Chapter 13: Probability.

Exercise 13.5

1. A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of

(i) 5 successes? (ii) at least 5 success? (iii) at most 5 successes?

Solution:

Let X denote the number of success of getting odd numbers in an experiment of 6 trials.

Probability of getting an odd number in a single throw of a die is, $p=\frac{3}{6}=\frac{1}{2}$

$$\therefore q=1-p=\frac{1}{2}$$

X has a binominal distribution

Thus, $P(X=x)={}^{n}C_{n-1}q^{n-x}p^{x}$, where n = 0, 1, 2 ……. N

$$={}^{6}C_{x}\left(\frac{1}{2}\right)^{6-x}\left(\frac{1}{2}\right)^{x}$$

$$={}^{6}C_{x}\left(\frac{1}{2}\right)^{6}$$

(i) P (5 success) = P (X = 5)

$$={}^{6}C_{5}\left(\frac{1}{2}\right)^{6}$$

$$=6.\frac{1}{64}=\frac{3}{32}$$

(ii) P (at least 5 success) = $P(X\geq 5)$

$$=P(X=5)+P(X=6)$$

$$={}^{6}C_{5}\left(\frac{1}{2}\right)^{6}+{}^{6}C_{6}\left(\frac{1}{2}\right)^{6}$$

$$=6.\frac{1}{64}+1.\frac{1}{64}$$

$$=\frac{7}{64}$$

(iii) P (at most 5 success) = $P(X\le 5)$

$$=1-P(X>5)$$

$$=1-P(X=6)$$

$$=1-{}^{6}C_{6}\left(\frac{1}{2}\right)^{6}$$

$$=1-\frac{1}{64}=\frac{63}{64}$$

2. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two success

Solution:

Let X denote the number of times of getting doublets in an experiment of throwing two 3 dice simultaneously four times.

Probability of getting doublets in a single throw of the pair of dice is

$$p=\frac{6}{36}=\frac{1}{6}$$

$$\therefore q=1-p=1-\frac{1}{6}=\frac{5}{6}$$

Clearly, X has the binomial distribution with $n=4,\ p=\frac{1}{6}\ and\ q=\frac{5}{6}$

$\therefore P(X=r)={}^{n}C_{x}q^{n-x}p^{x}$, where x = 0, 1, 2, 3 ……n

$$={}^{4}C_{x}\left(\frac{5}{6}\right)^{4-x}\left(\frac{1}{6}\right)^{x}$$

$$={}^{6}C_{x}\frac{5^{4-x}}{6^{4}}$$

$\therefore$ P (2 successes) = $P(X=2)$

$$={}^{4}C_{2}\frac{5^{4-2}}{6^{4}}$$

$$=6\frac{25}{1296}=\frac{25}{216}$$

3. There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution:

Let X denote the number of defective items in a sample of 10 items drawn successively

$$\Rightarrow p=\frac{5}{100}=\frac{1}{20}$$

$$\therefore q=1-\frac{1}{20}=\frac{19}{20}$$

X has a binomial distribution with n = 10 and $p=\frac{1}{20}$

$P(X=x)={}^{n}C_{x}q^{n-x}p^{x}$, where x = 0, 1, 2 …… n

$$={}^{10}C_{x}\left(\frac{19}{20}\right)^{10-x}\left(\frac{1}{20}\right)^{x}$$

P (not more than 1 defective item) = $P(X\le 1)$

$$=P(X=0)+P(X=1)$$

$$= {}^{10}C_0\left(\frac{19}{20}\right)^{10}\left(\frac{1}{20}\right)^{0} + {}^{10}C_1\left(\frac{19}{20}\right)^{9}\left(\frac{1}{20}\right)^{1}$$

$$=\left(\frac{19}{20}\right)^{10} + 10\left(\frac{19}{20}\right)^{9}\left(\frac{1}{20}\right)$$

$$=\left(\frac{19}{20}\right)^{9}\left[\frac{19}{20}+\frac{10}{20}\right]$$

$$=\left(\frac{29}{20}\right)\left(\frac{19}{20}\right)^{9}$$

4. Five cards are drawn successively with replacement from a well – shuffled deck of 52 cards. What is the probability that

(i) All the five cards are spades?

