Chapter 4: Determinants

Exercise 4.4

1. Write Minors and Cofactors of the elements of following determinants

i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Solution:

i) Given, determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

minor of element a_{ij} is M_{ij}

$\therefore M_{11} = 3$

$M_{12} = 0$

$M_{21} = -4$

$M_{22} = 2$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^{2}(3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^{3}(0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^{3}(-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^{4}(2) = 2$

ii) Given, determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

minor of element a_{ij} is M_{ij}

$\therefore M_{11} = d$

$M_{12} = b$

$M_{21} = c$

$M_{22} = a$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = b$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$

2. i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Solution:

i) Given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

minor of element a_{ij} is M_{ij}

$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$$M_{21}=\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}=0$$

$$M_{22}=\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}=1$$

$$M_{23}=\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}=0$$

$$M_{31}=\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix}=0$$

$$M_{32}=\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}=0$$

$$M_{33}=\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}=1$$

Cofactor of a_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$

$$A_{11}=(-1)^{1+1}M_{11}=1$$

$$A_{12}=(-1)^{1+2}M_{12}=0$$

$$A_{13}=(-1)^{1+3}M_{13}=0$$

$$A_{21}=(-1)^{2+1}M_{21}=0$$

$$A_{22}=(-1)^{2+2}M_{22}=1$$

$$A_{23}=(-1)^{2+3}M_{23}=0$$

$$A_{31}=(-1)^{3+1}M_{31}=0$$

$$A_{32}=(-1)^{3+2}M_{32}=0$$

$$A_{33}=(-1)^{3+3}M_{33}=1$$

ii) The given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Minor of element a_{ij} is M_{ij}

$$M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11$$

$$M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$$

$$M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 4 \end{vmatrix} = 3 - 0 = 3$$

$$M_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$$

$$M_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

$$M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$M_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$$

$$M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$$

$$M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$

$$A_{11} = (-1)^{1+1} M_{11} = 11$$

$$A_{12} = (-1)^{1+2} M_{12} = -6$$

$$A_{13}=(-1)^{1+3}M_{13}=3$$

$$A_{21}=(-1)^{2+1}M_{21}=4$$

$$A_{22}=(-1)^{2+2}M_{22}=2$$

$$A_{23}=(-1)^{2+3}M_{23}=-1$$

$$A_{31}=(-1)^{3+1}M_{31}=-20$$

$$A_{32}=(-1)^{3+2}M_{32}=13$$

$$A_{33}=(-1)^{3+3}M_{33}=5$$

3. Using Cofactors of elements of second row, evaluate $\Delta=\begin{vmatrix}5&3&8\\2&0&1\\1&2&3\end{vmatrix}$

Solution:

Given determinant is $\Delta=\begin{vmatrix}5&3&8\\2&0&1\\1&2&3\end{vmatrix}$

Using definition of minors and cofactors

$$M_{21}=\begin{vmatrix}3&8\\2&3\end{vmatrix}=9-16=-7$$

$$\therefore A_{21}=(-1)^{2+1}M_{21}=7$$

$$M_{22}=\begin{vmatrix}5&8\\1&3\end{vmatrix}=15-8=7$$

$$\therefore A_{22}=(-1)^{2+2}M_{22}=7$$

$$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$$

$$\therefore A_{23} = (-1)^{2+3} M_{23} = -7$$

Since, Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors

$$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{33}A_{33}$$

$$= 2(7) + 0(7) + 1(-7)$$

$$= 14 - 7 = 7$$

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & xz \\ 1 & z & xy \end{vmatrix}$

Solution:

Given determinant is $\begin{vmatrix} 1 & x & yz \\ 1 & y & xz \\ 1 & z & xy \end{vmatrix}$

Using definition of minors and cofactors

$$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$$

$$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$$

$$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$$

$$\therefore A_{13} = (-1)^{1+3} M_{13} = (z - y)$$

$$A_{23} = (-1)^{2+3} M_{23} = -(z - x) = (x - z)$$

$A_{33} = (-1)^{3+3} M_{33} = (y-x)$

Since, Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors

$\therefore \Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$= yz(z-y) + zx(x-z) + xy(y-x)$

$= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$

$= (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y)$

$= z(x^2 - y^2) + z^2(y-x) + xy(y-x)$

$= (x-y)\left[zx + zy - z^2 - xy\right]$

$= (x-y)\left[z(x-z) + y(z-x)\right]$

$= (x-y)(y-z)(z-x)$

Thus, $\Delta = (x-y)(y-z)(z-x)$

5. For the matrices A and B, verify that $(AB)' = B'A'$ where

i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

Solution:

i) $AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$

$$\therefore (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Now $A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}, B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$\therefore B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Hence, proved that $(AB)' = B'A'$

ii) $AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$

$$\therefore (AB)' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

Now $A' = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, B' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$

$$\therefore B'A' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

Hence, proved that $(AB)' = B'A'$