Chapter: Vector Algebra.

Exercise 10.2

1. Compute the magnitude of vectors: $\vec{a}=\hat{i}+\hat{j}+\hat{k};\vec{b}=2\hat{i}-7\hat{j}-3\hat{k}$ and

$$\vec{c}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$$

Solution: The magnitude of a vector $\vec{p}=x\hat{i}+y\hat{j}+z\hat{k}$ is $|\vec{p}|=\sqrt{x^2+y^2+z^2}$

Hence, the magnitude of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ is

$$|\vec{a}|=\sqrt{(1)^2+(1)^2+(1)^2}$$
$$=\sqrt{3}$$

The magnitude of the vector $\vec{b}=2\hat{i}-7\hat{j}-3\hat{k}$ is

$$|\vec{b}|=\sqrt{(2)^2+(-7)^2+(-3)^2}$$
$$=\sqrt{4+49+9}$$
$$=\sqrt{62}$$

The magnitude of the vector $\vec{c}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$ is

$$|\vec{c}|=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2-\left(\frac{1}{\sqrt{3}}\right)^2}$$
$$=\sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$$
$$=1$$

2. Write two different vectors having same magnitude:

Solution: Consider two vectors $\vec{a}=\left(\hat{i}+2\hat{j}-3\hat{k}\right)$ and $\vec{b}=\left(2\hat{i}-\hat{j}+3\hat{k}\right)$

The magnitude of a vector $\vec{p}=x\hat{i}+y\hat{j}+z\hat{k}$ is $|\vec{p}|=\sqrt{x^2+y^2+z^2}$

Hence, the magnitudes of the above vectors are

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2}$$
$$= \sqrt{1+4+9}$$
$$= \sqrt{14}$$

and

$$|\vec{b}| = \sqrt{2^2 + 1^2 + (-3)^2}$$
$$= \sqrt{4+1+9}$$
$$= \sqrt{14}$$

Hence, the magnitudes are not equal.

Answer may vary. Student can give any example.

3. Write two different vectors having same direction

Solution: Suppose that two vectors are $\vec{p} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$ and $\vec{q} = \left(2\hat{i} - 4\hat{j} + 6\hat{k}\right)$

The directions of two vectors are same if and only if their direction ratios are in proportion.

It means the components of vectors must be in proportion.

Here the components of the vectors are $\langle 1,-2,3\rangle, \langle 2,-4,6\rangle$, and these triads are direction ratios of the base lines of above vectors

Since the direction ratios are proportional to each other, so that the vectors $\vec{p} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$ and $\vec{q} = \left(2\hat{i} - 4\hat{j} + 6\hat{k}\right)$ are in the same direction.

Observing the vectors $\vec{q} = 2\vec{p}$

Hence, the above two vectors are examples which are not equal but in the same direction.

Answer may vary, student can give any other example too.

4. Find the value of x and y so that the vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ are equal

Solution: Two vectors are equal if each component of one vector must be equal to the corresponding component of the second vector.

It gives, $2\hat{i}+3\hat{j}=x\hat{i}+y\hat{j} \Rightarrow x=2, y=3$

5. Find the scalar and vector components of the vector with initial points $(2,1)$ and terminal point $(-5,7)$

Solution: Suppose that the points are $A(2,1)$ and $B(-5,7)$

Hence,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB}-\overrightarrow{OA}\\ &=(-5-2)\hat{i}+(7-1)\hat{j}\\ &=-7i+6j\end{aligned}$$

The scalar and vector components of the vector with initial points $(2,1)$ and terminal point $(-5,7)$ are $\langle -7,6\rangle$ and $-7\hat{i},6\hat{j}$

6. Find the sum of the vectors $\vec{a}=\hat{i}-2\hat{j}+\hat{k}, \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\bar{c}=\hat{i}-6\hat{j}-7\hat{k}$.