(ii) Only 3 cards are spades?

(iii) None is a spade?

Solution:

Let X represent the number of spade cards among the five cards drawn.

In a well shuffled deck of 52 cards, there are 13 spades cards.

$$\Rightarrow p=\frac{13}{52}=\frac{1}{4}$$

$$\therefore q = 1-\frac{1}{4}=\frac{3}{4}$$

X has a binomial distribution with n = 5 and $p=\frac{1}{4}$

$P(X=x)={}^{n}C_x q^{n-x}p^{x}$, where x = 0, 1, 2 ……. N

$$= {}^{5}C_x\left(\frac{3}{4}\right)^{5-x}\left(\frac{1}{4}\right)^{x}$$

(i) P (all five cards are spades) = P (X = 5)

$$= {}^5C_5 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^5$$

$$= 1.\frac{1}{1024} = \frac{1}{1024}$$

(ii) P (only 3 card are spades) = P (X = 3)

$$= {}^5C_3 \left(\frac{3}{4}\right)^2 . \left(\frac{1}{4}\right)^3$$

$$= 10.\frac{9}{6}.\frac{1}{64}$$

$$= \frac{45}{512}$$

(iii) P (none is a spade) = P (X = 0)

$$= {}^5C_0 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^0$$

$$= 1.\frac{243}{1024} = \frac{243}{1024}$$

5. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. What is the probability that out of 5 such bulbs

(i) None

(ii) Not more than one

(iii) More than one

(iv) At least one

Will fuse after 150 days of use

Solution:

Let X represents the number of bulbs that will fuse after 150 days of use in an experiment of 5 trials

Given, p = 0.05

$\therefore q = 1 - p = 1 - 0.05 = 0.95$

X has a binomial distribution with n = 5 and p = 0.05

$\therefore P(X = x) = {}^{n}C_{x} q^{n-x} p^{x}$, where x = 0, 1, 2 ….. n

$= {}^{5}C_{x} (0.95)^{5-x} . (0.05)^{x}$

(i) P (none) = P (X = 0)

$= {}^{5}C_{0} (0.95)^{5} . (0.05)^{0}$

$= 1 \times (0.95)^{5}$

$= (0.95)^{5} = 0.7737$

(ii) P (not more than one) = $P(X \le 1)$

$= P(X = 0) + P(X = 1)$

$= {}^{5}C_{0} \times (0.95)^{5} . (0.05)^{0} + {}^{5}C_{1} (0.95)^{4} . (0.05)^{1}$

$= 1 \times (0.95)^{5} + 5 \times (0.95)^{4} . (0.05)$

$= (0.95)^{5} + (0.25)(0.95)^{4}$

$= (0.95)^{4} + [0.95 + 0.25]$

$= (0.95)^{4} \times 1.2$

= 0.977

(iii) P (more than 1) = P (X>1)

$= 1 - P(X \le 1)$

= 1 – P (not more than 1)

= $1 - (0.95)^{4} \times 1.2$

= 0.02

(iv) P (at least one) = $P(X \geq 1)$

$=1-P(X<1)$

$=1-P(X=0)$

$=1-{}^{5}C_{0}(0.95)^{5}\times(0.05)^{0}$

$=1-1\times(0.95)^{5}$

$=1-(0.95)^{5}$

$=0.2263$

6. A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

Solution:

Let X denote the number of balls marked with the digit 0 among the 4 balls drawn.

