**Chapter 4: Determinants**

**Exercise 4.5**

1.  Find adjoint of each of the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Since, Cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then, $A_{11}4, A_{12} = -3, A_{13} = -2, A_{22} = 1$

$\therefore adjA = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$

$= \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

2.  Find adjoint of each of the matrices $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = \begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2 + 10) = -12$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5$$

$$A_{23} = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0 - 2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix} = -5 - 6 = -11$$

$$A_{32} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5 - 4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5$$

Thus, $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

3.  Verify $A(adjA) = (adjA)A = |A|_I \cdot \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

**Solution:**

$$A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Here,

$$|A| = -12 - (-12)$$

$$= -12 + 12 = 0$$

$$\therefore |A|I = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+i} M_{ij}$

Then,

$$A_{11} = -6, A_{12} = 4, A_{21} = -3, A_{22} = 2$$

$$\therefore adjA = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

Now,

$$A(adjA) = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Also,

$$(adjA)A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

$$\begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $A(adjA) = (adjA)A = |A|I$

4.    Verify $A(adjA) = (adjA)A = |A|I$ $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

**Solution:**

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$|A| = 1(0-0) + 1(9+2) + 2(0-0)$$

$$= 11$$

$$\therefore |A|I = 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$$A_{11} = 0, A_{12} = -(9+2) = -11, A_{13} = 0$$

$$A_{21} = -(-3-0) = 3, A_{22} = 3-2 = 1, A_{22} = -(0+1) = -1$$

$$A_{31} = 2-0 = 2, A_{32} = -(-2-6) = 8, A_{33} = 0+3 = 3$$

$$\therefore adjA = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now,

$$A(adjA) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Also,

$$(adjA)\,A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Therefore, $A(adjA) = (adjA)\,A = A = |A|\,I$

5.    Find the inverse of each of the matrices (if it exists) $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Here,

$|A| = -2 + 15 = 13$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$A_{11} = 2,\, A_{12} = 3,\, A_{21} = -5,\, A_{22} = -1$

$$\therefore adjA = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{13} \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

6. Find the inverse of each of the matrices (if it exists) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Here,

$$|A| = 1(10-0) - 2(0-0) + 3(0-0) = 10$$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$$A_{11} = 10-0, A_{12} = -(0-0) = 0, A_{13} = 0-0 = 0$$

$$A_{21} = -(10-0) = -10, A_{22} = 5-0 = 5, A_{23} = -(0-0) = 0$$

$$A_{31} = 8-6 = 2, A_{32} = -(4-0) = -4, A_{33} = 2-0 = 2$$

$$\therefore adjA = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

7. Find the inverse of each of the matrices (if it exist). $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Here,

$|A| = 1(-3-0) - 0 + 0 = -3$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$A_{11} = -3 - 0 = -3, A_{12} = -(-3-0) = 3, A_{13} = 6 - 15 = -9$

$A_{22} = -(0-0) = 0, A_{22} = -1-0 = -1, A_{22} = -(2-0) = -2$

$A_{31} = 0 - 0 = 0, A_{32} = -(0-0) = 0, A_{33} = 3 - 0 = 3$

$\therefore adjA = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

$\therefore A^{-1} = \dfrac{1}{|A|} adjA = \dfrac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

8. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Here,

$|A| = 2(-1-0) - 1(4-0) + 3(8-7)$

$= 2(-1) - 1(4) + 3(1)$

$= -2 - 4 + 3$

$= -3$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$A_{11} = -1 - 0 = -1, A_{12} = -(4-0) = -4, A_{13} = 8 - 7 = 1$

$A_{22} = -(1-6) = 5, A_{22} = 2 + 21 = 23, A_{23} = -(4+7) = -11$

$A_{31} = 0 + 3 = 3, A_{22} = -(0-12) = 12, A_{33} = -2 - 4 = -6$

$\therefore adjA = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$

9. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Expanding along C1

$|A| = 1(8-6) - 0 + 3(3-4) = 2-3 = -1$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$A_{11} = 8-6, A_{12} = -(0+9) = -9, A_{13} = 0-6 = -6$

