Chapter: Vector Algebra.

Miscellaneous Exercise.

1. Write down a unit vector in XY-plane, making an angle of 30^0 with the positive direction of x-axis

Solution:

Unit vector is $\vec{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$, where θ is angle with positive X axis

$$\vec{r} = \cos 30^0\hat{i} + \sin 30^0\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

2. Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

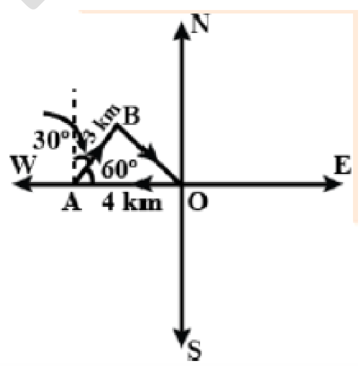
Solution:

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

3. A girl walks 4 km towards west, then she walks 3 km in a direction 30^0 east of north and stops. Determine the gril's displacement from her initial point of departure

Solution:



$$\overrightarrow{OA} = -4\hat{i}$$

$$\overrightarrow{AB}=\hat{i}\left|\overrightarrow{AB}\right|\cos 60^{0}+\hat{j}\left|\overrightarrow{AB}\right|\sin 60^{0}$$

$$=\hat{i}3\times\frac{1}{2}+\hat{j}3\times\frac{\sqrt{3}}{2}$$

$$=\frac{3}{2}\hat{i}+\frac{3\sqrt{3}}{2}\hat{j}$$

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$$

$$=\left(-4\hat{i}\right)+\left(\frac{3}{2}\hat{i}+\frac{3\sqrt{3}}{2}\hat{j}\right)$$

$$=\left(-4+\frac{3}{2}\right)\hat{i}+\frac{3\sqrt{3}}{2}\hat{j}$$

$$=\left(\frac{-8+3}{9}\right)\hat{i}+\frac{3\sqrt{3}}{9}\hat{j}$$

$$=\frac{-5}{2}\hat{i}+\frac{3\sqrt{3}}{9}\hat{j}$$

4. If $\vec{a}=\vec{b}+\vec{c}$, then is it true that $|\vec{a}|=|\vec{b}|+|\vec{c}|$? Justify your answer.

Solution:

In ΔABC, $\overrightarrow{CB}=\vec{a},\overrightarrow{CA}=\vec{b},\overrightarrow{AB}=\vec{c}$

$\vec{a}=\vec{b}+\vec{c}$, by triangle law of addition for vectors.

$|\vec{a}|<|\vec{b}|+|\vec{c}|$, by triangle inequality law of lengths

Hence, it's not true that $|\vec{a}|=|\vec{b}|+|\vec{c}|$

5. Find the value of x for which $x\left(\hat{i}+\hat{j}+\hat{k}\right)$ unit vectro

Solution:

$$\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right|=1$$

$$\Rightarrow \sqrt{x^2+x^2+x^2}=1$$

$$\Rightarrow \sqrt{3x^2}=1$$

$$\Rightarrow \sqrt{3}x=1$$

$$x=\pm\frac{1}{\sqrt{3}}$$

6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

Solution:

$$\vec{c}=\vec{a}+\vec{b}+\vec{b}=(2+1)\hat{i}+(3-2)\hat{j}+(-1+1)\hat{k}=3\hat{i}+\hat{j}$$

$$|\vec{c}|=\sqrt{3^2+1^2}=\sqrt{9+1}=\sqrt{10}$$

$$\therefore \vec{c}=\frac{\vec{c}}{|\vec{c}|}=\frac{(3\hat{i}+\hat{j})}{\sqrt{10}}$$

So, vector of magnitude 5 and parallel to the resultant of $\vec{a}$ and $\vec{b}$ is

$$\pm 5(\hat{c})=\pm 5\left(\frac{1}{\sqrt{10}}(3\hat{i}+\hat{j})\right)\pm\frac{3\sqrt{10}}{2}\hat{i}\pm\frac{\sqrt{10}}{2}\hat{j}$$

7. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$

Solution:

$$2\vec{a}-\vec{b}+3\vec{c}=2(\hat{i}+\hat{j}+\hat{k})-(2\hat{i}-\hat{j}+3\hat{k})+3(\hat{i}-2\hat{j}+\hat{k})$$

$$=2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}$$

$$=3\hat{i}-3\hat{j}+2\hat{k}$$

$$\left|2\vec{a}-\vec{b}+3\vec{c}\right|=\sqrt{3^2+(-3)^2+2^2}=\sqrt{9+9+4}=\sqrt{22}$$