X has a binominal distribution with n = 4 and $p=\frac{1}{10}$

$\therefore q=1-p=1-\frac{1}{10}=\frac{9}{10}$

$P(X=x)={}^{n}C_{x}q^{n-x}p^{x}, x=1,2.....n$

$={}^{n}C_{x}\left(\frac{9}{10}\right)^{4-x}\left(\frac{1}{10}\right)^{x}$

P (none marked with 0) = P(X= 0)

$={}^{4}C_{x}\left(\frac{9}{10}\right)^{4}\left(\frac{1}{10}\right)^{0}$

$=1.\left(\frac{9}{10}\right)^{4}$

$$=\left(\frac{9}{10}\right)^4$$

7. In an examination, 20 questions of true – false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tail, he answers 'false'. Find the probability that he answers at least 12 questions correctly

Solution:

Let X represent the number of correctly answered questions out of 20 questions.

Since "head" on a coin represents the true answer and "tail" represents the false answer, the correctly answered questions are Bernoulli trials.

$$\therefore p=\frac{1}{2}$$

$$\therefore q=1-p=1-\frac{1}{2}=\frac{1}{2}$$

X has a binominal distribution with n = 20 and $p=\frac{1}{2}$

$$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}, where\ x=0,1,2 \ldots\ldots n$$

$$={}^{20}C_{x}\left(\frac{9}{10}\right)^{20-x}\left(\frac{1}{2}\right)^{x}$$

$$={}^{20}C_{x}\left(\frac{1}{2}\right)^{20}$$

P (at least 12 questions answered correctly) = $P(X\geq 12)$

$$=P(X=12)+P(X=13)+\ldots.+P(X=20)$$

$$={}^{20}C_{12}\left(\frac{1}{2}\right)^{20}+{}^{20}C_{13}\left(\frac{1}{2}\right)^{20}+\ldots.+{}^{20}C_{20}\left(\frac{1}{2}\right)^{20}$$

$$=\left(\frac{1}{2}\right)^{20}.\left[{}^{20}C_{12}+{}^{20}C_{13}+\ldots\ldots+{}^{20}C_{20}\right]$$

8. Suppose X has a binominal distribution $B\left(6,\frac{1}{2}\right)$. Show that X = 3 is the most likely outcome. (Hint: P(X=3) is the maximum among all $P(x_i), x_i = 0,1,2,3,4,5,6$)

Solution:

X is the random variable whose binomial distribution is $B\left(6,\frac{1}{2}\right)$

Thus, n = 6 and $p=\frac{1}{2}$

$\therefore q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$

Then, $P(X=x) = {}^{n}C_{x}q^{n-x}p^{x}$

$= {}^{6}C_{x}\left(\frac{1}{2}\right)^{6-x}\cdot\left(\frac{1}{2}\right)^{x}$

$= {}^{6}C_{x}\left(\frac{1}{2}\right)^{6}$

Here, $P(X=x)$ will be maximum, if ${}^{6}C_{x}$ will be maximum

Then, ${}^{6}C_{0} = {}^{6}C_{6} = \frac{6!}{0!.6!} = 1$

${}^{6}C_{1} = {}^{6}C_{5} = \frac{6!}{1!.5!} = 6$

${}^{6}C_{2} = {}^{6}C_{4} = \frac{6!}{2!.4!} = 15$

${}^{6}C_{3} = \frac{6!}{3!.3!} = 20$

The value of ${}^{6}C_{3}$ is maximum

Thus, for X = 3, P (X = x) is maximum

Therefore, X = 3 is the most likely outcome

9. On a multiple choice examination with three possible answer for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

Solution:

Let X represent the number of correct answers by guessing in the set of 5 multiple choice questions.