Solution: The sum of two vectors is defined as another vector whose components are equal to the sum of the corresponding components of the vectors

The given vectors are

$$\begin{aligned}\vec{a}&=\hat{i}-2\hat{j}+\hat{k},\\ \vec{b}&=-2\hat{i}+4\hat{j}+5\hat{k}\\ \bar{c}&=\hat{i}-6\hat{j}-7\hat{k}\end{aligned}$$

The sum of the above vectors is defined as

$$\vec{a}+\vec{b}+\vec{c}=\left(\hat{i}-2\hat{j}+\hat{k}\right)+\left(-2\hat{i}+4\hat{j}+5\hat{k}\right)+\left(\hat{i}-6\hat{j}-7\hat{k}\right)$$
$$=(1-2+1)\hat{i}+(-2+4-6)\hat{j}+(1+5-7)\hat{k}$$
$$=0\hat{i}-4\hat{j}-1\hat{k}$$
$$=-4\hat{j}-\hat{k}$$

Therefore, the required vector is $-4\hat{j}-\hat{k}$

7. Find the unit vector in the direction of the vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$

Solution: The unit vector in the direction of the vector $\vec{p}=x\hat{i}+y\hat{j}+z\hat{k}$ is $\frac{\vec{p}}{|\vec{p}|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

The magnitude of the given vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ is

$$|\vec{a}|=\sqrt{1^2+1^2+2^2}$$
$$=\sqrt{1+1+4}$$
$$=\sqrt{6}$$

The unit vector in the direction of $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ is

$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}$$
$$=\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}$$
$$=\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$$

8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$ where P and Q are two points $(1,2,3)$ and $(4,5,6)$ respectively

Solution: Given that $\overrightarrow{OP}=\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{OQ}=4\hat{i}+5\hat{j}+6\hat{k}$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$$
$$=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$
$$=3\hat{i}+3\hat{j}+3\hat{k}$$
$$=3\left(\hat{i}+\hat{j}+\hat{k}\right)$$

The unit vector in the direction of the vector $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$ is $\frac{\vec{p}}{|\vec{p}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

The magnitude of the vector is

$$\left|\overrightarrow{PQ}\right| = \sqrt{3^2 + 3^2 + 3^2}$$
$$= 3\sqrt{3}$$

Therefore, the unit vector is

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

9. For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$

Solution: The given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

The sum of two vectors is

$$\vec{a} + \vec{b} = (2-1)\hat{i} + (-1+1)\hat{j} - (2-1)\hat{k}$$
$$= 1\hat{i} + 0\hat{j} + 1\hat{k}$$
$$= \hat{i} + \hat{k}$$

$$\left|\vec{a} + \vec{b}\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

The unit vector in the direction of the vector $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$ is $\frac{\vec{p}}{|\vec{p}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

Therefore, the unit vector is

$$\frac{\left(\vec{a} + \vec{b}\right)}{\left|\vec{a} + \vec{b}\right|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

10. Find a vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units

Solution: The given vector is $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$

Magnitude of the above vector is

$$\begin{aligned}|\vec{a}| &= \sqrt{5^2+(-1)^2+2^2}\\ &= \sqrt{25+1+4}\\ &= \sqrt{30}\end{aligned}$$

The vector of magnitude k units in the direction of vector $\vec{p}=x\hat{i}+y\hat{j}+z\hat{k}$ is

$$\frac{k\vec{p}}{|\vec{p}|}=k\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right)$$

Therefore, the vector with magnitude 8 units in the direction of $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$ is $8\hat{a}$

$$\begin{aligned}8\hat{a} &= \frac{8\vec{a}}{|\vec{a}|}\\ &= \frac{8\left(5\hat{i}-\hat{j}+2\hat{k}\right)}{\sqrt{30}}\\ &= \frac{40}{\sqrt{30}}\hat{i}-\frac{8}{\sqrt{30}}\hat{j}+\frac{16}{\sqrt{30}}\hat{k}\end{aligned}$$

11. Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear

Solution: Suppose that the vectors are

$$\begin{aligned}\vec{a} &= 2\hat{i}-3\hat{j}+4\hat{k},\\ \vec{b} &= -4\hat{i}+6\hat{j}-8\hat{k}\end{aligned}$$

If two vectors $\vec{p},\vec{q}$ are collinear then $\vec{p}=k\vec{q}$ where k is constant.

