Chapter 3: Matrices.

Exercise Miscellaneous

1. Let $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, show that $(aI+bA)^n=a^nI+na^{n-1}bA,$ where I is the identity matrix of order 2 and $n\in N$

Solution:

Given, $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

By using the principle of mathematical induction

For n = 1

$P(1):(aI+bA)=aI+ba^{\circ}A=aI+bA$

Therefore, the result is true for n = 1

Let the result be true for n = k

That is, $P(k):(aI+bA)^k=a^kI+ka^{k-1}bA$

Now, we have to prove that the result is true for $n=k+1$

$$(aI+bA)^{k-1}=(aI+bA)^k(aI+bA)$$

$$=\left(a^kI+ka^{k-1}bA\right)(aI+bA)$$

$$=a^{k-1}+ka^kbAI+a^kbIA+ka^{k-1}b^2A^2$$

$$=a^{k-1}I+(k+1)a^kbA+ka^{k-1}b^2A^2 \qquad ....(1)$$

Now, $A^2=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$$

From (1), we have

$$(aI+bA)^{k+1}=a^{k+1}+(k+1)a^{k}bA+0$$

$$=a^{k+1}+(k+1)a^{k}bA$$

Thus, the result is true for $n=k+1$

By the principal of mathematical induction, we have

$$(aI+bA)^{n}=a^{n}I+na^{n-1}bA \text{ where } A=\begin{bmatrix}0&1\\0&0\end{bmatrix}, n\in N$$

2. If $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$, prove that $A^{n}\begin{bmatrix}3^{n-1}&3^{n-1}&3^{n-1}\\3^{n-1}&3^{n-1}&3^{n-1}\\3^{n-1}&3^{n-1}&3^{n-1}\end{bmatrix}, n\in N$

Solution:

Given, $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

By using the principles of mathematical induction

For n = 1, we have

$$P(1)=\begin{bmatrix}3^{1-1}&3^{1-1}&3^{1-1}\\3^{1-1}&3^{1-1}&3^{1-1}\\3^{1-1}&3^{1-1}&3^{1-1}\end{bmatrix}=\begin{bmatrix}3^{0}&3^{0}&3^{0}\\3^{0}&3^{0}&3^{0}\\3^{0}&3^{0}&3^{0}\end{bmatrix}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}=A$$

Thus, the result is true for n = 1

Let the result be true for n = k

$$P(k): A^{k}=\begin{bmatrix}3^{k-1}&3^{k-1}&3^{k-1}\\3^{k-1}&3^{k-1}&3^{k-1}\\3^{k-1}&3^{k-1}&3^{k-1}\end{bmatrix}$$

Now, we have to prove that the result is true for $n=k+1$

Now, $A^{k+1} = A.A^k$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \\ 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \\ 3.3^{k-1} & 3.3^{k-1} & 3.3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3.3^{(k+1)-1} & 3.3^{(k+1)-1} & 3.3^{(k+1)-1} \\ 3.3^{(k+1)-1} & 3.3^{(k+1)-1} & 3.3^{(k+1)-1} \\ 3.3^{(k+1)-1} & 3.3^{(k+1)-1} & 3.3^{(k+1)-1} \end{bmatrix}$$

Thus, the result is true for $n = k+1$

By the principal of mathematical induction, we have

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, n \in N$$

3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where n is any positive integer

Solution:

Given, $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

By using the principle of mathematical induction

For n = 1, we have

$$P(1): A^1 = \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} A$$

Thus, the result is true for n = 1

Let the result be true for n = k

$$P(k): A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}, n \in N$$

Now, we have to prove that the result is true for $n = k+1$

$$A^{k+1} = A^k.A$$

$$= \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1+2k)-4k & -4(1+2k)+4k \\ 3k+1-2k & -4k-1(1-2k) \end{bmatrix}$$

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ 1+k & 1-2(k+1) \end{bmatrix}$$

Thus, the result is true for $n = k+1$

By the principal of mathematical induction, we have

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}, n \notin N$$

4. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix

Solution:

Given, A and B are symmetric matrices. Therefore, we have

$A' = A \text{ and } B' = B$ ....(1)

Now, $(AB - BA)' = (AB)' - (BA)'$

$= B'A' - A'B'$

$= BA - AB$ [Using (1)]

$$=-(AB-BA)$$

$$\therefore (AB-BA)'=-(AB-BA)$$

Hence, $(AB-BA)$ is a skew- symmetric matrix

5. Show that the matrix $B'AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.

Solution:

Let A is a symmetric matrix, then $A'=A$ .....(1)

$$(B'AB)'=\{B'(AB)\}'$$

$$=(AB)'(B)'$$

$$=B'A'(B)$$

$$=B'(AB) \quad \text{[Using (1)]}$$

$$\therefore (B'AB)'=B'AB$$

Thus, if A is symmetric matrix, then $B'AB$ is a symmetric matrix.

