Chapter 13: Probability.

Exercise 13.1

1. Given that E and F are events such that $P(E)=0.6, P(F)=0.3$ and $P(E\cap F)=0.2$ then find the values of $P(E|F)$ and $P(F|E)$

Solution: Given $P(E)=0.6, P(F)=0.3$ and $P(E\cap F)=0.2$

The conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{0.2}{0.3}=\frac{2}{3}$$

Therefore, $P(E|F)=\frac{2}{3}$

The conditional probability states that

$$P(F|E)=\frac{P(E\cap F)}{P(E)}=\frac{0.2}{0.6}=\frac{1}{3}$$

Therefore, $P(F|E)=\frac{1}{3}$

2. Compute $P(A|B)$ if $P(B)=0.5$ and $P(A\cap B)=0.32$

Solution: Given, $P(B)=0.5$ and $P(A\cap B)=0.32$

The conditional probability states that

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{0.32}{0.5}=\frac{16}{25}$$

Therefore, $P(A|B)=\frac{16}{25}$

3. If $P(A)=0.8, P(B)=0.5$ and $P(B|\text{A})=0.4$ find

i) $P(A\cap B)$ ii) $P(A|B)$ iii) $P(A\cup B)$

Solution: Given, $P(A)=0.8, P(B)=0.5$ and $P(B|\text{A})=0.4$

The conditional probability states that

$$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}$$

(i) Hence, $0.4=\frac{P(A\cap B)}{0.8} \Rightarrow P(A\cap B)=0.32$

(ii) Use conditional probability

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$$
$$=\frac{0.32}{0.5}$$
$$=0.64$$

(iii) Use addition theorem on probability

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$=0.8+0.5-0.32$$
$$=0.98$$

Hence, $P(A\cup B)=0.98$

4. Evaluate $P(A\cup B)$, if $2P(A)=P(B)=\frac{5}{13}$ and $P(A|B)=\frac{2}{5}$

Solution: Given, $2P(A)=P(B)=\frac{5}{13}$

It implies that $P(A)=\frac{5}{26}$ and

Use the conditional probability $P(A|B)=\frac{P(A\cap B)}{P(B)}$

Substitute the appropriate values in the above equation.

$$\frac{2}{5}=\frac{P(A\cap B)}{\frac{5}{13}}$$

$$P(A\cap B)=\frac{5}{13}\times\frac{2}{5}$$

$$=\frac{2}{13}$$

Use the Addition theorem on probability

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{5}{26}+\frac{5}{13}-\frac{2}{13}$$

$$=\frac{5+10-4}{26}$$

$$=\frac{11}{26}$$

Therefore, $P(A\cup B)=\frac{11}{26}$

5. If $P(A)=\frac{6}{11}, P(B)=\frac{5}{11}$ and $P(A\cup B)=\frac{7}{11}$, find

i. $P(A\cap B)$

ii. $P(A|B)$

iii. $P(B|A)$

Solution: Given, $P(A)=\frac{6}{11}, P(B)=\frac{5}{11}$ and $P(A\cup B)=\frac{7}{11}$

i. Addition theorem of probability states that

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Substitute the appropriate values

$$\frac{7}{11}=\frac{6}{11}+\frac{5}{11}-P(A\cap B)$$

$$P(A\cap B)=\frac{4}{11}$$

Therefore, $P(A \cap B) = \frac{4}{11}$

ii. The conditional probability states that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{4}{11}}{\frac{5}{11}}$$

$$= \frac{4}{5}$$

Therefore, $P(A|B) = \frac{4}{5}$

iii. The conditional probability states that

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{4}{11}}{\frac{6}{11}}$$

$$= \frac{2}{3}$$

Therefore, $P(B|A) = \frac{2}{3}$

6. A coin is tossed three times, where

i. E : head on third toss, F : heads on first two tosses

ii. E : At least two heads, F : At most two heads

iii. E : At most two heads, F : At least one tail.

Find the value of $P(E|F)$ in each part.