Probability of getting a correct answer is $p=\frac{1}{3}$

$\therefore q=1-p=1-\frac{1}{3}=\frac{2}{3}$

Clearly, X has a binomial distribution with n = 5 and $p=\frac{1}{3}$

$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}$

$={}^{5}C_{x}\left(\frac{2}{3}\right)^{5-x}.\left(\frac{1}{3}\right)^{x}$

P (guessing more than 4 correct answers) = $P(X\geq 4)$

$=P(X=4)+P(X=5)$

$={}^{5}C_{4}\left(\frac{2}{3}\right).\left(\frac{1}{3}\right)^{4}+{}^{5}C_{5}\left(\frac{1}{3}\right)^{5}$

$=5.\frac{2}{3}.\frac{1}{81}+1.\frac{1}{243}$

$=\frac{10}{243}+\frac{1}{243}=\frac{11}{243}$

10. A person buys a lottery ticket in 50 lotteries, in each of which his change of winning a prize is $\frac{1}{100}$. What is the probability that he will in a prize (a) at least once (b) exactly once (c) at least twice?

Solution:

Let X represents the number of winning prizes in 50 lotteries

Clearly, X has a binominal distribution with n = 50 and $p=\frac{1}{100}$

$\therefore q=1-p=1-\frac{1}{100}=\frac{99}{100}$

$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}={}^{50}C_{x}\left(\frac{99}{100}\right)^{50-x}\left(\frac{1}{100}\right)^{x}$

(a) P (winning at least once) = $P(X\geq 1)$

$=1-P(X<1)$

$=1-P(X=0)$

$=1-{}^{50}C_0\left(\frac{99}{100}\right)^{50}$

$=1-1.\left(\frac{99}{100}\right)^{50}$

$=1-\left(\frac{99}{100}\right)^{50}$

(b) P (Winning exactly once) = P (X =1)

$={}^{50}C_1\left(\frac{99}{100}\right)^{49}\left(\frac{1}{100}\right)^{1}$

$=50\left(\frac{1}{100}\right)\left(\frac{99}{100}\right)^{49}$

$=\frac{1}{2}\left(\frac{99}{100}\right)^{49}$

(c) P (at least twice) = $P(X\geq 2)$

$=1-P(X<2)$

$=1-P(X\leq 1)$

$=1-[P(X=0)+P(X=1)]$

$=[1-P(X=0)]-P(X=1)$

$=1-\left(\frac{99}{100}\right)^{50}-\frac{1}{2}\left(\frac{99}{100}\right)^{49}$

$=1-\left(\frac{99}{100}\right)^{49}\left[\frac{99}{100}+\frac{1}{2}\right]$

$=1-\left(\frac{99}{100}\right)^{49}\left[\frac{149}{100}\right]$

$=1-\left(\frac{149}{100}\right)\left(\frac{99}{100}\right)^{49}$

11. Find the probability of getting 5 exactly twice in 7 thrown of a die

Solution:

Let X represents the number of times of getting 5 in 7 throws of the die

Probability of getting 5 in a single throw of the die, $p=\frac{1}{6}$

$\therefore q=1-p=1-\frac{1}{6}=\frac{5}{6}$

Clearly, X has the probability distribution with n = 7 and $p=\frac{1}{6}$

$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}={}^{7}C_{x}\left(\frac{5}{6}\right)^{7-x}\left(\frac{1}{6}\right)^{x}$

P (getting 5 exactly twice) = P (X =2)

$={}^{7}C_{2}\left(\frac{5}{6}\right)^{5}\left(\frac{1}{6}\right)^{2}$

$=21\left(\frac{5}{6}\right)^{5}.\frac{1}{36}$

$=\left(\frac{7}{12}\right)\left(\frac{5}{6}\right)^{5}$

12. Find the probability of throwing at most 2 sixes in 6 throws of a single die

Solution:

Let X represent the number of times of getting sixes in 6 throws of the die.

