Chapter 5: Continuity and differentiability.

Exercise 5.1

1. Prove that the function $f(x)=5x-3$ is continuous at $x=0, x=-3 \text{ and at } x=5$

Solution:

The given function is $f(x)=5x-3$

At $x=0, f(0)=5\times 0-3=3$

$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}(5x-3)=5\times 0-3=3$

$\therefore \lim_{x\to\infty} f(x)=f(0)$

Therefore, f is continuous at $x=0$

At $x=-3, f(-3)=5x(-3)-3=18$

$\lim_{x\to 3} f(x)=\lim_{x\to 3} f(5x-3)=5x(-3)-3=-18$

$\therefore \lim_{x\to 3} f(x)=f(-3)$

Therefore, f is continuous at $x=-3$

At $x=5, f(x)=f(5)=5\times 5-3=25-3=22$

$\lim_{x\to 5} f(x)=\lim_{x\to 5}(5x-3)=5\times 5-3=22$

$\therefore \lim_{x\to 5} f(x)=f(5)$

Therefore, is f continuous at $x=5$

2. Examine the continuity of the function at $f(x)=2x^2-1\ x=3$

Solution:

The given function is $f(x)=2x^2-1$

At $x=3,\ f(x)=f(3)=2\times3^2-1=17$

$$\lim_{x\to3} f(x)=\lim_{x\to3}(2x^2-1)=2\times3^2-1=17$$

$$\therefore \lim_{x\to3} f(x)=f(3)$$

Thus, f is continuous at $x=3$.

3. Examine the following functions for continuity

a) $f(x)=x-5$

b) $f(x)=\frac{1}{x-5},\ x\neq5$

c) $f(x)=\frac{x^2-25}{x+5},\ x\neq5$

d) $f(x)=|x-5|$

Solution:

a) The given function is $f(x)=x-5$

It is evident that f is defined at every real number k and its value at k is $k-5$

It is also observed that $\lim_{x\to k} f(x)=\lim_{x\to k} f(x-5)=k=k-5=f(k)$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f is continuous at every real number and therefore, it is a continuous function.

b) The given function is $f(x)=\frac{1}{x-5},\ x\neq5$ for any real number $k\neq5$, we obtain

$$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{1}{x-5}=\frac{1}{k-5}$$

Also, $f(k)=\frac{1}{k-5}$ $\qquad (As\ k\neq5)$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

c) The given function is $f(x)=\dfrac{x^2-25}{x+5},\ x\neq 5$

For any real number $c\neq -5$, we obtain

$$\lim_{x\to c} f(x)=\lim_{x\to c}\frac{x^2-25}{x+5}=\lim_{x\to c}\frac{(x+5)(x-5)}{x+5}=\lim_{x\to c}(x-5)=(c-5)$$

Also, $f(c)=\dfrac{(c+5)(c-5)}{c+5}=c(c-5)(as\, c\neq 5)$

$\therefore \lim_{x\to c} f(x)=f(c)$

Hence f is continuous at every point in the domain of f and therefore. It is continuous function.

d) The given function is $f(x)=|x-5|=\begin{cases}5-x, \; if \;\; x<5\\ x-5, if \;\; x\geq 5\end{cases}$

This function f is defined at all points of the real line.

Let c be a point on a real time. Then, $c<5$ or $c=5\; or\, c>5$

Case I: $c<5$

Then, $f(c)=5-c$

$\lim_{x\to c} f(x)=\lim_{x\to c}(5-x)=5-c$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all real numbers less than 5.

Case II : $c=5$

Then, $f(c)=f(5)=(5-5)=0$

$\lim_{x\to 5^-} f(x)=\lim_{x\to 5}(5-x)=(5-5)=0$

$\lim_{x\to 5^+} f(x)=\lim_{x\to 5}(x-5)=0$

$\therefore \lim_{x \to c^+} f(x) = \lim_{x \to c^+} (x) = f(c)$

Therefore, f is continuous at $x = 5$

Case III : $c > 5$

Then, $f(c) = f(5) = c - 5$

$\lim_{x \to c} f(x) = \lim_{x \to c} f(x-5) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

4. Prove that the function $f(x) = x^n$ is continuous at is a $x = n$ positive integer.

Solution:

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

Then, $\lim_{x \to n} f(n) = \lim_{x \to n} f(x^n) = n^n$

$\therefore \lim_{x \to n} f(x) = f(n)$

Therefore, f is continuous at n, where n is a positive integer.

