Chapter 5: Continuity and differentiability.

Exercise 5. Miscellaneous

1. Differentiate the function w.r.t x

$(3x^2 - 9x + 5)^9$

Solution:

Let $y = (3x^2 - 9x + 5)^9$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx} = (3x^2 - 9x + 5)^9$$

$$= 9(3x^2 - 9x + 5)^8 . \frac{d}{dx}(3x^2 - 9x + 5)$$

$$= 9(3x^2 - 9x + 5)^8 .(6x - 9)$$

$$= 9(3x^2 - 9x + 5)^8 .3(2x - 3)$$

$$= 27(3x^2 - 9x + 5)^8 .(2x - 3)$$

2. Differentiate the function w.r.t x

$\sin^3 x + \cos^6 x$

Solution:

Let $\mathrm{y} = \sin^3 x + \cos^6 x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^3 x) + \frac{d}{dx}(\cos^6 x)$$

$$= 3\sin^2 x . \frac{d}{dx}(\sin x) + 6\cos^5 x . \frac{d}{dx}(\cos x)$$

$$= 3\sin^2 x \cos x + 6\cos^5 x .(-\sin x)$$

$=3\sin x\cos x\left(\sin x-2\cos^4 x\right)$

3. Differentiate the function w.r.t x

$(5x)^{3\cos 2x}$

Solution:

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both sides, we obtain

$\log y=3\cos 2x\log 5x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=3\left[\log 5x.\frac{d}{dx}(\cos 2x)+\cos 2x.\frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[\log 5x(-\sin 2x).\frac{d}{dx}(2x)+\cos 2x.\frac{1}{5x}.\frac{d}{dx}(5x)\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[-2\sin x\log 5x+\frac{\cos 2x}{x}\right]$$

$$\Rightarrow\frac{dy}{dx}=3y\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

$$\therefore\frac{dy}{dx}=(5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

4. Differentiate the function w.r.t x

$\sin^{-1}\left(x\sqrt{x}\right),\ 0\le x\le 1$

Solution:

Let $y=\sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^{3}}}\times\frac{d}{dx}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x^{3}}}\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=\frac{1}{\sqrt{1-x^{3}}}\times\frac{3}{2}.x^{\frac{1}{2}}$$

$$=\frac{3\sqrt{x}}{2\sqrt{1-x^{3}}}$$

$$=\frac{3}{2}\sqrt{\frac{x}{1-x^{3}}}$$

5. Differentiate the function w.r.t x

$$\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},-2<x<2$$

Solution:

Let $y=\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right)-\left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^{2}}$$

$$\frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{2x+7}$$

$$=\frac{\sqrt{2x+7}\frac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$=\frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\,(2x+7)}-\frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$=-\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

6. Differentiate the function w.r.t x

$$\cot^{-1}\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right], 0<x<\frac{\pi}{2}$$

Solution:

Let $y=\cot^{-1}\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right]$............(1)

Then, $\left[\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}\right]$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)-(1-\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\frac{dy}{dx}=\frac{1}{2}$$

7. Differentiate the function w.r.t x

$$(\log x)^{\log x}, x>1$$

Solution:

Let $y=(\log x)^{\log x}$

Taking logarithm on both sides, we obtain

$$\log y=\log x.\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[\log x.\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x.\frac{1}{\log x}.\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

8. Differentiate the function w.r.t x

$\cos(a\cos x + b\sin x)$, for some constant a and b

Solution:

Let $y = \cos(a\cos x + b\sin x)$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x).\frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x).[a(-\sin x) + b\cos x]$$

$$= (a\cos x + b\sin x)\sin(a\cos x + b\sin x)$$

9. Differentiate the function w.r.t x

$(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking logarithm on both sides, we obtain

$$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$$

$$\Rightarrow \log y = (\sin x - \cos x)\log(\sin x - \cos x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x).\log(\sin x - \cos x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x).\frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x).\frac{d}{dx}\log(\sin x - \cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x).(\cos x + \sin x) + (\sin x - \cos x).\frac{1}{(\text{sinx} - \text{cosx})}\frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\left[(\cos x + \sin x).\log(\sin x - \cos x) + (\cos x + \sin x)\right]$$

$$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right]$$

10. Differentiate the function w.r.t x

$x^x + x^a + a^x + a^a$, for such fixed $a > 0$ and $x > 0$

Solution:

