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Chapter: 2. Inverse trigonometric functions.

Exercise Miscellaneous

1. Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Solution:

Consider

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]$$
$$=\cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right]$$
$$=\left(\frac{\pi}{6}\right)$$

2. Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Solution:

$$\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left[\tan\left(2\pi-\frac{5\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[\tan\left(\pi-\frac{5\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{6}\right)\right]$$

$$=\frac{\pi}{6}$$

3. Prove $2\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{24}{7}$

Solution:

Let $\sin^{-1}\frac{3}{5}=x$

$\Rightarrow \sin x=\frac{3}{5}$

$\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\frac{4}{5}$

$\therefore \tan x=\frac{3}{4}$

$\therefore x=\tan^{-1}\frac{3}{4}$

$\Rightarrow \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$

In $2\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{24}{7}$

L.H.S $2\sin^{-1}\frac{3}{5}=2\tan^{-1}\frac{3}{4}$

$=\tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right)$

$=\tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right)$

$=\tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$

$=\tan^{-1}\frac{24}{7}=R.H.S$

4. Prove $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$

Solution:

Let $\sin^{-1}\frac{8}{17}=x$

$\sin x=\frac{8}{17}$

$\Rightarrow \cos x=\sqrt{1-\left(\frac{8}{17}\right)^2}$

$=\sqrt{\frac{225}{289}}$

$=\frac{15}{17}$

$\therefore \tan x=\frac{8}{15}$

$\Rightarrow x=\tan^{-1}\frac{8}{15}$

$\therefore \sin^{-1}\frac{8}{17}=\tan^{-1}\frac{8}{15}$ ......(1)

Now, let $\sin^{-1}\frac{3}{5}=y$

$\Rightarrow \sin y=\frac{3}{5}$

$\Rightarrow \cos y=\sqrt{1-\left(\frac{3}{5}\right)^2}$

$=\sqrt{\frac{16}{25}}$

$=\frac{4}{5}$

$\therefore \tan y=\frac{3}{5}$

$$\Rightarrow y = \tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad ......(2)$$

$$\text{L.H. S } = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4} \qquad \text{[using (1) and (2)]}$$

$$= \tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}$$

$$= \tan^{-1}\left(\frac{32+45}{60-24}\right)$$

$$= \tan^{-1}\frac{77}{36} = R.\text{H.S}$$

5. Prove $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Solution:

Let $\cos^{-1}\frac{4}{5} = x$

$$\Rightarrow \cos x = \frac{4}{5}$$

$$\Rightarrow \sin x = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

$$\therefore \tan x = \frac{3}{4}$$

$$\Rightarrow x = \tan^{-1}\frac{3}{4}$$

$\therefore \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$ ....(1)

Now, let $\cos^{-1}\frac{12}{13} = y$

$\Rightarrow \cos y = \frac{12}{13}$

$\Rightarrow \sin y = \frac{5}{13}$

$\therefore \tan y = \frac{5}{12}$

$\Rightarrow y = \tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12}$ ....(2)

Let $\cos^{-1}\frac{33}{65} = z$

$\Rightarrow \cos\ z = \frac{33}{65}$

$\Rightarrow \cos\ z = \frac{56}{65}$

$\therefore \tan z = \frac{56}{33}$

$\Rightarrow z = \tan^{-1}\frac{56}{33}$

$\therefore \cos^{-1}\frac{33}{65} = \tan^{-1}\frac{56}{33}$ .....(3)

In $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

L.H.S $= \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}$

$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12}$ [Using (1) and (2)]

$= \tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}$

$= \tan^{-1}\frac{36+20}{48-15}$

$= \tan^{-1}\frac{56}{33}$

$= \tan^{-1}\frac{56}{33}$ [Using (3)]

= R. H. S

6. Prove $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

Solution:

Let $\sin^{-1}\frac{3}{5} = x$

$\Rightarrow \sin x = \frac{3}{5}$

$\Rightarrow \cos x = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\therefore \tan x = \frac{3}{4}$