Solution: The sample space S is $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

i. Given that E : head on third toss, F : heads on first two tosses

$$E = \{HHH, HTH, THH, TTH\} \text{ and } F = \{HHH, HHT\}$$

Hence, $E \cap F = \{HHH\}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{1}{4}$$

Therefore, $P(E|F)=\frac{1}{4}$

ii. Given that E : At least two heads, F : At most two heads

$E=\{HHH, HTH, THH, HHT\}$ and

$F=\{TTT, TTH, THT, HTT, HTH, THH, HHT\}$

Hence, $E\cap F=\{HHH, HTH, THH, HHT\}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{4}{7}$$

Therefore, $P(E|F)=\frac{4}{7}$

iii. Given that E : At most two tails, F : At least one tail.

$E=\{HHH, TTH, THT, HTT, THH, HTH, HHT\}$ and

$F=\{TTT, HHT, THH, HTH, TTH, THT, HTT\}$

Hence, $E\cap F=\{HHT, THH, HTH, TTH, THT, HTT\}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{6}{7}$$

Therefore, $P(E|F)=\frac{6}{7}$

7. Two coins are tossed once, where

i. E : tail appears on one coin, F : one coin shows head

ii. E : no tail appears, F : no head appears.

Find the value of $P(E|F)$ in each part.

Solution: The sample space S is $S=\{HH,HT,TH,TT\}$

i. Given that E : tail appears on one coin, F : one coin shows head

$$E=\{HT,TH\} \text{ And } F=\{HT,TH\}$$

Hence, $E\cap F=\{HT,TH\}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{2}{2}$$
$$=1$$

Therefore, $P(E|F)=1$

ii. Given that E : no tail appears, F : no head appears.

$$E=\{HH\} \text{ And } F=\{TT\}$$

Hence, $E\cap F=\{\ \}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{0}{1}$$
$$=0$$

Therefore, $P(E|F)=0$

8. A die is thrown three times, suppose that $E:$ Appear 4 on third toss, $F:$ 6, 5 appear on first two tosses. Then find $P(E|F)$

Solution: The number of possible cases for the event $E:$ Appear 4 on third toss:

First two tosses any one of first six numbers, so that $n(E)=6\times6=36$

For the event $F:$ 6, 5 appear on first two tosses

$$F=\{(6,5,1),(6,5,2),(6,5,3),(6,5,4),(6,5,5),(6,5,6)\}$$

Hence, $E\cap F=\{(6,5,4)\}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{1}{6}$$

Therefore, $P(E|F)=\frac{1}{6}$

9. Mother, Father and son line up at random for a family picture

$E:$ Son on one end, $F:$ Father in middle

Then find $P(E|F)$

Solution: Suppose that $A:$Mother, $B:$ Father, $C:$Son

Given that $E:$ Son on one end, $F:$ Father in middle

It implies that $E=\{ABC,BAC,CAB.CBA\}$ and $F=\{ABC,CBA\}$

Hence, $E\cap F=\{ABC,CBA\}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{2}{2}$$
$$=1$$

Therefore, $P(E|F)=1$

10. A black and a red dice are rolled.

    a. Find the conditional probability of obtaining a sum greater than 9, given that the black die results 5

    b. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution: Given that one black and one red dice thrown.

a. Suppose that E be the event of getting total 9 and F be the event of getting 5 on the black die

The set of favourable cases to the event E is

$$E=\{(B_4,R_6),(B_5,R_5),(B_5,R_6),(B_6,R_4),(B_6,R_5),(B_6,R_6)\}$$

The set of favourable cases to the event F is

$$F=\{(B_5,R_1),(B_5,R_2),(B_5,R_3),(B_5,R_4),(B_5,R_5),(B_5,R_6)\}$$

Hence, $E\cap F=\{(B_5,R_5),(B_5,R_6)\}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{2}{6}$$
$$=\frac{1}{3}$$

Therefore, $P(E|F)=\frac{1}{3}$

b. Suppose that E be the event of getting sum 8 and F be the event of getting number less than 4 on the red dice

The set of favourable cases to the event E is

$$E=\{(B_2,R_6),(B_3,R_5),(B_4,R_4),(B_5,R_3),(B_6,R_2)\}$$

The set of favourable cases to the event F is

$$F=\begin{Bmatrix}(B_1,R_1),(B_2,R_1),(B_3,R_1),(B_4,R_1),(B_5,R_1),(B_6,R_1),\\(B_1,R_1),(B_2,R_1),(B_3,R_1),(B_4,R_2),(B_5,R_2),(B_6,R_2),\\(B_1,R_3),(B_2,R_3),(B_3,R_3),(B_4,R_3),(B_5,R_3),(B_6,R_3)\end{Bmatrix}$$

