Chapter: 2. Inverse trigonometric functions.

Exercise 2.2

1. Prove $3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right), x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

Solution: Suppose that $x=\sin\theta \Rightarrow \theta=\sin^{-1}x$

We have,

$$\begin{aligned}\sin^{-1}\left(3x-4x^3\right)&=\sin^{-1}\left(3\sin\theta-4\sin^3\theta\right)\\&=\sin^{-1}\left(\sin 3\theta\right)\\&=3\theta\\&=3\sin^{-1}x\end{aligned}$$

Therefore, $3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right)$

2. Prove that $3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right), x\in\left[\frac{1}{2},1\right]$

Solution: Let $x=\cos\theta \Rightarrow \cos^{-1}x=\theta$

Consider the right hand side

$$\begin{aligned}\cos^{-1}\left(4x^3-3x\right)&=\cos^{-1}\left(4\cos^3\theta-3\cos\theta\right)\\&=\cos^{-1}\left(\cos 3\theta\right)\\&=3\theta\\&=3\cos^{-1}x\end{aligned}$$

Therefore, $3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right)$

3. Prove $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

Solution: Consider the expression $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$

Use the formula $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}.\frac{7}{24}}$$

$$=\tan^{-1}\frac{48+77}{264-14}$$

$$=\tan^{-1}\left(\frac{125}{250}\right)$$

$$=\tan^{-1}\left(\frac{1}{2}\right)$$

Therefore, $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\left(\frac{1}{2}\right)$

4. Prove $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution:

L.H.S $=2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

$$=\tan^{-1}\frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{1}{\left(\frac{3}{4}\right)}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}$$

$$=\tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right)$$

$$=\tan^{-1}\frac{31}{17}$$

$= \text{R. H. S}$

5. Write the function in the simplest from: $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution:

Given, $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$

Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2-1}}{x}=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$=\frac{\theta}{2}$$

$$=\frac{1}{2}\tan^{-1}x$$

6. Write the function in the simplest form: $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x|>1$

Solution:

Given, $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x|>1$

Put $x = \operatorname{cosec}\theta \Rightarrow \theta = \operatorname{cosec}^{-1} x$

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\operatorname{cosec}^2\theta - 1}}$$

$$= \tan^{-1}\left(\frac{1}{\cot\theta}\right)$$

$$= \tan^{-1}(\tan\theta)$$

$$= \theta$$

$$= \operatorname{cosec}^{-1} x$$

$$= \frac{\pi}{2} - \sec^{-1} x$$

7. Write the function in the simplest from: $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x < \pi$

Solution:

Given, $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x < \pi$

$$= \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

8. Write the function in the simple form: $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), 0 < x < \pi$

Solution:

Given, $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

$$= \tan^{-1}\left(\frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}}\right)$$

$$= \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$$

$$= \tan^{-1}(1) - \tan^{-1}(\tan x)$$

$$= \frac{\pi}{4} - x$$

9. Write the function in the simplest form: $\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}, |x| < a$

Solution:

Given, $\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}$

Put $x = a\sin\theta$

$$\Rightarrow \frac{x}{a} = \sin\theta \Rightarrow \sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \tan^{-1}\frac{x}{\sqrt{a^2 - x^2}} = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1 - \sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta)$$

$= \theta$

$= \sin^{-1}\dfrac{x}{a}$

10. Write the function in the simplest form: $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right), a > 0; \dfrac{-a}{\sqrt{3}} \le x \le \dfrac{a}{\sqrt{3}}$

Solution:

Given, $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$

Put $x = a\tan\theta$

$\Rightarrow \dfrac{x}{a} = \tan\theta$

$\Rightarrow \theta = \tan^{-1}\dfrac{x}{a}$

$= \tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right) = \tan^{-1}\left(\dfrac{3a^2.a\tan\theta - a^3\tan^3\theta}{a^3 - 3a.a^2\tan^2\theta}\right)$

$= \tan^{-1}\left(\dfrac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)$

$= 3\theta$

$= 3\tan^{-1}\dfrac{x}{a}$

11. Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right]$

Solution:

Let $\sin^{-1}\dfrac{1}{2} = x$

$\Rightarrow \sin x = \dfrac{1}{2} = \sin\left(\dfrac{\pi}{6}\right)$

$\therefore \sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{6}$

$$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right] = \tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\cos\frac{\pi}{3}\right]$$

$$= \tan^{-1}\left[2\times\frac{1}{2}\right]$$

$$= \tan^{-1}1$$

$$= \frac{\pi}{4}$$

12. Find the value of $\cot\left(\tan^{-1}a + \cot^{-1}a\right)$

Solution:

Given, $\cot\left(\tan^{-1}a + \cot^{-1}a\right)$

$$= \cot\left(\frac{\pi}{2}\right)$$

$$= 0$$

13. Find the value of $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0 \text{ and } xy < 1$

Solution:

Let $x = \tan\theta$

$$\Rightarrow \theta = \tan^{-1}x$$

$$\therefore \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}\left(\sin 2\theta\right)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

Let $y=\tan\phi$

$\Rightarrow \phi=\tan^{-1} y$

$\therefore \cos^{-1}\frac{1-y^2}{1+y^2}=\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)$

$=\cos^{-1}(\cos 2\phi)$

$=2\phi$

$=2\tan^{-1} y$

$\therefore \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$=\tan\frac{1}{2}\left[2\tan^{-1}x+2\tan^{-1}y\right]$

$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$=\frac{x+y}{1-xy}$

14. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x

Solution:

Given, $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$

$\Rightarrow \frac{1}{5}\times x+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$

$\Rightarrow \frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \qquad ....(1)$

Now, let $\sin^{-1}\frac{1}{5}=y$

$$\Rightarrow \sin^{-1}\frac{1}{5}=y$$

$$\Rightarrow \sin y=\frac{1}{5}$$

$$\Rightarrow \cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}=\frac{2\sqrt{6}}{5}$$

$$\Rightarrow y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$$

$$\therefore \sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \qquad ....(2)$$

Let $\cos^{-1}x=z$

$$\Rightarrow \cos z=x$$

$$\Rightarrow \sin z=\sqrt{1-x^2}$$

$$\Rightarrow z=\sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$\therefore \cos^{-1}=\sin^{-1}\left(\sqrt{1-x^2}\right) \qquad ....(3)$$

From (1), (2) and (3) we have

$$\frac{x}{5}+\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=1$$

$$\Rightarrow \frac{x}{5}+\frac{2\sqrt{6}}{5}\sqrt{1-x^2}=1$$

$$\Rightarrow x+2\sqrt{6}\sqrt{1-x^2}=5$$

On squaring both sides, we get

$$(4)(6)\left(1-x^2\right)=25+x^2-10x$$

$$\Rightarrow 24-24x^2=25+x^2-10x$$

$$\Rightarrow 25x^2-10x+1=0$$

$$\Rightarrow (5x-1)^2=0$$

$$\Rightarrow (5x-1)=0$$

$$\Rightarrow x=\frac{1}{5}$$

15. If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of x

Solution:

Given, $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$\Rightarrow 4-2x^2=3$

$\Rightarrow 2x^2=4-3=1$

$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$

16. Find the values of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Solution:

Given, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$\Rightarrow \sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left[\sin\left(\pi-\frac{2\pi}{3}\right)\right]$

$=\sin^{-1}\left(\sin\frac{\pi}{3}\right)$

$=\frac{\pi}{3}$

17. Find the values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Solution:

Given, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$\Rightarrow \tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right]$

$=\tan^{-1}\left[-\tan\left(\pi-\frac{\pi}{4}\right)\right]$

$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)$

$$= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]$$

$$= -\frac{\pi}{4}$$

18. Find the values of $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

Solution:

Let $\sin^{-1}\frac{3}{5}=x$

$$\Rightarrow \sin x=\frac{3}{5}$$

$$\Rightarrow \cos x=\sqrt{1-\sin^2 x}=\frac{4}{5}$$

$$\Rightarrow \sec x=\frac{5}{4}$$

$$\therefore \tan x=\sqrt{\sec^2 x-1}=\sqrt{\frac{25}{16}-1}=\frac{3}{4}$$

$$\therefore x=\tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4} \qquad ....(i)$$

Now, $\therefore \cot^{-1}\frac{3}{2}=\tan^{-1}\frac{2}{3} \qquad ....(ii)$

Thus, $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

$$= \tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right) \qquad \text{[using (i) and (ii)]}$$

$$= \tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}.\frac{2}{3}}\right)$$

$$= \tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$= \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}$$

19. Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Solution:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{-7\pi}{6}\right)$$

$$= \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right]$$

$$= \frac{5\pi}{6}$$

20. Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Solution:

Let $\sin^{-1}\left(-\frac{1}{2}\right) = x$

$$\Rightarrow \sin x = \frac{-1}{2}$$

$$= -\sin\frac{\pi}{6}$$

$$= \sin\left(\frac{-\pi}{6}\right)$$

Range of the principal value of $\sin^{-1}x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$$

$$= \sin\left(\frac{3\pi}{6}\right)$$

$$= \sin\left(\frac{\pi}{2}\right) = 1$$