Chapter: Vector Algebra.

Exercise 10.4

1. Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Solution:

We have, $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)=19\hat{j}+19\hat{k}$$

$$\therefore\left|\vec{a}\times\vec{b}\right|=\sqrt{(19)^2+(19)^2}=\sqrt{2\times(19)^2}=19\sqrt{2}$$

2. Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

Solution:

$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$\vec{a}+\vec{b}=4\hat{i}+4\hat{j},\vec{a}-\vec{b}=2\hat{i}+4\hat{k}$

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\sqrt{16^2+(-16)^2+(-8)^2}$$

$$=\sqrt{2^2\times8^2+2^2\times8^2+8^2}$$

$$=8\sqrt{2^2+2^2+1}=8\sqrt{9}=8\times3=24$$

So, the unit vector is $=\pm\frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$

$=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}=\pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}$

3. If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the component of $\vec{a}$

Solution:

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ Put

$|\vec{a}|=1.\cos\frac{\pi}{3}=\frac{a_1}{|\vec{a}|}$

$\Rightarrow\frac{1}{2}=a_1$

$\cos\frac{\pi}{4}=\frac{a_2}{|a_1|}$

$\Rightarrow\frac{1}{\sqrt{2}}=a_2$

$\cos\theta=\frac{a_3}{|\vec{a}|}$

$\Rightarrow a_3=\cos\theta$

$\Rightarrow\sqrt{a_1^2+a_2^2+a_3^2}=1$

$\Rightarrow\left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\theta=1$

$\Rightarrow\frac{1}{4}+\frac{1}{2}+\cos^2\theta=1$

$$\Rightarrow \frac{3}{4}+\cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1-\frac{3}{4}=\frac{1}{4}$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$$

So, $\theta = \frac{\pi}{3}$ and components of $\vec{a}$ are $\left(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2}\right)$

4. Show that $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$

Solution:

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$$

$$=(a-b)\times\vec{a}+(a-b)\times b$$

$$=a\times\vec{a}-\vec{b}\times\vec{d}+a\times\vec{b}-\vec{b}\times\vec{b}$$

$$=0+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}=0$$

$$=2\left(\vec{a}\times\vec{b}\right)$$

5. Find λ and μ if $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$

Solution:

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow \hat{i}(6\mu-27\lambda)-\hat{j}(2\mu-27)+\hat{k}(2\lambda-6)=0\hat{i}+0\hat{j}+0\hat{k}$$

$6\mu-27\lambda=0$

$2\mu-27=0$

$2\lambda-6=0$

$\lambda=3$

$\mu=\frac{27}{3}$

6. Given that $\vec{a}.\vec{b}=0$ and $\vec{a}\times\vec{b}=0$. What can you conclude about $\vec{a}$ and $\vec{b}$?

Solution:

$\vec{a}.\vec{b}=0$

i) $|\vec{a}|=0\,or\,|\vec{b}|=0\,or\,\vec{a}\perp\vec{b}$ (if $|\vec{a}|\neq 0$ and $|\vec{b}|\neq 0$)

$\vec{a}\times\vec{b}=0$

ii) $|\vec{a}|=0$ $or\,|\vec{b}|=0\,or\,\vec{a}\,\|\,\vec{b}$ (if $|\vec{a}|\neq 0$ and $|\vec{b}|\neq 0$)

But $\vec{a}$ and $\vec{b}$ cannot be parallel and perpendicular at same time.

So, $|\vec{a}|=0$ or $|\vec{b}|=0$

7. Let the vectors $\vec{a},\vec{b},\vec{c}$ gives as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k},b_1\hat{i}+b_2\hat{j}+b_3\hat{k},c_1\hat{i}+c_2\hat{j},c_3\hat{k}$. Then show that $=\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Solution:

$\left(\vec{b}+\vec{c}\right)=\left(b_1+c_1\right)\hat{i}+\left(b_2+c_2\right)\hat{j}+\left(b_3+c_3\right)\hat{k}$

Now, $\vec{a}\times\left(\vec{b}+\vec{c}\right)\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$

$=i\left[a_2\left(b_3+c_3\right)-a_3\left(b_2+c_2\right)\right]-\hat{j}\left[a_1\left(b_3+c_3\right)-a_3\left(b_1+c_1\right)\right]+\hat{k}\left[a_1\left(b_2+c_2\right)-a_2\left(b_1+c_1\right)\right]$

$= i\left[a_2b_3 - a_2c_3 - a_3b_2 - a_3c_2\right] + \hat{j}\left[-a_1b_3 + a_1c_3 + a_3b_1 + a_3c_1\right] + \hat{k}[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i\left[a_2b_3 - a_3b_2\right] + \hat{j}\left[b_1a_3 - a_2b_3\right] + k\left[a_1b_2 - a_2b_1\right]$$

$$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \hat{i}\left[a_2c_3 - a_3c_2\right] + \hat{j}\left[a_3c_1 - a_1c_3\right] + \hat{k}\left[ab_2 - a_2b\right]$$

$$\left(\vec{a}\times\vec{b}\right) + \left(\vec{a}\times\vec{c}\right) = \hat{i}\left[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2\right] + \hat{j}\left[b_1a_3 + a_3c_1 - a_1b_3 - a_i\}3\right]$$

$$+\hat{k}\left[a,b + a_1c, -a,b_1 - a,c_1\right]$$

$$\vec{a}\times\left(\vec{b}+\vec{c}\right) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$$

Hence, proved

8. If either $\vec{a} = 0$ or $b = 0$, then $\vec{a}\times b = 0$. Is the converse true? Justify your answer with an example

Solution:

Let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}, \vec{a}\times\vec{b} = \vec{0}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} = \hat{i}\left(24-24\right) - \hat{j}\left(16-16\right) + \hat{k}\left(12-12\right) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$$\therefore \vec{a} \neq 0$$

$\left|\vec{b}\right| = \sqrt{4^2+6^2+8^2} = \sqrt{116}$

$\therefore \vec{b} \neq 0$

Hence, converse of the statement need not be ture

9. Find the area of the triangle with vertices $A(1,1,2), B(2,3,5)$ and $C(1,5,5)$

Solution:

$$\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$$

$$\text{Area } = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

So, area of ΔABC is $\dfrac{\sqrt{61}}{2}$ sq units

10. Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$\therefore \left|\vec{a} \times \vec{b}\right| = \sqrt{20^2 + 5^2 + 5^2} = \sqrt{400+25+25} = 15\sqrt{2}$$

So, area of parallelogram is $15\sqrt{2}$ sq units

11. Let the vector $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$

Solution:

$$|\vec{a}\times\vec{b}|=1$$

$$\Rightarrow |\vec{a}||\vec{b}||\sin\theta|=1$$

$$\Rightarrow |\vec{a}||\vec{b}||\sin\theta|=1$$

$$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$$

$$\Rightarrow \sin\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$ respectively is

A) $\frac{1}{2}$ B) 1 C) 2 D) 4

Solution:

$$\overrightarrow{AB}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{BC}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=\hat{k}(-2)=-2\hat{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(-2)^2}=2$$

So, area of the required rectangle is 2 square units