Thus, required unit vector is $\dfrac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|}=\dfrac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}}=\dfrac{3}{\sqrt{22}}\hat{i}-\dfrac{3}{\sqrt{22}}\hat{j}+\dfrac{2}{\sqrt{22}}\hat{k}$

8. Show that the points $A(1,-2,-8), B(5,0,-2)$ and $C(11,3,7)$ are collinear, and find the ratio in which B divides AC

Solution:

$$\overrightarrow{AB}=(5-1)\hat{i}+(0+2)\hat{j}+(-2+8)\hat{k}=4\hat{i}+2\hat{j}+6\hat{k}$$

$$\overrightarrow{BC}=(11-5)\hat{i}+(3-0)\hat{j}+(7+2)\hat{k}=6\hat{i}+3\hat{j}+9\hat{k}$$

$$\overrightarrow{AC}=(11-1)\hat{i}+(3+2)\hat{j}+(7+8)\hat{k}=10\hat{i}+5\hat{j}+15\hat{k}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{4^2+2^2+6^2}=\sqrt{16+4+36}=\sqrt{56}=2\sqrt{14}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{6^2+3^2+9^2}=\sqrt{36+9+81}=\sqrt{126}=3\sqrt{14}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{10^2+5^2+15^2}=\sqrt{100+25+225}=\sqrt{350}=5\sqrt{14}$$

$$\therefore \left|\overrightarrow{AC}\right|=\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|$$

So, this points are collinear

Let B divides AC in the ratio $\lambda:1$. $\overrightarrow{OB}=\dfrac{\lambda\overrightarrow{OC}+\overrightarrow{OA}}{(\lambda+1)}$

$$\Rightarrow 5\hat{i}-2\hat{k}=\frac{\lambda\left(11\hat{i}+3\hat{j}+7\hat{k}\right)+\left(\hat{i}-2\hat{j}-8\hat{k}\right)}{\lambda+1}$$

$$\Rightarrow(\lambda+1)\left(5\hat{i}-2\hat{k}\right)=11\lambda\hat{i}+3\lambda\hat{j}+7\lambda\hat{k}+\hat{i}-2\hat{j}-8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i}-2(\lambda+1)\hat{k}=(11\lambda+1)\hat{i}+(3\lambda-2)\hat{j}+(7\lambda-8)\hat{k}$$

$\Rightarrow \lambda = \frac{2}{3}$

So, the required ratio is 2:3

9. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\left(2\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-3\vec{b}\right)$ externally in the ratio 1:2 Also, show that P is the mid point of the line segment RQ

Solution:

$\overrightarrow{OP} = 2\vec{a}+\vec{b}, \overrightarrow{OQ} = \vec{a}-3\vec{b}$

$$\overrightarrow{OR} = \frac{2\left(2\vec{a}+\vec{b}\right)-\left(\vec{a}-3\vec{b}\right)}{2-1} = \frac{4\vec{a}+2\vec{b}-\vec{a}-3\vec{b}}{1} = 3\vec{a}+5\vec{b}$$

So, the position vector of R is $3\vec{a}+5\vec{b}$

Position vector of midpoint of $RQ = \frac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$= \frac{\left(a\sqrt{6}\right)+\left(3\vec{a}+5\vec{b}\right)}{2}$$

$= 2\vec{a}+\vec{b}$

$= \overrightarrow{OP}$

Thus, P is midpoint of line segment RQ

10. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution:

Diagonal of a parallelogram is $\vec{a}+\vec{b}$

$\vec{a}+\vec{b}=(2+1)\hat{i}+(-4-2)\hat{j}+(5-3)\hat{k}=3\hat{i}-6\hat{j}+2\hat{k}$

So, the unit vector parallel to diagonal is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{3^2+(-6)^2+2^2}}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4}}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{7}=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -4 & 3\\ 1 & -2 & -3\end{vmatrix}$$

$$=\hat{i}(12+10)-\hat{j}(-6-5)+\hat{k}(-4+4)$$

$$=22\hat{i}+11\hat{j}$$

$$=11(\hat{i}+\hat{j})$$

$$\therefore |\vec{a}\times\vec{b}|=11\sqrt{2^2+1^2}=11\sqrt{5}$$

So, area of parallelogram is $11\sqrt{5}$ sq units

11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$.