Let A is a skew – symmetric matrix

Then, $A'=A$

$$(B'AB)'=[B'(AB)]'=(AB)'(B)'$$

$$=(B'A')B=B'(-A)B$$

$$=-B'AB$$

$$\therefore (B'AB)'=-B'AB$$

Thus, A is skew – symmetric matrix then $B'AB$ is a skew – symmetric matrix

Therefore, if A is a symmetric or skew – symmetric matrix, then $B'AB$ is a symmetric or skew – symmetric matrix accordingly.

6. Solve system of linear equations, using matrix method.

$$2x - y = -2$$
$$3x + 4y = 3$$

Solution:

The given system of equation can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Now, $|A| = 8 + 3 = 11 \neq 0$

Thus, A is non – singular. Therefore, its inverse exists

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Thus, $x = \frac{-5}{11}$ and $y = \frac{12}{11}$

7. For what values of $x, \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$?

Solution:

$$\text{Given, } \begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}1+4+1 & 2+0+0 & 0+2+2\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}6 & 2 & 4\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=0$$

$$\Rightarrow \left[6(0)+2(2)+4(x)\right]=0$$

$$\Rightarrow [4+4x]=[0]$$

$$\therefore 4+4x=0$$

$$\Rightarrow x=-1$$

Thus, the required value of x is – 1

8. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I=0$

Solution:

$$\text{Given, } A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$\therefore A^2=A.A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}3(3)+1(-1) & 3(1)+1(2)\\ -1(3)+2(-1) & -1(1)+2(2)\end{bmatrix}$$

$$=\begin{bmatrix}9-1 & 3+2\\ -3-2 & -1+4\end{bmatrix}=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$$

$$L.H.S=A^2-5A+7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0 = R.H.S$$

$$\therefore A^2 - 5A + 7I = 0$$

9. Find X, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$

Solution:

Given, $\begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} x-2 & -10 & 2x-8 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \left[x(x-2) - 40 + 2x - 8\right] = 0$$

$$\Rightarrow \left[x^2 - 2x - 40 + 2x - 8\right] = [0]$$

$$\Rightarrow \left[x^2 - 48\right] = [0]$$

$$\Rightarrow x^2 - 48 = 0$$

$$\Rightarrow x^2 = 48$$

$$\Rightarrow x = \pm 4\sqrt{3}$$

10. A manufacture produces three products X, Y, Z which he sells in two markets. Annual sales are indicated below

Market	Products		
I	10000	2000	18000
II	6000	20000	8000

a) If unit sale prices of X, Y and Z are Rs 2.50, Rs 1.50 and Rs 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

b) If the unit costs of the above three commodities are Rs 2.00, Rs 1.00 and 50 paise respectively. Find the gross profit.

Solution:

a. Here, the total revenue in market I can be represented in the form of matrix as

$$\begin{bmatrix}10000 & 2000 & 18000\end{bmatrix}\begin{bmatrix}2.50\\1.50\\1.00\end{bmatrix}$$

$= 10000\times 2.50 + 2000\times 1.50 + 18000\times 1.00$

$= 25000 + 3000 + 18000$

$= 46000$

And, the total revenue in market II can be represented in the form of a matrix as

$$\begin{bmatrix}6000 & 20000 & 8000\end{bmatrix}\begin{bmatrix}2.50\\1.50\\1.00\end{bmatrix}$$

$= 6000\times 2.50 + 20000\times 1.50 + 8000\times 1.00$

$= 15000 + 30000 + 8000$

$= 53000$

Thus, the total revenue in market I is Rs 46000 and the same in market II is Rs 53000

b. Here, the total cost prices of all the products in the market I can be represented in the form of a matrix as

$$\begin{bmatrix}10000 & 2000 & 180000\end{bmatrix}\begin{bmatrix}2.50\\1.00\\0.50\end{bmatrix}$$

$= 10000\times 2.00 + 2000\times 1.00 + 18000\times 0.50$

$= 20000 + 2000 + 9000 = 31000$

As, the total revenue in market I is Rs 46000, the gross profit in this market is

Rs 46000 –Rs 31000 = Rs 150000

The total cost prices of all the products in market II can be represented in the form of a matrix as

$$\begin{bmatrix} 6000 & 2000 & 8000 \end{bmatrix} \begin{bmatrix} 2.00 \\ 1.00 \\ 0.50 \end{bmatrix}$$