5. Is the function f defined by $f(x) = \begin{cases} x, if\ x \le 1 \\ 5, if\ x > 1 \end{cases}$

Continuous at $x = 0$? At $x = 1$?. At $x = 2$?

Solution:

The given function f is $f(x) = \begin{cases} x,\ if\ x \le 1 \\ 5, if\ x > 1 \end{cases}$

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At $x=0$

It is evident that f is defined at 0 and its value of 0 is 0

Then, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x=0$

At $x=1$

f is defined at 1 and its value at is 1.

The left hand limit of at f i $x=1$ s,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$

The right hand limit of at f is $x=1$,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} f(5)$

$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

Therefore, is f is not continuous at $x=1$

At $x=2$

f is defined at 2 and its value at 2 is 5.

Then, $\lim_{x \to 2} f(x) = \lim_{x \to 2} f(5) = 5$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Therefore f is continuous at $x=2$

6. Find all points of discontinuous of f, where f is defined by

$$f(x) = \begin{cases} 2x+3, & if \ x \leq 2 \\ 2x-3 & if \ x > 2 \end{cases}$$

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Solution:

The given function f if $f(x)=\begin{cases}2x+3, & if\ \ x\le 2\\ 2x-3 & if\ x>2\end{cases}$

It is evident that the given function f is defined at all the points of the real time.

Let c be a point on the real line. Then, three cases arise.

I. $c<2$

II. $c>2$

III. $c=2$

Case (i) $c<2$

Then, $f(x)=2x+3$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x+3)=2c+3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x<2$

Case (ii) $c>2$

Then, $f(c)=2c-3$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(2x-3)=2c-3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x>2$

Case (iii) $c=2$

Then, the left hand limit of f at $x=2$ is

$$\lim_{x\to 2} f(x)=\lim_{x\to 2^-}(2x+3)=2\times 2+3=7$$

The right hand limit of f at is $x=2$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x+3)=2\times 2-3=1$$

It is observed that the left and right hand limit of f at $x=2$ do not coincide.

Hence, $x=2$ is the only point of discontinuity of f.

7. Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases} |x|+3, & if\ x\le -3 \\ -2x, & if\ -3<x<3 \\ 6x+2 & if\ x\ge 3 \end{cases}$$

Solution:

The given function f is defined at all the points at the real line.

Let c be a point on the real line.

Case I:

If $c<-3$, then $f(c)=-c+3$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-x+3)=-c+3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x<-3$

Case II :

If , $c=-3$ then $f(-3)=-(-3)+3=6$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(-x+3)=-(-3)+3=6$$

$$\therefore \lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} f(-2x)=2x(-3)=6$$

$$\therefore \lim_{x\to 3} f(x)=f(-3)$$

Therefore, f is continuous at $x=-3$

Case III :

If , $-3 < c < 3$ then $f(c) = -2c$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to 3c} (-2x) = -2c$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore, f is continuous in (-3, 3)

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} f(-2x) = -2 \times 3 = 6$

The right hand limit of f at $x = 3$ is

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^-} f(6x + 2) = 6 \times 3 + 2 = 20$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Case V:

If $c > 3$, then $f(c) = 6c + 2$ and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (6x + 2) = 6c + 2$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$

Therefore f is continuous at all points x, such that $x > 3$.

Hence, $x = 3$ is the only point of discontinuity of f .

8. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \frac{|x|}{x}, & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$

Solution:

The given function f is $f(x) = \begin{cases} \frac{|x|}{x}, & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$

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It is known that, $x<0 \Rightarrow |x|=-x$ and $x>0 \Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)\begin{cases}\dfrac{|x|}{x}=\dfrac{-x}{x}=-1 & if\ \ x<0\\ 0, & if\ x=0\\ \dfrac{|x|}{x}=\dfrac{x}{x}=1 & if\ x>0\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<0$, then $f(c)=1$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore f is continuous at all points $x<0$

Case II:

If $c=0$, then the left hand limit of f at $x=0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-1)=-1$$

The right hand limit of f at $x=0$ is

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(1)=1$$

It is observed that the left and right hand limit of f at $x=0$ do not coincide.

