Chapter: 1. Relations and functions

Exercise 1.3

1. Let $f:\{1,3,4\}\to\{1,2,5\}$ and the function $g:\{1,2,5\}\to\{1,3\}$ be given by $f=\{(1,2),(3,5),(4,1)\}$ and $g=\{(1,3),(2,3),(5,1)\}$ write down the $g\circ f$

Solution: Given that the functions $f:\{1,3,4\}\to\{1,2,5\}$ and $g:\{1,2,5\}\to\{1,3\}$ defined as

$$f=\{(1,2),(3,5),(4,1)\} \text{ And } g=\{(1,3),(2,3),(5,1)\}$$

The function $g\circ f$ is defined from the domain of the function f to the range of the function g

It means $g\circ f:\{1,3,4\}\to\{1,3\}$

Therefore, $g\circ f:\{(1,3),(3,1),(4,3)\}$

2. Let $f:R\to R, g:R\to R$ and $h:R\to R$ are any three functions. Show that $(f+g)\circ h=f\circ h+g\circ h$ and $(f\cdot g)\circ h=(f\circ h)\cdot(g\circ h)$

Solution: The compound function is defined as $f\circ g(x)=f(g(x))$

We want to prove that $(f+g)\circ h=f\circ h+g\circ h$

Consider the left hand side of the relation $(f+g)\circ h=f\circ h+g\circ h$

$$\begin{aligned}(f+g)\circ h&=(f+g)(h(x))\\&=f(h(x))+g(h(x))\\&=f\circ h(x)+g\circ h(x)\end{aligned}$$

Therefore, $(f+g)\circ h=f\circ h+g\circ h$

We want to prove that $(f\cdot g)\circ h=(f\circ h)\cdot(g\circ h)$

Consider the left hand side of the relation $(f\cdot g)\circ h=(f\circ h)\cdot(g\circ h)$

$$\begin{aligned}(f\cdot g)\circ h &= (f\cdot g)(h(x))\\ &= f(h(x))\cdot g(h(x))\\ &= f\circ h(x)\cdot g\circ h(x)\end{aligned}$$

Therefore, $(f\cdot g)\circ h=(f\circ h)\cdot(g\circ h)$

3. Find gof and fog, if

a. $f(x)=|x|$ and $g(x)=|5x-2|$

b. $f(x)=8x^3$ and $g(x)=x^{\frac{1}{3}}$

Solution:

a) The given functions are $f(x)=|x|$ and $g(x)=|5x-2|$

Finding $g\circ f(x)$

$$g\circ f(x)=g(f(x))=g(|x|)=|5|x|-2|$$

Finding $f\circ g(x)$

$$f\circ g(x)=f(g(x))=f(|5x-2|)=|5x-2|$$

Therefore, $g\circ f(x)=|5|x|-2|,\ f\circ g(x)=|5x-2|$

b) The given functions are $f(x)=8x^3$ and $g(x)=x^{\frac{1}{3}}$

Finding $g\circ f(x)$

$$g\circ f(x)=g(f(x))=g(8x^3)=(8x^3)^{\frac{1}{3}}=2x$$

Finding $f\circ g(x)$

$$f\circ g(x)=f(g(x))=f\left(x^{\frac{1}{3}}\right)=8\left(x^{\frac{1}{3}}\right)^3=8x$$

Therefore, $g\circ f(x)=2x,\ f\circ g(x)=8x$

4. If $f(x)=\frac{4x+3}{6x-4}, x\neq\frac{2}{3}$, then show that $fof(x)=x$, for all $x\neq\frac{2}{3}$. And then find the inverse of the function $f(x)$?