$A_{21} = -(-4+4) = 0, A_{22} = 4-6 = -2, A_{23} = -(-2+3) = -1$

$A_{31} = 3-4 = -1, A_{32} = -(-3-0) = 3, A_{33} = 2-0 = 2$

$\therefore adjA = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \end{bmatrix}$

$\therefore A^{-1} = \dfrac{1}{|A|} adjA = -1 \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

10. Find the inverse of each of the matrices (if it exists). $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ a & \sin a & -\cos a \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ a & \sin a & -\cos a \end{bmatrix} a$

Here,

$$|A| = 1\left(-\cos^2 a - \sin^2 a\right) = -\left(\cos^2 a + \sin^2 a\right) = -1$$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$$A_{11} = -\cos^2 a - \sin^2 a = -1, A_{12} = 0, A_{13} = 0$$

$$A_{21} = 0, A_{22} = -\cos a, A_{23} = -\sin a$$

$$A_{31} = 0, A_{32} = -\sin a, A_{33} = \cos a$$

$$\therefore adjA = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -1 \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos a & -\sin a \\ 0 & -\sin a & \cos a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & \sin a & -\cos a \end{bmatrix}$$

11. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

**Solution:**

Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

Here,

$$|A| = 15 - 14 = 1$$

Since, cofactor of $a_{ij}$ is $A_{ij} = (-1)^{i+j} M_{ij}$

Then,

$A_{11} = 5, A_{12} = -2, A_{21} = -7, A_{22} = 3$

$\therefore adjA = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$\therefore A^{-1} = \dfrac{1}{|A|} adjA = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

Now, let $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

Here

$|B| = 54 - 56 = -2$

$\therefore adjB = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$\therefore B^{-1} = \dfrac{1}{|B|} adjB = -\dfrac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} -\dfrac{9}{2} & 4 \\ \dfrac{7}{2} & -3 \end{bmatrix}$

Now,

$B^{-1}A^{-1} = \begin{bmatrix} -\dfrac{9}{2} & 4 \\ \dfrac{7}{2} & -3 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} -\dfrac{45}{2}-8 & \dfrac{63}{2}+12 \\ \dfrac{35}{2}+6 & -\dfrac{49}{2}-9 \end{bmatrix} = \begin{bmatrix} -\dfrac{61}{2} & \dfrac{87}{2} \\ \dfrac{47}{2} & -\dfrac{67}{2} \end{bmatrix}$ .............(1)

Then,

$$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$$

$$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

Therefore, we have $|AB| = 67 \times 61 - 87 \times 47 = 4087 - 4089 = -2$

Also,

$$\therefore adj(AB) = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \frac{1}{|AB|} adj(AB) = -\frac{1}{2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{67}{2} & \dfrac{87}{2} \\ \dfrac{47}{2} & -\dfrac{67}{2} \end{bmatrix} \dots\dots\dots\dots(2)$$

From (1) and (2), we have

$$(AB)^{-1} = B^{-1}A^{-1}$$

Hence proved

12.    If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 71 = 0$. Hence find $A^{-1}$

**Solution:**

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = A.A$$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $A^2 - 5A + 7I = 0$

$$\Rightarrow A^2 - 5A = -7I$$

$$\Rightarrow A.A(A^{-1}) - 5AA^{-1} = -7IA^{-1} \qquad \text{[Multiplying by } A^{-1}\text{]}$$

$$\Rightarrow A(AA^{-1}) - 5I = -7A^{-1}$$

$$\Rightarrow AI - 5I = -7A^{-1}$$

$$= A^{-1} = -\frac{1}{7}(A - 5I)$$

$$= A^{-1} = \frac{1}{7}(5I - A)$$

$$= \frac{1}{7}\left( \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \right)$$

$$= \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

13.  For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ find the number a and b such that $A^2 + aA + bI = 0$

**Solution:**

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Now,

$$A^2 + aA + bI = 0$$

$$\Rightarrow (AA)A^{-1} + aAA^{-1} + bIA^{-1} = 0 \qquad \text{[Multiplying by } A^{-1}\text{]}$$

$$\Rightarrow A(AA^{-1}) + aI + b(IA^{-1}) = 0$$

$$\Rightarrow AI + aI + bA^{-1} = 0$$

$$\Rightarrow A + aI = -bA^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{b}(A + aI)$$