$= 6000 \times 2.00 + 20000 \times 1.00 + 8000 \times 0.50$

$= 12000 + 20000 + 4000$

$= 36000$

Since the total revenue in market II is Rs 53000, the gross profit in this market is Rs 53000 – Rs 36000 = Rs 170000

11. Find the matrix X so that $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

Solution:

Given, $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

Here, X has to be a 2×2 matrix

Now, let $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Thus, we have $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a+4c & 2a+5c & 3a+6c \\ b+4d & 2b+5d & 3b+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

Comparing the corresponding elements of two matrices, we have

$a + 4c = -7, \quad 2a + 5c = -8 \quad 3a + 6c = -9$

$b + 4d = 2, \quad 2b + 5d = 4, \quad 3b + 6d = 6$

Now, $a + 4c = -7 \Rightarrow a = -7 - 4c$

$\therefore 2a + 5c = -8 \Rightarrow -14 - 8c + 5c = -8$

$\Rightarrow -3c = 6$

$\Rightarrow c=-2$

$\therefore a=-7-4(-2)=-7+8=1$

Now, $b+4d=2 \Rightarrow b=2-4d$

$\therefore 2b+5d=4 \Rightarrow 4-8d+5d=4$

$\Rightarrow -3d=0$

$\Rightarrow d=0$

$\therefore b=2-4(0)=2$

$\therefore a=1, b=2, c=-2, d=0$

Thus, the required matrix X is $\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

12. If A and B are square matrices of the same order such that AB = BA, then prove by induction that $AB^n=B^nA$. Further, prove that $(AB)^n=A^nB^n$ for all $n \in N$

Solution:

Given, A and B are square matrices of the same order such that AB = BA

For n = 1, we have

$P(1): AB=BA$ [Given]

$\Rightarrow AB^1=B^1A$

Therefore, the result is true for n = 1

Let the result be true for n = k

$P(k): AB^k=B^kA$ ....(1)

Now, we have to prove that the result is true for $n=k+1$

$AB^{k+1}=AB^k.B$

$=(B^kA)B$ [By (1)]

$= B^k(AB)$

$= B^k(BA)$

$= (B^k B)A$

$= B^{k+1}A$

Therefore, the result is true for $n = k+1$

By the principle of mathematical induction, we have $AB^n = B^n A, n \in N$

13. Choose the correct answer in the following questions.

If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2 = I$ then

$(A) 1+\alpha^2+\beta\gamma = 0$

$(B) 1-\alpha^2+\beta\gamma = 0$

$(C) 1-\alpha^2-\beta\gamma = 0$

$(D) 1+\alpha^2-\beta\gamma = 0$

Solution:

Given, $A\begin{bmatrix} \alpha & \beta \\ y & -\alpha \end{bmatrix}$

$$\therefore A^2 = A.A = \begin{bmatrix} \alpha & \beta \\ y & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ y & -\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2+\beta\gamma & \alpha\beta-\alpha\beta \\ \alpha\gamma-\alpha\gamma & \beta\gamma+\alpha^2 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2+\beta\gamma & 0 \\ 0 & \beta\gamma+\alpha^2 \end{bmatrix}$$

Now, $A^2 = I$

$$\Rightarrow \begin{bmatrix} \alpha^2+\gamma & 0 \\ 0 & \beta\gamma+\alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$$\alpha^2+\beta\gamma=1$$

$$\Rightarrow \alpha^2+\beta\gamma-1=0$$

$$\Rightarrow 1-\alpha^2-\beta\gamma=0$$

14. If the matrix A is both symmetric and skew symmetric, then

(A) A is diagonal matrix

(B) A is a zero matrix

(C) A is a square matrix

(D) None of these

Solution:

If A is both symmetric and skew – symmetric matrix, then

$$A'=\text{A and A}'=-\text{A}$$

$$A'=A$$

$$A'=-A$$

$$\Rightarrow A=-A$$

$$\Rightarrow A+A=0$$

$$\Rightarrow 2A=0$$

$$\Rightarrow A=0$$

15. If A is square matrix such that $A^2=A$, then $(I+A)^3-7A$ is equal to

(A) A (B) $I-A$ (C) I (D) 3A

Solution:

$$(I+A)^3-7A=I^3+A^3+3I^2A+3A^2I-7A$$

$$=I+A^3+3A+3A^2-7A$$

$$=I+A^2.A+3A+3A-7A \qquad [A^2=A]$$

$$= I + A.A - A$$

$$= I + A^2 - A$$

$$= I + A - A$$

$$= \mathrm{I}$$

$$\therefore (I + A)^3 - 7A = I$$