Solution:

Given that the function is $f(x)=\frac{4x+3}{6x-4}, x\neq\frac{2}{3}$

Consider the compound function $f\circ f(x)$

$$f\circ f(x)=f(f(x))$$
$$=f\left(\frac{4x+3}{6x-4}\right)$$

Again apply the function

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

Take the denominator common and cancel out the common denomiantor

$$=\frac{16x+12+18x-12}{24x+18-24x+16}$$
$$=\frac{34x}{34}$$
$$=x$$

Therefore, $f\circ f(x)=x$

Since $f\circ f(x)=x$, the inverse of the function $f(x)$ is it self

Therefore the inverse of the function $f(x)$ is $f(x)$

5. State with reason whether following functions have inverse

(i) A function $f:\{1,2,3,4\}\to\{10\}$ defined as $f=\{(1,10),(2,10),(3,10),(4,10)\}$

(ii) A function $g:\{5,6,7,8\}\to\{1,2,3,4\}$ defined as

$g=\{(5,4),(6,3),(7,4),(8,2)\}$

(iii) A function $h:\{2,3,4,5\}\to\{7,9,11,13\}$ defined as

$h=\{(2,7),(3,9),(4,11),(5,13)\}$

Solution:

i) The function $f:\{1,2,3,4\}\to\{10\}$ is defined as

$$f=\{(1,10),(2,10),(3,10),(4,10)\}$$

The function is not one to one, so it is not bijective

Hence the function is not invertible.

ii) A function $g:\{5,6,7,8\}\to\{1,2,3,4\}$ defined as $g=\{(5,4),(6,3),(7,4),(8,2)\}$

The function $g(x)$ is not one to one because $g(5)=g(7)=4$

Hence, the function is not invertible

iii) A function $h:\{2,3,4,5\}\to\{7,9,11,13\}$ defined as

$h=\{(2,7),(3,9),(4,11),(5,13)\}$

The function $h(x)$ is one to one because each element of the domain has unique image and all the elements of codomain of the function has preimages

Hence the function is onto

Therefore, the function is invertible.

6. Show that $f:[-1,1]\to R$, given by $f(x)=\dfrac{x}{x+2}$ is one – one. Find the inverse of the function $f:[-1,1]\to$ Range of f.

Solution: The given function $f:[-1,1]\to R$ is defined as $f(x)=\dfrac{x}{x+2}$

Suppose that $f(x)=f(y)$. It implies that $\dfrac{x}{x+2}=\dfrac{y}{y+2}$

Cross multiply

$$\begin{aligned} xy+2x&=xy+2y \\ 2x&=2y \\ x&=y \end{aligned}$$

Hence, the function is one to one.

Suppose that $f(x)=y$

$$\begin{aligned}\frac{x}{x+2}&=y\\ x&=xy+2y\\ x(1-y)&=2y\\ x&=\frac{2y}{1-y}\end{aligned}$$

Suppose that $g(x)=\frac{2x}{1-x}$

Consider $f\circ g(x)$

$$f\circ g(x)=f\left(\frac{2x}{1-x}\right)=\frac{\frac{2x}{1-x}}{\frac{2x}{1-x}+2}=\frac{\frac{2x}{1-x}}{\frac{2x+2-2x}{1-x}}=\frac{2x}{2}=x$$

And $g\circ f(x)=x$, hence $f(x),g(x)$ are inverses to each other

The inverse of the function $f(x)=\frac{x}{x+2}$ is $g(x)=\frac{2x}{1-x}$

7. Consider $f:R\to R$ given by $f(x)=4x+3$. Show that $f(x)$ is invertible. Find the inverse of $f(x)$

Solution:

The given function $f:R\to R$ is given by $f(x)=4x+3$

Suppose that $f(x)=f(y)$

$$\begin{aligned}f(x)&=f(y)\\ 4x+3&=4y+3\\ 4x&=4y\\ x&=y\end{aligned}$$

Hence, the function is one to one

Suppose that $y \in R$, and let $y = 4x + 3$

It implies $x = \frac{y-3}{4} \in R$

Hence, $f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y$

Therefore $f(x)$ is onto

Since the function is bijective function, f^{-1} exists

The function $g : \mathrm{R} \to \mathrm{R}$ defined as $g(x) = \frac{x-3}{4}$

Consider the compound function $(gof)(x)$

$$g \circ f(x) = g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = \frac{4x}{4} = x$$

And consider the compound function

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = y$$

Hence, $f \circ g(x) = g \circ f(x) = x$, so that $f(x), g(x)$ are inverse functions to each

