Chapter 4: Determinants

Exercise. Miscellaneous

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of

Solution:

Given, $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

$$= x\left(x^2-1\right)-\sin\theta\left(-x\sin\theta-\cos\theta\right)+\cos\theta\left(-\sin\theta+x\cos\theta\right)$$

$$= x^3-x+x\sin^2\theta+\sin\theta\cos\theta-\sin\theta\cos\theta+x\cos^2\theta$$

$$= x^3-x+x\left(\sin^2\theta+\cos^2\theta\right)$$

$$= x^3-x+x$$

$$= x^3$$

Therefore, Δ is independent of θ.

2. Without expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

Solution:

L.H.S $= \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Applying $R_1 = aR_1, R_2 \to bR_2$ $and R_3 \to cR_3$

$$=\frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

$$=\frac{1}{abc}abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$ [Taking out abc from C_3]

$$=\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

Applying $C_1 \leftrightarrow C_3$ *and* $C_2 \leftrightarrow C_3$

$$=\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

=R.H.S

Hence proved

3. Evaluate $=\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\cos\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Solution:

Given, $\Delta=\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\cos\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along C_3

$$\Delta=-\sin\alpha\left(-\sin\alpha\sin^2\beta+\cos^2\beta\sin a\right)+\cos\alpha\left(\cos\alpha\cos^2\beta+\cos\alpha\sin^2\beta\right)$$

$$=\sin^2\alpha(1)+\cos^2\alpha(1)$$

$$=1$$

4. If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Show that either $a+b+c=0$ or $a=b=c$

Solution:

Given, $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1

$$\Delta = 2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right]$$

$$= 2(a+b+c)\left[-b^2-c^2+2bc-bc+ba+ac-a^2\right]$$

$$= 2(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right] = 0$$

$$\Rightarrow Either\ a+b+c=0, or\ ab+bc+ca-a^2-b^2-c^2=0$$

Now, $ab+bc+ca-a^2-b^2-c^2=0$ b

$$\Rightarrow -2ab-2bc-2ca+2a^3+2b^3+2c^3=0$$

$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$

$$\Rightarrow (a-b)=(b-c)^2=(c-a)^2=0 \quad \left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$$

$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$

$$\Rightarrow a=b=c$$

Therefore, if $\Delta=0$, then either $a+b+c=0$ or $a=b=c$

5. Solve the equations $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0, a\neq 0$

Solution:

Given, $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$

Applying $R_1 \to R_1+R_2+R_3$

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix}=0$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & a & x \\ x & x & a \end{vmatrix}=0$$

Expanding along R_1

$\Rightarrow a^2(3x+a)=0$

But $a \neq 0$

Therefore, we have

$3x+a=0$

$\Rightarrow x=-\frac{a}{3}$

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution:

Given, $\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking out a, b and c from C_1, C_2 and C_3

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

$$= 2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$

$$\Delta = 2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3

$$\Delta = 2ab^2c\left[a(c-a)+a(a+c)\right]$$

$$= 2ab^2c\left[ac-a^2+a^2+ac\right]$$

$$= 2ab^2c(2ac)$$

$$= 4a^2b^2c^2$$

Hence proved

7. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ verify that

(i) $[adjA]^{-1} = adj\left(A^{-1}\right)$

(ii) $\left(A^{-1}\right)^{-1} = A$

Solution:

Given, $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$\therefore |A| = 1(15-1)+2(-10-1)+1(-2-3) = 14-22-5 = -13$

Now, $A_{11} = 14, A_{12} = 11, A_{13} = -5$

$A_{21} = 11, A_{22} = 4, A_{23} = -3$

$A_{31} = -5, A_{32} = -3, A_{33} = -1$

$$\therefore adjA = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA)$$

$$= -\frac{1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

$(i)\ |adjA| = 14(-4-9)-11(-11-15)-5(-33+20)$

$= 14(-13)-11(-26)-5(-13)$

$= -182+286+65 = 169$

$$\text{Here, } adj(adjA) = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$\therefore [adjA]^{-1} = \frac{1}{|adjA|}(adj(adjA))$$

$$= \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & 65 \end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Now, $A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$