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u, x^a = v, a^x = w$ and $a^a = s$

$\therefore y = u + v + w + s$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx}.................(1)$$

$u = x^x$

$\Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x \log x$

Differentiating both side with respect to x we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x.1 + x.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + \log x) \ldots\ldots\ldots.(2)$$

$v = x^a$

$$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \ldots\ldots\ldots.(3)$$

$w = a^x$

$\Rightarrow \log w = \log a^x$

$\Rightarrow \log w = x \log a$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w}.\frac{dw}{dx} = \log a.\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \ldots\ldots\ldots.(4)$$

$s = a^a$

Since a is constant, a^a is also constant

$$\therefore \frac{ds}{dx} = 0 \ldots\ldots\ldots.(5)$$

From 1,2,3,4 and 5, we obtain

$$\frac{dy}{dx} = x^x\left(1+\log x\right)+ax^{a-1}+a^x\log a+0$$

$$= x^x\left(1+\log x\right)+ax^{a-1}+a^x\log a$$

11. Differentiate the function w.r.t x

$x^{x^2-3}+\left(x-3\right)^{x^2}$, for $x>3$

Solution:

Let $y = x^{x^2-3}+\left(x-3\right)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = \left(x-3\right)^{x^2}$

$\therefore y = u+v$

Differentiating both sides with respect to x we obtain

$$\frac{dv}{dx} = \frac{du}{dx}+\frac{dv}{dx}\text{.............}(1)$$

$u = x^{x^2-3}$

$\therefore \log u = \log\left(x^{x^2-3}\right)$

$\log u = \left(x^2-3\right)\text{logx}$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x.\frac{d}{dx}\left(x^2-3\right)+\left(x^2-3\right).\frac{d}{dx}\left(\log x\right)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x.2x+\left(x^2-3\right).\frac{1}{3}$$

$$\Rightarrow \frac{du}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x}+2\times\log x\right]$$

Also,

$$v=(x-3)^{x^2}$$

$$\therefore \log v=\log(x-3)^{x^2}$$

$$\Rightarrow \log v=x^2\log(x-3)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\log(x-3)\frac{d}{dx}(x^2)+x^2\frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\log(x-3)2x+x^2.\frac{1}{x-3}.\frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx}=v\left[2x\log(x-3)+\frac{x^2}{x-3}.1\right]$$

$$\Rightarrow \frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Substituting the expression of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)x^2\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

12. Find $\frac{dy}{dx}$, if $y=12(1-\cos t), x=10(t-\sin t), \frac{\pi}{2}<t<\frac{\pi}{2}$

$-\frac{\pi}{2}<t<\frac{\pi}{2}$

Solution:

It is given that $y=12(1-\cos t), x=10(t-\sin t)$

$$\therefore \frac{dx}{dt}=\frac{d}{dt}[10(t-\sin t)]=10\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dx}=\frac{d}{dx}\left[12\left(1-\cos t\right)\right]=12\frac{d}{dt}\left(1-\cos t\right)=12.\left[0-\left(-\sin t\right)\right]=12\sin t$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10\left(1-\cos t\right)}=\frac{12.2\sin\frac{t}{2}.\cos\frac{t}{2}}{10.2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

13. Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},-1\le x\le 1$

Solution:

It is given that $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1\left(\sqrt{1-x^2}\right)}}.\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}.\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=0$$

14. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for $-1<x<1$, prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$

Solution:

It is given that, $x\sqrt{1+y}+y\sqrt{1+x}=0$

$$x\sqrt{1+y}=-y\sqrt{1+x}$$

Squaring both sides, we obtain

$x^2(1+y)=y^2(1+x)$

$\Rightarrow x^2+x^2y=y^2+xy^2$

$\Rightarrow x^2-y^2=xy^2-x^2y$

$\Rightarrow x^2-y^2=xy(y-x)$

$\Rightarrow (x+y)(x-y)=xy(y-x)$

$\therefore x+y=-xy$

$\Rightarrow (1+x)y=-x$

$\Rightarrow y=\frac{-x}{(1+x)}$

Differentiating both sides with respect to x, we obtain

$y=\frac{-x}{(1+x)}$

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

Hence, proved

15. If $(x-a)^2+(y-b)^2=c^2$, for some c > 0, prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b

Solution:

It is given that, $(x-a)^2+(y-b)^2=c^2$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\frac{d}{dx}(x-a)+2(y-b)\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a).1+2(y-b)\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \quad \text{...........(1)}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$=-\frac{\left[(y-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(y-b)\right]}{(y-b)^2}$$

$$=-\left[\frac{(y-b)-(x-a)\frac{dy}{dx}}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a)\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \quad \text{[using (1)]}$$

$$=-\left[\frac{(y-b)^2+(x+a)^2}{(y-b)^2}\right]$$

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[\left(1+\frac{(x-a)^2}{(y-b)^2}\right)\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= -\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{c^2}{(y-b)^3}} = \frac{\frac{c^2}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

= - c, which is constant and is independent of a and b

Hence, proved

16. If $\cos y = x\cos(a+y)$ with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Solution:

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx} = [\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y \frac{dy}{dx} = \cos(a+y)\frac{d}{dx}(x) + x.\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y = \frac{dy}{dx}\cos(a+y) + x.[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \quad ....(1)$$

Since $\cos y = x\cos(a+y), x = \frac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to $\left[\frac{\cos y}{\cos(a+y)}\sin(a+y)-\sin y\right]\frac{dy}{dx}=\cos(a+y)$

$$\Rightarrow\left[\cos y.\sin(a+y)-\sin y.\cos(a+y)\right]\frac{dy}{dx}=\cos^2(a+y)$$

$$\Rightarrow\sin(a+y-y)\frac{dy}{dx}=\cos^2(a+b)$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos^2(a+b)}{\sin a}$$

Hence, proved

17. If $x=a(\cos t+t\sin t)\,and\ y=a(\sin t-t\cos t)$, find $\frac{d^2y}{dx^2}$

Solution:

It is given that, $x=a(\cos t+t\sin t)\,and\ y=a(\sin t-t\cos t)$

$$\therefore\frac{dx}{dt}=a.\frac{d}{dt}(\cot+t\sin t)$$

$$=a\left[-\sin t+\sin t.\frac{d}{dt}(t)+t.\frac{d}{dt}(\sin t)\right]$$

$$=a[-\sin t+\sin t+t\cos t]=at\cos t$$

$$\frac{dy}{dx}=a.\frac{d}{dt}(\sin t-t\cos t)$$

$$=a\left[\cos t-\left\{\cos t.\frac{d}{dt}(t)+t.\frac{d}{dt}(\cos t)\right\}\right]$$

$$=a\left[\cos t-\{\cos t-t\sin t\}\right]=at\sin t$$

$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{at\sin t}{at\cos t}=\tan t$$

Then, $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(\tan t)=\sec^2 t\frac{dt}{dx}$

$$=\sec^2 t\frac{1}{a\,t\cos t}\quad\left[\frac{dx}{dt}=at\cos t\Rightarrow\frac{dt}{dx}=\frac{1}{at\cos t}\right]$$

$$=\frac{\sec^3 t}{at},0<t<\frac{\pi}{2}$$

18. If $f(x)=|x|^3$, show that $f''(x)$ exists for all real x, and find it

Solution:

It is known that, $|x|=\begin{cases}x, & if\ x\geq 0\\ -x, & if\ x<0\end{cases}$

Therefore, when $x\geq 0$, $f(x)=|x|^3=x^3$

In this case, $f'(x)=3x^2$ and hence, $f''(x)=6x$

When $x<0$, $f(x)=|x|^3=(-x^3)=x^3$

In this case, $f'(x)=3x^2$ and hence $f''(x)=6x$

Thus, for $f(x)=|x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x)=\begin{cases}6x, & if\ x\geq 0\\ -6x, & if\ x<0\end{cases}$$

19. Using mathematical induction prove that $\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers n

Solution:

To prove: $P(n):\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers n

For n = 1,

$P(1): \frac{d}{dx}(x) = 1 = 1.x^{1-1}$

$\therefore p(n)$ is true for n = 1

Let p(k) is true for some positive integer k

That is, $p(k): \frac{d}{dx}(x^k) = kx^{k-1}$

It is to be proved that $p(k+1)$ is also true

Consider $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x.x^k)$

$x^k \frac{d}{dx}(x) + x.\frac{d}{dx}(x^k)$

$= x^k.1 + x.k.x^{k-1}$

$= x^k + kx^k$

$= (k+1)x^k$

$= (k+1).x^{(k+1)-1}$

Thus, P (k+1) is true whenever P(k) is true

Therefore, by the principal of mathematical induction, the statement P(n) is true for every

positive integer n

Hence, proved

20. Using the fact that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines

Solution:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\frac{d}{dx}(A+B)=\cos B\frac{d}{dx}(\sin A)+\sin A\frac{d}{dx}(\cos B)+\sin B\frac{d}{dx}(\cos A)+\cos A\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\frac{d}{dx}(A+B)=\cos B.\cos A\frac{d}{dx}+\sin A(-\sin B)\frac{dB}{dx}+\sin B(-\sin A)\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$

$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

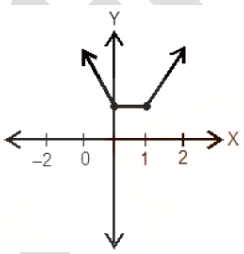
21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer

Solution:

Consider $f(x)=|x|+|x+1|$

Since modulus function is everywhere continuous and sum of two continuous function is also continuous

Differentiability of $f(x)$:Graph of $f(x)$ shows that $f(x)$ is everywhere derivable except possible at x= 0 and x = 1



At x =0, Left hand derivative

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{(|x|+|x-1|)-(1)}{x}=\lim_{x\to 0^-}\frac{|(-x)-(x-1)|-1}{x}=\lim_{x\to 0^-}\frac{-2x}{x}=-2$$

Right hand derivative =

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{(|x|+|x-1|)-(1)}{x}=\lim_{x\to 0^+}\frac{(-x)-(x-1)-1}{x}=\lim_{x\to 0^-}\frac{0}{x}=0$$

Since $L.H.D \neq R.H.D$ $f(x)$ is not derivable at x = 0

At x = 1

L.H.D

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{(|x|+|x-1|)}{x-1}=\lim_{x\to 1^-}\frac{(x)-(x-1)-1}{x-1}=\lim_{x\to 1^-}\frac{0}{x-1}=0$$

R.H.D

$$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{(|x|+|x-1|-1)}{x-1}=\lim_{x\to 1^+}\frac{(x)-(x-1)-1}{x-1}=\lim_{x\to 1^-}\frac{2(x-1)}{x-1}=2$$

Since $L.H.D \neq R.H.D$ $f(x)$ is not derivable at x = 1

$\therefore f(x)$ is continuous everywhere but not derivable at exactly two points

22. If $y=\begin{bmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{bmatrix}$, prove that $\frac{dy}{dx}=\begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$

Solution:

$$y=\begin{bmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{bmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

Then, $\frac{dy}{dx}=\frac{d}{dx}[(mc-nb)f(x)]-\frac{d}{dx}[(lc-na)g(x)]+\frac{d}{dx}[(lb-ma)h(x)]$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$$

Thus, $\frac{dy}{dx} = \begin{bmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{bmatrix}$

23. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1,$ show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Solution:

It is given that, $y = e^{a\cos^{-1}x}$

Taking logarithm on both sides, we obtain

$\log y = a\cos^{-1}x \log e$

$\log y = a\cos^{-1}x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{y}\frac{dy}{dx} = ax\frac{1}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$

By squaring both the sides, we obtain

$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$

$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$

$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$

Again, differentiating both sides with respect to x, we obtain

$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x) + (1-x^2)\times 2\frac{dy}{dx}\frac{d^2y}{dx^2} = a^2.2y\frac{dy}{dx}$

$$\Rightarrow x\frac{dy}{dx}+\left(1-x^2\right)\frac{d^2y}{dx^2}=a^2.y \qquad \left[\frac{dy}{dx}\neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

Hence, proved