Solution:

Let a vector be equally inclined to OX, OY and OZ at an angle α

So, the direction cosines of the vector are $\cos\alpha, \cos\alpha$ and $\cos\alpha$

$$\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$$

$$\Rightarrow 3\cos^2\alpha=1$$

$$\Rightarrow \cos\alpha=\frac{1}{\sqrt{3}}$$

So, the DCs of the vector are $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$

12. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d}=15$

Solution:

$$\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$$

$$\vec{d}.\vec{a}=0\Rightarrow d_1+4d_2+2d_3=0$$

$$\vec{d}.\vec{b}=0\Rightarrow 3d_1-2d_2+7d_3=0$$

$$\vec{c}.\vec{d}=15\Rightarrow 2d_1-d_2+4d_3=15$$

Solving these equations, we get $d_1=\frac{160}{3},d_2=-\frac{5}{3},d_3=-\frac{70}{3}$

$$\therefore\vec{d}=\frac{160}{3}\hat{i}+\frac{5}{3}\hat{j}+\frac{70}{3}\hat{k}=\frac{1}{3}\left(160\hat{i}+5\hat{j}+70\hat{k}\right)$$

13. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ

Solution:

$$\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

So, unit vector along $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$ is $\left(\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\right)$

$$\left(\hat{i}+\hat{j}+\hat{k}\right).\left(\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\right)=1$$

$$\Rightarrow\frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\Rightarrow \sqrt{\lambda^2+4\lambda+44}=\lambda+6$$

$$\Rightarrow \lambda^2+4\lambda+44=(\lambda+6)^2$$

$$\Rightarrow \lambda^2+4\lambda+44=\lambda^2+12\lambda+36$$

$$\Rightarrow 8\lambda=8$$

$$\Rightarrow \lambda=1$$

14. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined $\vec{a},\vec{b}$ and $\vec{c}$

Solution:

$$\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$$

$$|\vec{a}|=|\vec{b}|=|\vec{c}|$$

Let $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b},\vec{c}$ at angles $\theta_1,\theta_2,\theta_3$ respectively.

$$\cos\theta_1=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_2=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{\vec{a}.\vec{b}+\vec{b}.\vec{b}+\vec{c}.\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_3=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{\vec{a}.\vec{c}+\vec{b}.\vec{c}+\vec{c}.\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Since, $|\vec{a}|=|\vec{b}|=|\vec{c}|\Rightarrow \cos\theta_1=\cos\theta_2=\cos\theta_3$

So, $\theta_1=\theta_2=\theta_3$

15. Prove that, $(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$ if and only if $\vec{a},\vec{b}$ are perpendicular, givne $\vec{a}\neq\vec{0},\vec{b}\neq\vec{0}$

Solution:

$$(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$$

$$\Rightarrow \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=|\vec{a}|^2+|\vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2$$

$$\Rightarrow 2\vec{a}.\vec{b}=0$$

$$\Rightarrow \vec{a}.\vec{b}=0$$

So $\vec{a}$ and $\vec{b}$ are perpendicular

16. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\vec{b}\geq 0$ only when

A) $0<\theta<\frac{\pi}{2}$ B) $0\leq\theta\leq\frac{\pi}{2}$ C) $0<\theta<\pi$ D) $0\leq\theta\leq\pi$

Solution:

$$\therefore \vec{a}\vec{b}\geq 0$$

$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta\geq 0$$

$$\Rightarrow \cos\theta\geq 0 \quad \because \left[|\vec{a}|\geq 0 \text{ and } |\vec{b}|\geq 0\right]$$

$$0\leq\theta\leq\frac{\pi}{2}$$

$$\vec{a}.\vec{b}\geq 0 \text{ if } 0\leq\theta\leq\frac{\pi}{2}$$

So the right answer is B

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

A) $\theta=\frac{\pi}{4}$ B) $\theta=\frac{\pi}{3}$ C) $\theta=\frac{\pi}{2}$ D) $\theta=\frac{2\pi}{3}$

Solution:

$|\vec{a}|=|\vec{b}|=1$

$|\vec{a}+\vec{b}|=1$

$\Rightarrow(\vec{a}+\vec{b})(\vec{a}+\vec{b})=1$

$\Rightarrow\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=1$

$\Rightarrow|\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2=1$

$\Rightarrow1^2+2|\vec{a}||\vec{b}|\cos\theta+1^2=1$

$\Rightarrow1^2+21.1\cos\theta+1=1$

$\Rightarrow\cos\theta=-\frac{1}{2}$

$\Rightarrow\theta=\frac{2\pi}{3}$

So, $\vec{a}+\vec{b}$ is unit vector if $\theta=\frac{2\pi}{3}$

The correct answer is D

18. The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ is

A) 0 B) -1 C) 1 D) 3

Solution:

$\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$

$= \hat{i}.\hat{i} + \hat{j}.(-\hat{j}) + \hat{k}.\hat{k}$

$= 1 - 1 + 1$

$= 1$

The correct answer is C

19. If θ is the angel between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\,\vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to

A) 0 B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) n

Solution:

$|\vec{a}\vec{b}| = |\vec{a} \times \vec{b}|$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow \cos\theta = \sin\theta$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4}$

The correct answer is B