Therefore, f is not continuous at $x=0$.

Case III:

If $c>0$, $f(c)=1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(1) = 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of f.

9. Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \frac{x}{|x|}, & if\ x < 0 \\ -1 & if\ x \geq 0 \end{cases}$

Solution:

The given function f is $f(x) = \begin{cases} \frac{x}{|x|}, & if\ x < 0 \\ -1 & if\ x \geq 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{x}{|x|}, & if\ x < 0 \\ -1 & if\ x \geq 0 \end{cases} \Rightarrow f(x) = -1\ for\ all\ x \in R$$

Let c be any real number. Then $\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is continuous function.

Hence, the given function has no point of discontinuity.

10. Find all the points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x+1, & if\ \ x \geq 1 \\ x^2+1, & if\ \ x < 1 \end{cases}$$

Solution:

The given function f is $f(x)=\begin{cases} x+1, & if\ \ x\geq 1 \\ x^2+1, & if\ \ x<1 \end{cases}$

The given function f is defined at all the points of the real lime.

Let c be a point on the real time.

Case I :

If $c<1$, then $f(c)=c^2+1$ and $\lim_{x\to c} f(x)=\lim_{x\to c} f(x^2+1)=c^2+1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<1$

Case II :

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of f at $x=1$ is

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2+1)=1^2+1=2$

The right hand limit of f at $x=1$ is

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x^2+1)=1^2+1=2$

$\therefore \lim_{x\to 1} f(x)=f(c)$

Therefore, f is continuous at $x=1$

Case III:

If $c>1$, then $f(c)=c+1$

$\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>1$.

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Hence, the given function f has no points of discontinuity.

11. Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases} x^3-3, & if\ \ x\le 2 \\ x^2+1, & if\ \ x>2 \end{cases}$$

Solution:

The given function f is $f(x)=\begin{cases} x^3-3, & if\ \ x\le 2 \\ x^2+1, & if\ \ x>2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c<2$, then $f(c)=c^3-3$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^3-3\right)=c^3-3$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such at that $x<2$

Case II :

If $c=2$, then $f(c)=f(2)=2^3-3=5$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^-}\left(x^3-3\right)=2^3-3=5$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\left(x^2+1\right)=2^2+1=5$

$\therefore \lim\limits_{x\to 2} f(x)=f(2)$

Therefore, f is continuous at $x=2$

Case III :

If $c>2$, then $f(c)=c^2+1$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2+1\right)=c^2+1$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x>2$

Thus, the given function f is continuous at every point on the real time.

Hence, f has no point of discontinuity.

12. Find all points of discontinuity of f, where f is define by $f(x)=\begin{cases} x^{10}-1, & if\ x\le 1 \\ x^2, & if\ x>1 \end{cases}$

Solution:

The given function f is $f(x)=\begin{cases} x^{10}-1, & if\ x\le 1 \\ x^2, & if\ x>1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I :

If $c<1$, then $f(c)=c^{10}-1$ *and* $\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^{10}-1\right)=c^{10}-1$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x<1$

Case II :

If $c=1$, then the left hand limit of f at $x=1$ is

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=10^{10}-1=1-1=0$$

The right hand limit of f at $x=1$ is ,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^2\right)=1^2=1$$

It is observed that the left and right hand limit of f at $x=1$ do not coincide.

Therefore, f is not continuous at $x=1$

Case III :

If $c>1$, then $f(c)=c^2$

$\lim_{x\to c} f(x)=\lim_{x\to c}(x^2)=c^2$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore f is continuous at al points x, such that $x>1$

Thus, from the above observation, it can be concluded that $x=1$ is the only point of discontinuity of f.

13. Is the function define by $f(x)=\begin{cases} x+5, & if\ x\le 1 \\ x-5, & if\ x>1 \end{cases}$ a continuous function?

