Chapter: Vector Algebra.

Exercise 10.3

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$

Solution: Given that $|\vec{a}|=\sqrt{3},|\vec{b}|=2,\vec{a}\cdot\vec{b}=\sqrt{6}$

Hence, $\sqrt{6}=\sqrt{3}\times 2\times\cos\theta$

It implies that

$$\cos\theta=\frac{\sqrt{6}}{\sqrt{3}\times 2}$$
$$=\frac{1}{\sqrt{2}}$$
$$\theta=45^\circ$$

2. Find the angle between the vectors $i-2\hat{j}+3\hat{k}$ and $3i-2\hat{j}+\hat{k}$

Solution:

$$|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$|\vec{b}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{9+4+1}=\sqrt{14}$$

$$\vec{a}.\vec{b}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)\left(3\hat{i}-2\hat{j}+\hat{k}\right)$$

$$=1.3+(-2)(-2)+3.1$$

$$=3+4+3$$

$$=10$$

$$\therefore 10=\sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta=\frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

3. Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$

Ans:

$\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Projection of $\vec{a}$ and $\vec{b}$ is $\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{1+1}}\{1.1+(-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$

4. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$

Ans:

$\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ *and* $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{7^2 + (-1)^2 + 8^2}}\{1(7) + 3(-1) + 7(8)\} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

5. Show that each of the given three vectors is a unit vector, which are mutually perpendicular to each other $\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}), \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$

Ans:

$$\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

$$\vec{c}=\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)=\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}$$

$$|\vec{a}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$

$$|\vec{b}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}}=1$$

$$|\vec{c}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=1$$

So, each of the vectors is a unit vector

$$\vec{a}.\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(\frac{-6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$

$$\vec{b}.\vec{c}=\frac{3}{7}\times\frac{6}{7}+\left(\frac{-6}{7}\right)\times\frac{2}{7}+\frac{2}{7}\times\left(\frac{-3}{7}\right)=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$

$$\vec{c}.\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\left(\frac{-3}{7}\right)\times\frac{6}{7}=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$$

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)=8$ and $|\vec{a}|=8|\vec{b}|$

Ans:

$$\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)=8$$

$$\Rightarrow \vec{a}\vec{a}-\vec{a}.\vec{b}+\vec{b}.\vec{a}-\vec{b}\vec{b}=8$$

$$\Rightarrow |\vec{a}|^2-|\vec{b}|^2=8$$

$$\Rightarrow \left(8|\vec{b}|\right)^2-|\vec{b}|^2=8$$

$$\Rightarrow 64|\vec{b}|^2-|\vec{b}|^2=8$$

$$\Rightarrow 63|\vec{b}|^2 = 8$$

$$\Rightarrow |\vec{b}|^2 = \frac{8}{63}$$

$$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}}$$

$$\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}| = 8|\vec{b}| = \frac{8\times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$$

7. Evaluate the product $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$

Ans:

$$(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$$

$$= 3\vec{a}.2\vec{a} + 3\vec{a}.7\vec{b} - 5\vec{b}.2\vec{a} - 5\vec{b}.7\vec{b}$$

$$= 6\vec{a}\vec{a} + 21\vec{a}\vec{b} - 10\vec{a}\vec{b} - 35\vec{b}\vec{b}$$

$$= 6|\vec{a}|^2 + 11\vec{a}\vec{b} - 35|\vec{b}|^2$$

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60^0 and their scalar product is $\frac{1}{2}$

Ans:

Let θ be angle between $\vec{a}$ and $\vec{b}$

$$|\vec{a}| = |\vec{b}|, \vec{a}.\vec{b} = \frac{1}{2}, \text{ and } \theta = 60^0$$

$$\therefore \frac{1}{2} = |\vec{a}||\vec{b}|\cos 60^0$$

$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2 = 1$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

9. Find $|\vec{x}|$, if for a unit vector $\vec{a},(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=12$

Ans:

$$(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=12$$

$$\Rightarrow \vec{x}.\vec{x}+\vec{x}\vec{a}-\vec{a}\vec{x}-\vec{a}.\vec{a}=12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12$$

$$\Rightarrow |\vec{x}|^2 = 13$$

$$\therefore |\vec{x}| = \sqrt{13}$$

10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}, \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the values of λ