Hence, $E\cap F=\{(B_5,R_3),(B_6,R_2)\}$

Conditional probability states that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$
$$=\frac{n(E\cap F)}{n(F)}$$
$$=\frac{2}{18}$$
$$=\frac{1}{9}$$

Therefore, $P(E|F)=\frac{1}{9}$

11. A fair die is rolled. Consider the events $E=\{1,3,5\},F=\{2,3\},G=\{2,3,4,5\}$ then find the following

i) $P(E|F)$ and $P(F|E)$

ii) $P(E|G)$ and $P(G|E)$

iii) $P(E\cup F|G)$ and $P(E\cap F|G)$

Solution: Given that $E=\{1,3,5\},F=\{2,3\},G=\{2,3,4,5\}$

i) Conditional probability states that $P(E|F)=\frac{P(E\cap F)}{P(F)}$

Here $E\cap F=\{3\}$.

It implies that

$$P(E|F)=\frac{P(E\cap F)}{P(F)}$$

$$=\frac{n(E\cap F)}{n(F)}$$

$$=\frac{1}{2}$$

Therefore, $P(E|F)=\frac{1}{2}$

Conditional probability states that $P(F|E)=\frac{P(E\cap F)}{P(E)}$

Here $E\cap F=\{3\}$.

It implies that

$$P(F|E)=\frac{P(E\cap F)}{P(E)}$$

$$=\frac{n(E\cap F)}{n(E)}$$

$$=\frac{1}{3}$$

Therefore, $P(F|E)=\frac{1}{3}$

ii) Conditional probability states that $P(E|G)=\frac{P(E\cap G)}{P(G)}$

Here $E\cap G=\{3,5\}$.

It implies that

$$P(E|\text{G})=\frac{P(E\cap G)}{P(G)}$$

$$=\frac{n(E\cap G)}{n(G)}$$

$$=\frac{2}{4}$$

Therefore, $P(E|G)=\frac{1}{2}$

Conditional probability states that $P(G|E)=\frac{P(E\cap G)}{P(E)}$

Here $E\cap G=\{3,5\}$.

It implies that

$$P(G|E)=\frac{P(E\cap G)}{P(E)}$$
$$=\frac{n(E\cap G)}{n(E)}$$
$$=\frac{2}{3}$$

Therefore, $P(G|E)=\frac{2}{3}$

iii) Conditional probability states that $P(E\cup F|G)=\frac{P(E\cup F\cap G)}{P(G)}$

Here $E\cup F=\{1,2,3,5\}E\cup F\cap G=\{2,3,5\}$.

It implies that

$$P(E\cup F|G)=\frac{P(E\cup F\cap G)}{P(G)}=\frac{n(E\cup F\cap G)}{n(G)}=\frac{3}{4}$$

Therefore, $P(E\cup F|G)=\frac{3}{4}$

Conditional probability states that $P(E\cap F|G)=\frac{P(E\cap F\cap G)}{P(G)}$

Here $E\cap F\cap G=\{3\}$

It implies that

$$P(E\cap F|G)=\frac{P(E\cap F\cap G)}{P(G)}=\frac{n(E\cap F\cap G)}{n(G)}=\frac{1}{4}$$

Therefore, $P(E\cap F|G)=\frac{1}{4}$

12. Assume that each born child is equally like to be a boy or a girl. If a family had two children. What is the conditional probability that both are girls given that

i) The youngest is a girl

ii) At least one is girl.

Solution: Suppose that B: boy and G: girl.

