{x-2}{x-3}=\frac{y-2}{y-3}$

Cross multiply

$$\begin{aligned} xy-3x-2y+6 &= xy-2x-3y+6 \\ -3x-2y &= -2x-3y \\ -x &= -y \\ x &= y \end{aligned}$$

Hence, $f(x)=f(y) \Rightarrow x=y$

Therefore, the function $f(x)=\left(\dfrac{x-2}{x-3}\right)$ is one to one.

Suppose that $y \in B = R-\{1\}$, it means y is a real number other than 1.

The function $f(x)$ is onto if there exists $x \in A$ such that $f(x)=y$

It implies $\dfrac{x-2}{x-3}=y$

Simplify

$$\begin{aligned} x-2 &= xy-3y \\ xy-x &= 3y-2 \\ x(y-1) &= 3y-2 \\ x &= \frac{3y-2}{y-1} \end{aligned}$$

For any value of $y \neq 1$, there exists a real number $x=\dfrac{3y-2}{y-1}$ such that $f(x)=y$

Hence, the function is one to one and onto.

Therefore, the function is bijective function.

11. Let $f: R \to R$ be defined as $f(x)=x^4$. Choose the correct answer

(A) $f(x)$ is one-one onto

(B) $f(x)$ is many-one onto

(C) $f(x)$ is one-one but not onto

(D) $f(x)$ is neither one-one nor onto

Solution: The given function is $f(x)=x^4$ defined on the set of real numbers

Observing that $f(-2)=f(2)$ does not implies that $-2=2$

Hence, the function is not one to one

And there is no real number whose fourth root of a negative number

So that the function is not onto.

Hence, the function is neither one to one nor onto.

This is matching with the option (D)

12. Let $f:R\to R$ be defined as $f(x)=3x$. Choose the correct answer.

A) $f(x)$ is one to one and onto

B) $f(x)$ is many to one

C) $f(x)$ is one to one but not onto

D) $f(x)$ is Neither one to one nor onto.

Solution: Consider the function $f(x)=3x$ defined on the real numbers.

Suppose that $f(x)=f(y)$

It implies that $3x=3y\Rightarrow x=y$

It concludes that the function $f(x)=3x$ is one to one.

For any real number y in co-domain R, there exists $\frac{y}{3}$ in R such that

$$f\left(\frac{y}{3}\right)=3\left(\frac{y}{3}\right)=y$$

Hence, the function is onto

Hence, the function is both one to one and onto

Therefore, this is matching with the option (A)