Ans:

$$\vec{a}+\lambda\vec{b}=\left(2\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-\hat{i}+2\hat{j}+\hat{k}\right)=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$$

$$\left(\vec{a}+\lambda\vec{b}\right).\vec{c}=0$$

$$\Rightarrow \left[(2+-)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right].\left(3\hat{i}+\hat{j}\right)=0$$

$$\Rightarrow 3(2-\lambda)+(2+2\lambda)+0(3+\lambda)=0$$

$$\Rightarrow 6-3\lambda+2+2\lambda=0$$

$$\Rightarrow -\lambda+8=0$$

$$\Rightarrow \lambda=8$$

11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$

Ans:

$$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$$

$$|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2=0$$

12. If $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$, then what can be concluded above the vector $\vec{b}$?

Ans:

$$\vec{a}.\vec{a}=0 \Rightarrow |\vec{a}|^2=0 \Rightarrow |\vec{a}|=0$$

$\therefore \vec{a}$ is the zero vector. Thus, any vector $\vec{b}$ can satisfy $\vec{a}.\vec{b}=0$

13. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

Ans:

$$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\left(\vec{a}\vec{b}+\vec{b}\vec{c}+\vec{c}\vec{a}\right)$$

$$\Rightarrow 0=1+1+1+2\left(\vec{a}\vec{b}+\vec{b}\vec{c}+\vec{c}\vec{a}\right)$$

$$\Rightarrow \left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\frac{-3}{2}$$

14. If either vectors $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}.\vec{b}=\vec{0}$. But the converse need not be true. Justify your answer with an example

Ans:

$\vec{a} = 2\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 3\hat{j} - 6\hat{k}$

$\vec{a}.\vec{b} = 2.3 + 4.3 + 3(-6) = 6 + 12 - 18 = 0$

$|a| = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$

$\therefore \vec{a} \neq \vec{0}$

$|\vec{b}| = \sqrt{3^2 + 3^2 + (-6)^2} = 54$

$\therefore \vec{b} \neq 0$

So, it is clear from above example that the converse of the given statement need not be True.

15. If the vertices A,B,C of a triangle ABC are $(1,2,3),(-1,0,0),(0,1,2)$, respectively, then find $\angle ABC$. [$\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Ans:

$\overrightarrow{BA} = \{1-(-1)\hat{i} + (2-1)\hat{j} + (3-0)\hat{k}\} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{BC} = \{0-(-1)\hat{i} + (1-0)\hat{j} + (2-0)\hat{k}\} = \hat{i} + \hat{j} + 2\hat{k}$

$\overrightarrow{BA}.\overrightarrow{BC} = (2\hat{i} + 2\hat{j} + 3\hat{k})(\hat{i} + \hat{j} + 2\hat{k}) = 2\times1 + 2\times1 + 3\times2 = 2 + 2 + 6 = 10$

$|\overrightarrow{BA}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$

$|\overrightarrow{BC}| = \sqrt{1+1+2^2} = \sqrt{6}$

$\overrightarrow{BA}.\overrightarrow{BC} = |\overrightarrow{BA}||\overrightarrow{BC}|\cos(\angle ABC)$

$\therefore 10 = \sqrt{17} \times \sqrt{6}\cos(\angle ABC)$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17} \times \sqrt{6}}$$

$$\Rightarrow (\angle ABC) = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

16. Show that the points $A(1,2,7), B(2,6,3)$ and $C(3,10,-1)$ are collinear.

Ans:

$$\overrightarrow{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = 2\sqrt{33}$$

$$\left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Hence, the given points are collinear

17. Show that the vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ from the vertices of a right angled triangle

Ans:

$$\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k}, \overrightarrow{OB} = \hat{i} - 3\hat{j} - 5\hat{k}, \overrightarrow{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$\overrightarrow{CA}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$

$\left|\overrightarrow{AB}\right|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$

$\left|\overrightarrow{BC}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$

$\left|\overrightarrow{AC}\right|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$

$\therefore\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{AC}\right|^2=6+35=41=\left|\overrightarrow{AB}\right|^2$

Hence, ΔABC is a right triangle.

18. If $\vec{a}$ is a nonzero vector of magnitude ‘a’ and λ a nonzero scalar. Then $\lambda\vec{a}$ is unit vector if

A) $\lambda=1$ B) $\lambda=-1$ C) $a=|\lambda|$ D) $a=\frac{1}{|\lambda|}$

Ans:

$|\lambda\vec{a}|=1$

$\Rightarrow|\lambda||\vec{a}|=1$

$\Rightarrow|\vec{a}|=\frac{1}{|\lambda|}$

$\Rightarrow a=\frac{1}{|\lambda|}$