Sample space for the experiment is $S=\{(BB),(BG),(GB),(GG)\}$

i) Let E be the event of getting family having both are girl children

$$E=\{GG\}$$

Let F be the event of getting a family having youngest is a girl

$$F=\{(GG),(GB)\}$$

Hence, $E\cap F=\{(GG)\}$

The conditional probability states that $P(E|F)=\frac{P(E\cap F)}{P(F)}$

$$\begin{aligned}P(E|F)&=\frac{P(E\cap F)}{P(F)}\\&=\frac{n(E\cap F)}{n(F)}\\&=\frac{1}{2}\end{aligned}$$

Therefore, $P(E|F)=\frac{1}{2}$

ii) Let E be the event of getting family having both are girl children

$$E=\{GG\}$$

Let F be the event of getting at least one girl

$$F=\{(GG),(GB),(BG)\}$$

Hence, $E \cap F = \{(GG)\}$

The conditional probability states that $P(E|F) = \frac{P(E \cap F)}{P(F)}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$
$$= \frac{n(E \cap F)}{n(F)}$$
$$= \frac{1}{3}$$

Therefore, $P(E|F) = \frac{1}{3}$

13. An instructor has a question bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions If a question is selected at random from the question bank, what is the probability that it will be an easy question that it is a multiple choice questions?

Solution: The given data can be tabulated as below

	True/False(T)	Multiple(M)	Total
Easy(E)	300	500	800
Difficult(D)	200	400	600
Total	500	900	1400

We want to find the probability getting an easy question that it is a multiple choice question.

Conditional probability states that $P(E|M) = \frac{P(E \cap M)}{P(M)}$

$$P(E|M) = \frac{P(E \cap M)}{P(M)} = \frac{n(E \cap M)}{n(M)} = \frac{500}{900} = \frac{5}{9}$$

Therefore, $P(E|M) = \frac{5}{9}$

14. Given that the two numbers appearing on throwing the two dice are different. Find the probability of the event " The sum of the numbers on the dice is 4"

Solution: Let E be the event of getting different numbers on two dice

The number of favourable cases to the event E is $n(E)=36-6(\text{doublet})=30$

Let F be the event of getting the sum of the numbers on the dice is 4.

$$F=\{(1,3),(2,2),(3,1)\}$$

Hence, $E\cap F=\{(1,3),(3,1)\}$

Conditional probability states that $P(F\mid E)=\dfrac{P(E\cap F)}{P(E)}$

$$P(F\mid E)=\frac{P(E\cap F)}{P(E)}$$
$$=\frac{2}{30}$$
$$=\frac{1}{15}$$

Therefore, $P(E\mid F)=\dfrac{1}{15}$

15. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event the coin shows a tail, given that at least one die shows as 3

Solution: The sample space of the experiment is

$$S=\begin{Bmatrix}(1,H),(1,T),(2,H),(2,T),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\(4,H),(4,T),(5,H),(5,T),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{Bmatrix}$$

Here $n(S)=20$

Let E be the event of showing tail on the coin

$$E=\{(1,T),(2,T),(4,T),(5,T)\}$$

Here, $n(E)=4$

Let F be the event of getting 3 at least on one dice

$$F=\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(6,3)\}$$

Here, $n(F)=7$

Consider $E\cap F=\varphi$

Hence, $n(E\cap F)=0$

Conditional probability states that $P(E|F)=\dfrac{P(E\cap F)}{P(F)}$

Hence,

$$\begin{aligned}P(E|F)&=\frac{P(E\cap F)}{P(F)}\\&=\frac{n(E\cap F)}{n(F)}\\&=\frac{0}{7}\\&=0\end{aligned}$$

Therefore, the probability of the event the coin shows a tail, given that at least one die shows as 3 is zero.

16. If $P(A)=\frac{1}{2}, P(B)=0$ then the value of $P(A|B)$

A) 0

B) $\frac{1}{2}$

C) Not defined

D) 1

Solution: Given $P(A)=\frac{1}{2}, P(B)=0$

Conditional probability states that $P(A|B)=\frac{P(A\cap B)}{P(B)}$

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$
$$=\frac{n(A\cap B)}{n(B)}$$
$$=\text{Not defined}$$

This is matching with the option (C)

17. If A and B are the events such that $P(A|B)=P(B|A)$ then

A) $A\neq B$

B) $A=B$

C) $A\cap B$ is an empty set

D) $P(A)=P(B)$

Solution: Given $P(A|B)=P(B|A)$

Conditional probability states that $P(A|B)=\frac{P(A\cap B)}{P(B)}$

It implies that $P(A)=P(B)$

This is matching with the option (D)