Other

Therefore, the function $f(x)$ invertible and the inverse of function $f(x)$ is given by

$f^{-1}(x) = g(x) = \frac{x-3}{4}$

8. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that $f(x)$ is invertible with the inverse f^{-1} given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Solution:

The given function $f:R_{+}\to[4,\infty)$ is defined as $f(x)=x^2+4$

Check the function $f(x)=x^2+4$ is “one to one”

Suppose that $f(x)=f(y)$

It implies that

$$x^2+4=y^2+4\Rightarrow x^2=y^2$$

Function is defined on the set of all positive real numbers, hence $x^2=y^2\Rightarrow x=y$

Therefore, the function $f(x)=x^2+4$ is one to one.

Check the function $f(x)=x^2+4$ is “onto”

Suppose that $y\in[4,\infty)$ and suppose that $y=x^2+4$

It implies that $x^2=y-4\geq 0\Rightarrow x=\sqrt{y-4}$

Hence,

$$\begin{aligned} f(x)&=f\left(\sqrt{x-4}\right)\\ &=\left(\sqrt{x-4}\right)^2+4\\ &=x-4+4\\ &=x \end{aligned}$$

Therefore, the function $f(x)$ is one to one and onto.

Since the function $f(x)$ is bijective, it is invertible

The function $g(x)$ is $g:[4,\infty)\to R_{+}$ defined as $g(x)=\sqrt{x-4}$

The composite function

$$
\begin{aligned}
g \circ f(x) &= g(f(x)) \\
&= g(x^2+4) \\
&= \sqrt{(x^2+4)-4} \\
&= \sqrt{x^2} \\
&= x
\end{aligned}
$$

The composite function

$$
\begin{aligned}
f \circ g(x) &= f(g(x)) \\
&= f\left(\sqrt{x-4}\right) \\
&= \left(\sqrt{x-4}\right)^2 + 4 \\
&= x-4+4 \\
&= x
\end{aligned}
$$

Therefore, $gof(x) = fog(x) = I_R$

Hence, the function $f(x)$ is invertible and its inverse is $f^{-1}(x) = \sqrt{x-4}$

9. Consider $f: R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that $f(x)$ is invertible with $f^{-1}(y) = \left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$

Solution:

Given that the function $f: R_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$

Let y be an arbitrary element of $[-5, \infty)$

$$
\begin{aligned}
y &= 9x^2 + 6x - 5 \\
y &= (3x+1)^2 - 1 - 5 \\
&= (3x+1)^2 - 6 \\
y + 6 &= (3x+1)^2 \\
3x + 1 &= \sqrt{y+6}
\end{aligned}
$$

It implies that $x=\left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$

Therefore, $f(x)$ is onto and range of the function $f(x)$ is $[-5,\infty)$

Consider the compound function

$(gof)(x)=g\left(9x^2+6x-5\right)=g\left((3x+1)^2-6\right)=\sqrt{(3x+1)^2-6+6}-1=x$

and $(fog)(y)=f(g(y))=\left(\sqrt{y+6}\right)^2-6=y+6-6=y$

Therefore, the inverse of the function $f^{-1}(y)=g(y)=\left(\frac{\sqrt{y+6}-1}{3}\right)$

10. Let $f:x\to y$ be an invertible function. Show that f has unique inverse

Solution:

Let $f:x\to y$ be an invertible function

Suppose f has two inverses $g_1(x)$ and $g_2(x)$

For all $y\in Y$

$$fog_1(y)=I_y(y)=fog_2(y)$$

It implies

$$f(g_1(y))=f(g_2(y))$$
$$g_1(y)=g_2(y)$$
$$g_1=g_2$$

11. Consider $f:\{1,2,3\}\to\{a,b,c\}$ given by $f(1)=a,\ f(2)=b\ \ f(3)=c$. Find f^{-1} and show that $\left(f^{-1}\right)^{-1}=f$

Solution: For the function $f:\{1,2,3\}\to\{a,b,c\}$ is given by $f(1)=a,\ f(2)=b$, and $f(3)=c$