$\Rightarrow x = \tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$ ....(1)

Now, let $\cos^{-1}\frac{12}{13} = y$

$$\Rightarrow \cos = \frac{12}{13}$$

$$\Rightarrow \sin y = \frac{12}{13}$$

$$\therefore \tan y = \frac{5}{12}$$

$$\Rightarrow y = \tan^{-1}\frac{5}{12}$$

$$\therefore \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12} \qquad ......(2)$$

$$\text{Let } \sin^{-1}\frac{56}{65} = z$$

$$\Rightarrow \sin z = \frac{56}{65}$$

$$\Rightarrow \cos z = \frac{33}{65}$$

$$\therefore \tan z = \frac{56}{33}$$

$$\Rightarrow z = \tan^{-1}\frac{56}{33}$$

$$\therefore \sin^{-1}\frac{56}{65} = \tan^{-1}\frac{56}{33} \qquad ......(3)$$

$$\text{In } \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$$

$$\text{L.H.S} = \therefore \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$$

$$= \therefore \tan^{-1}\frac{5}{12} + \tan^{-}\frac{3}{4} \qquad \text{[Using (1) and (2)]}$$

$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}.\frac{3}{4}}\right)$$

$$= \tan^{-1}\left(\frac{20+36}{48-15}\right)$$

$$= \tan^{-1}\left(\frac{56}{33}\right)$$

$$= \sin^{-1}\frac{56}{33} = R.H.S \qquad \text{[Using (3)]}$$

7. Prove $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Solution:

Let $\sin^{-1}\frac{5}{13} = x$

$$\Rightarrow \sin x = \frac{5}{13}$$

$$\Rightarrow \cos x = \frac{12}{13}$$

$$\therefore \tan x = \frac{5}{12}$$

$$\Rightarrow x = \tan^{-1}\frac{5}{12} \qquad .....(1)$$

$$\therefore \sin^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12} \qquad ......(1)$$

Let $\cos^{-1}\frac{3}{5} = y$

$$\Rightarrow \cos y = \frac{3}{5}$$

$$\Rightarrow \sin y = \frac{4}{5}$$

$$\therefore \tan y = \frac{4}{3}$$

$$\Rightarrow y = \tan^{-1}\frac{4}{3}$$

$$\therefore \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{4}{3} \qquad ....(2)$$

Using (1) and (2) we have

$$\text{R.H.S } = \sin^{-1}\frac{5}{12} + \cos^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{5}{12} + tam^{-1}\frac{4}{3}$$

$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right)$$

$$= \tan^{-1}\left(\frac{15+48}{36-20}\right)$$

$$= \tan^{-1}\frac{63}{16}$$

= L. H. S

8. Prove $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Solution:

Let $= \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

$$= \tan^{-1}\left(\frac{7+5}{35-1}\right) + \tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$= \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right)$$

$$= \tan^{-1}1$$

$$= \frac{\pi}{4} = R.H.S$$

9. Prove $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in [0,1]$

Solution:

Let $x = \tan^2\theta$

$$\Rightarrow \sqrt{x} = \tan\theta$$

$$\Rightarrow \theta = \tan^{-1}\sqrt{x}$$

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

$$\text{R.H.S} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$= \frac{1}{2}\times 2\theta$$

$$= \theta$$

$$= \tan^{-1}\sqrt{x}$$

$= L.H.S$

10. Prove $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2}, x\in\left(0,\frac{\pi}{4}\right)$

Solution:

Here, $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)^2}$ (by rationalizing)

$=\frac{(1+\sin x)(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$

$=\frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x}=\frac{1+\cos x}{\sin x}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$=\cot\frac{x}{2}$

$\therefore$ L.H.S $=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$=\cot^{-1}\left(\cot\frac{x}{2}\right)$

$=\frac{x}{2}=R.H.S$

11. Prove $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, -\frac{1}{\sqrt{2}}\le x\le 1$

Solution:

Put $x=\cos 2\theta$

$\Rightarrow \theta=\frac{1}{2}\cos^{-1}x$

L.H.S $=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2 2\theta}-\sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 2\theta}+\sqrt{2\sin^2 \theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}1-\tan^{-1}(\tan\theta)$$

$$= \frac{\pi}{4}-\theta = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = R.H.S$$

12. Prove $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution:

L.H.S $= \frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$

$$= \frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \qquad ....(1)$$

Now, let $\cos^{-1} = \frac{1}{3} = x$

$$\Rightarrow \cos x = \frac{1}{3}$$

$$\Rightarrow \sin x=\sqrt{1-\left(\frac{1}{3}\right)^{2}}=\frac{2\sqrt{2}}{3}$$

$$\therefore x=\sin^{-1}\frac{2\sqrt{2}}{3}$$

$$\Rightarrow \cos^{-1}\frac{1}{3}$$

$$\Rightarrow \sin^{-1}\frac{2\sqrt{2}}{3}$$

$$\therefore L.H.S=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}=R.H.S$$

13. Solve $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

Solution:

Given, $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^{2}x}\right)=\tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \frac{2\cos x}{1-\cos^{2}x}=2\operatorname{cosec} x$$

$$\Rightarrow \frac{2\cos x}{\sin^{2}x}=\frac{2}{\sin x}$$

$$\Rightarrow \cos x=\sin x$$

$$\Rightarrow \tan x=1$$

$$\Rightarrow x=\frac{\pi}{4}$$

14. Solve $\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x,(x>0)$

Solution:

Given, $\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$

$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x$

$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$

$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$

$\Rightarrow x=\tan\frac{\pi}{6}$

$\therefore x=\frac{1}{\sqrt{3}}$

15. Solve $\sin\left(\tan^{-1}x\right),|x|<1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

Solution:

Let $\tan y=x$

$\Rightarrow \sin y=\frac{x}{\sqrt{1+x^2}}$

Let $\tan^{-1}x=y$

$\therefore y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\Rightarrow \tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\therefore \sin\left(\tan^{-1}x\right)=\sin\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)=\frac{x}{\sqrt{1+x^2}}$

16. Solve $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

Solution:

$$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1}x=\frac{\pi}{2}-\sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1}x=\cos^{-1}(1-x)$$

Let $\sin^{-1}x=\theta$

$$\Rightarrow \sin\theta=x$$

$$\Rightarrow \cos\theta=\sqrt{1-x^2}$$

$$\therefore \theta=\cos^{-1}\left(\sqrt{1-x^2}\right)$$

$$\therefore \sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)$$

From equation (1), we have

$$-2\cos^{-1}\left(\sqrt{1-x^2}\right)=\cos^{-1}(1-x)$$

Put $x=\sin y$

$$-2\cos^{-1}\left(\sqrt{1-\sin^2 y}\right)=\cos^{-1}(1-\sin y)$$

$$\Rightarrow -2\cos^{-1}(\cos y)=\cos^{-1}(1-\sin y)$$

$$\Rightarrow -2y=\cos^{-1}(1-\sin y)$$

$$\Rightarrow 1-\sin y=\cos(-2y)=\cos 2y$$

$$\Rightarrow 1-\sin y=1-2\sin^2 y$$

$$\Rightarrow 2\sin^2 y-\sin y=0$$

$$\Rightarrow \sin y(2\sin y - 1) = 0$$

$$\Rightarrow \sin y = 0 \; or \; \frac{1}{2}$$

$$\therefore x = 0 \; or \; x = \frac{1}{2}$$

But, where $x = \frac{1}{2}$, does not satisfy the equation

Thus, x = 0 is the only solution

17. Solve $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{-3\pi}{4}$

Solution:

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$$

$$= \tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x(x+y) - y(x-y)}{y(x+y)}}{\frac{y(x+y) + x(x-y)}{y(x+y)}}\right]$$

$$= \tan^{-1}\left[\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right]$$

$$= \tan^{-1}\left[\frac{x^2 + y^2}{x^2 + y^2}\right] = \tan^{-1} 1 = \frac{\pi}{4}$$