Probability of getting six in a single throw of die, $p=\frac{1}{6}$

$\therefore q=1-p=1-\frac{1}{6}=\frac{5}{6}$

Clearly, X has a binomial distribution with n = 6

$\therefore P\left(X=x\right)={}^{n}C_{x}q^{n-x}p^{x}={}^{6}C_{x}\left(\frac{5}{6}\right)^{6-x}\left(\frac{1}{6}\right)^{x}$

P (at most 2 sixes) $= P(X \le 2)$

$= P(X=0)+P(X=1)+P(X=2)$

$= {}^6C_0\left(\frac{5}{6}\right)^6 + {}^6C_1\left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right) + {}^6C_2\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)^2$

$=1.\left(\frac{5}{6}\right)^6 + 6.\frac{1}{6}.\left(\frac{5}{6}\right)^5 + 15.\frac{1}{36}\left(\frac{5}{6}\right)^4$

$=\left(\frac{5}{6}\right)^6 + \left(\frac{5}{6}\right)^5 + \frac{5}{12}.\left(\frac{5}{6}\right)^4$

$=\left(\frac{5}{6}\right)^4\left[\left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right) + \left(\frac{5}{12}\right)\right]$

$=\left(\frac{5}{6}\right)^4\left[\frac{25}{36}+\frac{5}{6}+\frac{5}{12}\right]$

$=\left(\frac{5}{6}\right)^4\left[\frac{25+30+15}{36}\right]$

$=\frac{70}{36}.\left(\frac{5}{6}\right)^4 = \frac{35}{18}.\left(\frac{5}{6}\right)^4$

$=0.9377$

13. It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

Solution:

Let X denote the number of times of selecting defective articles in a random sample space of 12 articles

Clearly, X has a binominal distribution with n = 12 and p = 10% = $\frac{10}{100} = \frac{1}{10}$

$\therefore q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

$$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}={}^{12}C_{x}\left(\frac{9}{10}\right)^{12-x}\left(\frac{1}{10}\right)^{x}$$

P (Selecting 9 defective articles) $={}^{12}C_{9}\left(\frac{9}{10}\right)^{3}\left(\frac{1}{10}\right)^{9}$

$$=220.\frac{9^{3}}{10^{3}}.\frac{1}{10^{9}}$$

$$=\frac{22\times 9^{3}}{10^{11}}$$

14. In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs none is defective is

(A) 10^{-1} (B) $\left(\frac{1}{2}\right)^{5}$ (C) $\left(\frac{9}{10}\right)^{5}$ (D) $\frac{9}{10}$

Solution:

Let X denote the number of defective bulbs out of a sample of 5 bulbs

Probability of getting a defective bulb, $p=\frac{10}{100}=\frac{1}{10}$

$$\therefore q=1-p=1-\frac{1}{10}=\frac{9}{10}$$

Clearly, X has binomial distribution with n = 5 and $p=\frac{1}{10}$

$$\therefore P(X=x)={}^{n}C_{x}q^{n-x}p^{x}={}^{5}C_{x}\left(\frac{9}{10}\right)^{5-x}\left(\frac{1}{10}\right)^{x}$$

P (None of the bulbs is defective) = P (X = 0)

$$={}^{5}C_{0}\left(\frac{9}{10}\right)^{5}$$

$$=1.\left(\frac{9}{10}\right)^{5}=\left(\frac{9}{10}\right)^{5}$$

15. The probability that a student is not a swimming is $\frac{1}{5}$. The probability that out of five students, four are swimmers is

(A) ${}^5C_4\left(\frac{4}{5}\right)^4\frac{1}{5}$ (B) $\left(\frac{4}{5}\right)^4\frac{1}{5}$ (C) ${}^5C_1\frac{1}{5}\left(\frac{4}{5}\right)^4$ (D) None of these

Solution:

Let X denote the number of students, out of 5 students, who are swimmers.

Probability of students who are not swimmers, $q=\frac{1}{5}$

$$\therefore p=1-q=1-\frac{1}{5}=\frac{4}{5}$$

Clearly, X has a binomial distribution with n = 5 and $p=\frac{4}{5}$

$$\therefore P(X=x)={}^nC_x q^{n-x}p^x={}^5C_x\left(\frac{1}{5}\right)^{5-x}\left(\frac{4}{5}\right)^x$$

P (four students are swimmers) $=P(X=4)={}^5C_4\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^4$