Consider

$$\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}=-2\left(2\hat{i}-3\hat{j}+4\hat{k}\right)=-2\vec{a}$$

Therefore, the vectors $\vec{a},\vec{b}$ are collinear

12. Find the direction cosines of the vector $\hat{i}+2\hat{j}+3\hat{k}$

Solution:

Suppose that $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

The magnitude of the vector $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ is

$$\begin{aligned}|\vec{a}| &= \sqrt{1^2+2^2+3^2}\\ &= \sqrt{1+4+9}\\ &= \sqrt{14}\end{aligned}$$

So, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$

13. Find the direction cosines of the vector joining the point $A(1,2,-3)$ and $B(-1,-2,1)$ directed from A to B

Solution: The position vectors of the given points are

$$\overrightarrow{OA}=\hat{i}+2\hat{j}-3\hat{k},\ \overrightarrow{OB}=-\hat{i}-2\hat{j}+\hat{k}$$

Hence, the vector

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB}-\overrightarrow{OA}\\ &= -\hat{i}-2\hat{j}+\hat{k}-\left(\hat{i}+2\hat{j}-3\hat{k}\right)\\ &= -2\hat{i}-4\hat{j}+4\hat{k}\end{aligned}$$

The magnitude of the vector $\overrightarrow{AB}$ is

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= \sqrt{(-2)^2+(-4)^2+4^2}\\ &= \sqrt{4+16+16}\\ &= 6\end{aligned}$$

Therefore, the direction cosines of the vector $\overrightarrow{AB}$ are $\left\langle -\frac{2}{6},-\frac{4}{6},\frac{4}{6}\right\rangle = \left\langle -\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right\rangle$

14. Show that the vectors $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axes OX, OY and OZ

Solution: The given vector is $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and its magnitude is $|\vec{a}|=\sqrt{3}$

Hence, the direction cosines of the above vector are $\left\langle\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$

Suppose that the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ makes angles α,β,γ with coordinate axes then

$$\cos\alpha=\frac{1}{\sqrt{3}},\cos\beta=\frac{1}{\sqrt{3}},\cos\gamma=\frac{1}{\sqrt{3}}$$

It implies that all the angles are equal

Therefore, the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the coordinate axes

15. Find the position vector of a point R which divides the line joining two point P and Q whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively in the ratio $2:1$

i) Internally

ii) Externally

Solutions: The position vectors are $\overrightarrow{OP}=\hat{i}+2\hat{j}-\hat{k}$ and $\overrightarrow{OQ}=-\hat{i}+\hat{j}+\hat{k}$

i) Suppose that a point A divides the line segment joining P,Q in the ratio $2:1$ internally

$$\overrightarrow{OA}=\frac{m\vec{b}+n\vec{a}}{m+n}$$
$$=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1}$$
$$=\frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-\hat{k}\right)}{3}$$
$$=\frac{-i+4j+k}{3}$$

ii) Suppose that the point B divides the line segment joining the points P and Q in the ratio $2:1$ externally.

The position vector of B is

$$\overrightarrow{OB} = \frac{m\vec{b} - n\vec{a}}{m-n}$$
$$= \frac{2\left(-\hat{i} + \hat{j} + \hat{k}\right) - 1\left(\hat{i} + 2\hat{j} - \hat{k}\right)}{2-1}$$
$$= \frac{\left(-2\hat{i} + 2\hat{j} + 2\hat{k}\right) - \left(\hat{i} + 2\hat{j} - \hat{k}\right)}{1}$$
$$= \frac{-3i + 3k}{1}$$
$$= -3(i-k)$$

16. Find the position vector of midpoint of vector joining the points $P(2,3,4)$ and $Q(4,1,-2)$

Solution: Given that the position vectors are $\overrightarrow{OP} = 2i + 3j + 4k$ and $\overrightarrow{OQ} = 4i + j - 2k$

Suppose that the point R is midpoint of PQ then the position vector of R is

$$\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}$$
$$= \frac{\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) + \left(4\hat{i} + \hat{j} - 2\hat{k}\right)}{2}$$
$$= \frac{6i + 4j + 2k}{2}$$
$$= 3i + 2j + k$$