$$=\begin{bmatrix}-\frac{14}{13} & -\frac{11}{13} & \frac{5}{13}\\ -\frac{11}{3} & -\frac{4}{13} & \frac{3}{13}\\ \frac{5}{13} & \frac{3}{13} & \frac{1}{3}\end{bmatrix}$$

$$\therefore adj\left(A^{-1}\right)=\begin{bmatrix}-\frac{4}{169}-\frac{9}{169} & -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{33}{169}+\frac{20}{169}\\ -\left(-\frac{11}{69}-\frac{15}{169}\right) & -\frac{14}{169}-\frac{25}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right)\\ -\frac{33}{169}+\frac{20}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right) & \frac{56}{169}-\frac{121}{169}\end{bmatrix}$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Therefore, $[adjA]^{-1}=adj\left(A^{-1}\right)$

(ii) Since, $A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$

And $adjA^{-1}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$

Now, $|A^{-1}| = \left(\frac{1}{13}\right)^3 \left[-14\times(-13)+11\times(-26)+5\times(-13)\right] = \left(\frac{1}{13}\right)^3 \times(-169) = -\frac{1}{13}$

$$\therefore \left(A^{-1}\right) = \frac{adjA^{-1}}{|A^{-1}|}$$

$$= \frac{1}{\left(-\frac{1}{13}\right)} \times \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

= A

$$\therefore \left(A^{-1}\right)^{-1} = A$$

8. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Solution:

Given, $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ *and* $C_3 \to C_3 - C_1$

$$\Delta = 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along R_1

$$\Delta = 2(x+y)\left[-x^2 + y(x-y)\right]$$

$$= -2(x+y)\left(x^2 + y^2 - yx\right)\text{b}$$

$$= -2\left(x^3 + y^3\right)$$

9. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Solution:

Given, $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0` \\ 0 & 0 & x \end{vmatrix}$$

Expanding along C_1

$$\Delta = 1(xy - 0)$$

= xy

10. Using properties of determinants, prove that

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\alpha \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Solution:

Given, $\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\alpha \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$

Applying $R_2 \to R_2 - R_1 \; and \; R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta \\ \gamma-\alpha & \gamma^2-\alpha^2 & \alpha-\gamma \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 1 & \gamma+\alpha & -1 \end{vmatrix}$$

Appling $R_3 \to R_3 - R_2$

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 0 & \gamma-\beta & 0 \end{vmatrix}$$

Expanding along R_3

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\left[-(\gamma-\beta)(-\alpha-\beta-\gamma)\right]$$

$$= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)(\alpha+\beta+\gamma)$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

Hence proved

11. Using properties of determinants, prove that

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

Solution:

Given, $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1 \ and \ R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y-x & y^2-x^2 & p\left(y^3-x^3\right) \\ z-x & z^2-x^2 & p\left(z^3-x^3\right) \end{vmatrix}$$

$$=(y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^2 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 1 & z+x & p\left(z^2+x^2+xz\right) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\Delta = (y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 0 & z-y & p(z-y)(x+y+z) \end{vmatrix}$$

$$=(y-x)(z-x)(z-y)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 0 & 1 & p(x+y+z) \end{vmatrix}$$

Expanding along R_3

$$\Delta = (x-y)(y-z)(z-x)\left[(-1)(p)\left(xy^2+x^3+x^2y\right)+1+px^3+p(x+y+z)(xy)\right]$$

$$=(x-y)(y-z)(z-x)\left[-pxy^2-px^3-px^2y+1+px^3+px^2y+pxy^2+pxyz\right]$$

$$=(x-y)(y-z)(z-x)(1+pxyz)$$

Hence proved

12. Using properties of determinants, prove that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+ba+ca)$$

Solution:

Given, $\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - R_1$

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & a-b \\ 0 & a-c & 2c+a \end{vmatrix}$$

Expanding along C_1

$$\Delta = (a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right]$$

$$=(a+b+c)\left[4bc+2ab+2ac+a^2-a^2+ac+ba-bc\right]$$

$$=(a+b+c)(3ab+3bc+3ac)$$

$$=3(a+b+c)(ab+bc+ac)$$

Hence proved

13. Using properties of determinants, prove this $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}=1$

Solution:

Given, $\Delta=\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

Applying $R_2 \to R_1 - 2R_1$ *and* $R_3 \to R_3 - 3R_1$

$$\Delta=\begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we have

$$\Delta=\begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1

$$\Delta=1\begin{vmatrix} 1 & 2+p \\ 0 & 1 \end{vmatrix}$$

$$=1(1-0)$$

= 1

14. Using properties of determinants, prove that $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}=0$

Solution:

$$\text{Given, } \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$= \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta-\sin\alpha\sin\delta \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos\beta\cos\delta-\sin\beta\sin\delta \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta-\sin\gamma\sin\delta \end{vmatrix}$$

Applying $C_1 \to +C_1 + C_3$

$$\Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \cos\alpha\cos\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta-\sin\alpha\sin\delta \\ \cos\beta\cos\delta & \cos\beta\cos\delta & \cos\beta\cos\delta-\sin\beta\sin\delta \\ \cos\gamma\cos\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta-\sin\gamma\sin\delta \end{vmatrix}$$

Since, two columns C_1 *and* C_2 are identical

$\therefore \Delta = 0$

Hence proved.

15. Solve the system of the following equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}-\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Solution:

$$\text{Given, } \frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}+\frac{20}{z}=2$$

Let $\frac{1}{x}=p, \frac{1}{y}=q, \frac{1}{z}=r$

Then the given system of equations is as follows

$2p+3q+10r=4$

$4p-6q+5r=1$

$6p+9q+20r=1$

Let $A=\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X=\begin{bmatrix} p \\ q \\ r \end{bmatrix} \text{ and } B=\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Such that, this system can be written in the form of AX = B

Now, $|A|=2(120-45)-3(-80-30)+10(36+36)$

$=150+330+720$

$=1200$

Thus, A is non – singular

Therefore, its inverse exists.

Now, $A_{11}=75, A_{12}=110, A_{13}=72$

$A_{21}=150, A_{22}=100, A_{23}=0$

$A_{31}=75, A_{32}=100, A_{33}=-24$

$\therefore A^{-1}=\frac{1}{|A|}(adjA)$

$=\frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

Now, $X=A^{-1}B$

$$\Rightarrow \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\therefore p = \frac{1}{2}, \text{q} = \frac{1}{3} \text{ and } r = \frac{1}{5}$$

Thus, x =2, y=3 and z = 5

16. Choose the correct answer.

If a,b, c are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

A) 0 B) 1 C) X D) 2X

Solution:

Given, $\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+(a+c) \\ x+4 & x+5 & x+2c \end{vmatrix}$$ (Since a, b and c are in A.P., $2b = a+c$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$

$$\Delta = \begin{vmatrix} -1 & -1 & a-c \\ x+3 & x+4 & x+(a+c) \\ 1 & 1 & c-a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+a+c \\ 1 & 1 & c-a \end{vmatrix} = 0$$

17. Choose the correct answer.

If X, Y, Z are nonzero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

Given, $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$

$\therefore |A| = x(yz-0) = xyz \neq 0$

Now, $A_{11} = yz, A_{12} = 0, A_{13} = 0$

$A_{21} = 0, A_{22} = xz, A_{23} = 0$

$A_{31} = 0, A_{32} = 0, A_{33} = xy$

$$\therefore adjA=\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)$$

$$=\frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$=\begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xy}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$$

$$=\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

Thus (A) is the correct answer

18. Choose the correct answer.

Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0\le\theta\le 2n$, then

A) $Det(A)=0$ B) $Det(A)\in(2,\infty)$ C) $Det(A)\in(2,4)$

D) $Det(A)\in(2,4)$

Solution:

$$\text{Given, } A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$$

$$\therefore |A| = 1(1+\sin^2\theta) - \sin\theta(-\sin\theta+\sin\theta) + 1(\sin^2\theta+1)$$

$$=1+\sin^2\theta+\sin^2\theta+1$$

$$=2+2\sin^2\theta$$

$$=2(1+\sin^2\theta)$$

Now, $0 \le \theta \le 2\pi$

$$\Rightarrow 0 \le \sin\theta \le 1$$

$$\Rightarrow 0 \le \sin^2\theta \le 1$$

$$\Rightarrow 1 \le 1+\sin^2\theta \le 2$$

$$\Rightarrow 2 \le 2(1++\sin^2\theta) \le 4$$

$$\therefore Det\,(A)[2,4]$$

Thus, (D) is the correct answer