If we define $g:\{a,b,c\}\to\{1,2,3\}$ as $g(a)=1, g(b)=2, g(c)=3$

We have

$$f\circ g(a)=f\left(g(a)\right)=f(1)=a$$
$$f\circ g(b)=f\left(g(b)\right)=f(2)=b$$
$$f\circ g(c)=f\left(g(c)\right)=f(3)=c$$

$$g\circ f(1)=g\left(f(1)\right)=g(a)=1$$
$$g\circ f(2)=g\left(f(2)\right)=g(b)=2$$
$$g\circ f(3)=g\left(f(3)\right)=g(c)=3$$

Both compound functions are identity functions

Hence $f(x),g(x)$ are inverses to each other.

Therefore, $f^{-1}(x)=g(x)$

Suppose that the inverse of $g(x)$ is $h(x)$

Hence,

$$\begin{aligned}h(x)&=\{(1,a),(2,b),(3,c)\}\\&=f(x)\end{aligned}$$

It implies that

$$\begin{aligned}\left(f^{-1}(x)\right)^{-1}&=\left(g(x)\right)^{-1}\\&=h(x)\\&=f(x)\end{aligned}$$

Therefore, $\left(f^{-1}(x)\right)^{-1}=f(x)$

12. Let $f:X\to Y$ be an invertible function. Show that the inverse of f^{-1} is $f(x)$, it means $\left(f^{-1}(x)\right)^{-1}=f(x)$

Solution:

Let $f: X \to Y$ be an invertible function, there exists a function $g: Y \to X$ such that $g \circ f(x) = f \circ g(x) = x$.

It implies that $f^{-1}(x) = g(x)$

We know that $f^{-1} \circ f(x) = x$

It implies that the inverse of $f^{-1}(x)$ is $f(x)$

Therefore, $\left(f^{-1}(x)\right)^{-1} = f(x)$

13. If $f: R \to R$ is given by $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$, then $fof(x)$ is

A) $\frac{1}{x^3}$

B) x^3

C) x

D) $\left(3 - x^3\right)$

Solution:

The function $f: R \to R$ is defined as $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$

$$
\begin{aligned}
fof(x) &= f\left(f(x)\right) \\
&= f\left(3 - x^3\right)^{\frac{1}{3}} \\
&= \left[3 - \left(\left(3 - x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}} \\
&= \left[3 - \left(3 - x^3\right)\right]^{\frac{1}{3}} \\
&= x
\end{aligned}
$$

This is matching with the option (C)

14. Let $f: R-\left\{-\frac{4}{3}\right\} \rightarrow R$ be a function as $f(x)=\frac{4x}{3x+4}$.The inverse of the function $f \rightarrow R-\left\{-\frac{4}{3}\right\}$ is given by

A) $g(y)=\frac{3y}{3-4y}$

B) $g(y)=\frac{4y}{4-3y}$

C) $g(y)=\frac{4y}{4+3y}$

D) $g(y)=\frac{3y}{4-3y}$

Solution:

Consider the function $f: R-\left\{-\frac{4}{3}\right\} \rightarrow R$ defined as $f(x)=\frac{4x}{3x+4}$

Suppose that y be an arbitrary element of range of $f(x)$, there exists $x \in R-\left\{-\frac{3}{4}\right\}$ such that $y=f(x)$

It implies that

$$\begin{aligned} y &= \frac{4x}{3x+4} \\ 3xy+4y &= 4x \\ x(3y-4) &= -4y \\ x &= \frac{4y}{4-3y} \end{aligned}$$

The composite function

$$gof(x)=g(f(x))=g\left(\frac{4x}{3x+4}\right)=\frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)}=x$$

and $fog(y) = f(g(y)) = \frac{16}{12y+16-12x} = y$

Therefore, $g \circ f(x) = x, f \circ g(y) = y$

Hence the functions $f(x), g(x)$ are invertible and $f^{-1}(x) = g(x)$

The inverse of $f(x)$ is the map $g : \text{Range of } f \rightarrow R - \left\{-\frac{4}{3}\right\}$, which is given by

$g(y) = \frac{4y}{4-3y}$

This is matching with the option (B)