Now,

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Here,

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{b}\left(\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = -\frac{1}{b}\begin{bmatrix} 3+a & 2 \\ 1 & 1+a \end{bmatrix} = \begin{bmatrix} \dfrac{-3-a}{b} & -\dfrac{2}{b} \\ -\dfrac{1}{b} & \dfrac{-1-a}{b} \end{bmatrix}$$

Equating the corresponding elements of the two matrices

$$-\frac{1}{b} = -1 \Rightarrow b = 1$$

$$\frac{-3-a}{b} = 1 \Rightarrow -3-a \Rightarrow a = -4$$

Thus, -4 and 1 are the required values of a and b respectively

14.    For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ show that $A^3 - 6A^2 + 5A + 11I = 0$. Hence, $A^{-1}$

**Solution:**

Given, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 5A + 11I$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 2 & -5 & 15 \end{bmatrix} + 11\begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Thus, $A^2 - 6A^2 + 5A + 11I = 0$

Now,

$A^3 - 6A^2 + 5A + 11I = 0$

$\Rightarrow (AAA)A^{-1} - (AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0$         [Multiplying by $A^{-1}$ ]

$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) = -11(IA^{-1})$

$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$

$\Rightarrow A^{-1} = -\dfrac{1}{11}(A^2 - 6A + 5I)\ldots\ldots\ldots(1)$

Now,

$A^2 - 6A + 5I$

$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix}$

$\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

From equation (1), we get

$A^{-1} = -\dfrac{1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

$$= \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

15. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ verify that $A^3 - 6A^2 + 9A - 4I = 0$ and hence find $A^{-1}$

**Solution:**

Given, $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now,

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix} - \begin{bmatrix} 40 & -30 & 30 \\ -30 & 40 & -30 \\ 30 & -30 & 40 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 9A - 4I = 0$$

Now,

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$\Rightarrow (AAA)A^{-1} - 6(AA^{-1}) + 9AA^{-1} - 4IA^{-1} = 0 \qquad \text{[Multiplying by } A^{-1}\text{]}$$

$$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 9(AA^{-1}) = 4(IA^{-1})$$

$$\Rightarrow AAI - 6AI + 9I = 4A^{-1}$$

$$\Rightarrow A^2 - 6A + 9I = 4A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I) \ldots\ldots\ldots(1)$$

$$A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

From equation (1), we have

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

16.    Let A be nonsingular square matrix of order 3×3. Then $|adjA|$ is equal to

A) $|A|$          B) $|A|^2$          C) $|A|^3$          D) $3|A|$

**Solution:**

$$(adjA) = A = |A|I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow |(adjA)A| = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$\Rightarrow |(adjA\setminus =)||A| = |A|^3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|^3 (I)$$

$$\therefore |adjA| = |A|^2$$

Hence, (B) is the correct answer

17. If A is an invertible matrix of order 2, then $\det\left(A^{-1}\right)$ is equal to

    A) $\det(A)$      B) $\dfrac{1}{\det(A)}$      C) 1      D) 0

**Solution:**

Since A is an invertible matrix, $A^{-1}$ exists and $A^{-1} = \dfrac{1}{|A|}\,adjA$

As matrix A is order 2, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, $|A| = ad - bc$ and $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Now,

$A^{-1} = \dfrac{1}{|A|}\,adjA$

$\begin{bmatrix} \dfrac{d}{|A|} & \dfrac{-b}{|A|} \\ \dfrac{-c}{|A|} & \dfrac{a}{|A|} \end{bmatrix}$

$\therefore \left|A^{-1}\right| = \begin{vmatrix} \dfrac{d}{|A|} & \dfrac{-b}{|A|} \\ \dfrac{-c}{|A|} & \dfrac{a}{|A|} \end{vmatrix}$

$= \dfrac{1}{|A^2|} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

$= \dfrac{1}{|A^2|}(ad - bc)$

$= \dfrac{1}{|A^2|}|A|$

$$\frac{1}{|A|}$$

$$\therefore \det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

Hence, (B) is the correct answer