Solution:

The given function is $f(x)=\begin{cases} x+5, & if\ x\le 1 \\ x-5, & if\ x>1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I ;

If $c<1$, then $f(c)=c+5$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(x+5)=c+5$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<1$

Case II :

If $c=1$, then $f(1)=1+5=6$

The left hand limit of f at $x = 1$ is

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+5) = 1+5 = 6$$

The right hand limit of f at $x = 1$ is $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x-5) = 1-5 = -4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore f is not continuous at $x = 1$

Case III :

If $c > 1$, then $f(c) = c-5$ *and* $\lim_{x\to c} f(x) = \lim_{x\to c}(x-5) = c-5$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore f is continuous at all points x, such that $x > 1$

Thus, from the above observations, it can be concluded that $x = 1$ is the only point of discontinuity of f.

14. Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$$

Solution:

The given function is $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

The given function is defined at all the points of the interval $[0, 10]$.

Let c be a point in the interval $[0, 10]$

Case I ;

If , $0 \le c < 1$ then $f(c) = c+5$ $f(c) = 3$ *and* $\lim_{x\to c} f(x) = \lim_{x\to c}(3) = 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in the interval $[0,1)$

Case II :

If $c = 1$, then $f(3) = 3$

The left hand limit of f at $x = 1$ is

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3) = 3$

The left hand limit of f at $x = 1$ is $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3) = 3$

The right hand limit of f at $x = 1$ is $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4) = 4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore f is not continuous at $x = 1$

Case III :

If $c > 1$, then $f(c) = 4$ *and* $\lim_{x \to c} f(x) = \lim_{x \to c} (4) = 4$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore f is continuous at all points of the interval (1, 3)

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of f at $x = 3$ is $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$

The right hand limit of f at $x = 3$ is $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V :

If $3 < c \le 10$, then $f(c) = 5$ *and* $\lim_{x \to c} f(x) = \lim_{x \to c}(5) = 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at $x = 1$ *and* $x = 3$

15. Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & if\ x < 0 \\ 0, & if\ 0 \le x \le 1 \\ 4x. & if\ x > 1 \end{cases}$$

Solution:

The given function is $f(x) = \begin{cases} 2x, & if\ x < 0 \\ 0, & if\ 0 \le x \le 1 \\ 4x. & if\ x > 1 \end{cases}$

The given function is defined at all the points of the real line.

Let c be a point on the real line

Case I ;

If , $c < 0$ then $f(c) = 2c$, $\lim_{x \to c} f(x) = \lim_{x \to c}(2x) = 2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points , x such that x < 0

Case II :

If $c = 0$, then $f(c) = f(0) = 3$

The left hand limit of f at $x = 0$ is $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(2x) = 2 \times 0 = 0$

The right hand limit of f at $x = 0$ is $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(0) = 0$

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$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

Case III :

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(0) = 0$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore f is continuous at all points of the interval (0, 1)

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of f at $x = 1$ is $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(0) = 0$

The right hand limit of f at $x = 1$.is $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(4x) = 4 \times 1 = 4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case V :

If $c < 1$, then $f(c) = 4c$ *and* $\lim_{x \to c} f(x) = \lim_{x \to c}(4x) = 4c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Hence, f is not continuous only at $x = 1$

16. Discuss the continuity of the function f , where f is defined by

$$f(x) = \begin{cases} -2, & if \ x \le -1 \\ 2x, & if \ -1 < x \le 1 \\ 2, & if \ x > 1 \end{cases}$$

Solution:

The given function f is $f(x)=\begin{cases}-2, & if\ x\le -1\\ 2x, & if\ -1<x\le 1\\ 2, & if\ x>1\end{cases}$

The given function is defined at all point of the real time. Let c be a point on the real time.

Case I :

If $c<-1$, then $f(c=-2)$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2)=-2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x , such that $x<-1$

Case –II :

If $c=-1$, *then* $f(c)=f(-1)=-2$

The left hand limit of f *at* $x=-1$ is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(-2)=-2$

The right hand limit of f at $x=-1$ is ,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+} 2x(-1)=-2$

$\lim\limits_{x\to 1} f(x)=f(-1)$

Therefore , f is continuous at $x=-1$

Case III :

If $-1<c<1$, *then* $f(c)=2c$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x)=2c$

$\lim\limits_{x\to c} f(x)=f(c)$

Therefore , f is continuous at all points of the interval $(-1,1)$.

Case - IV :

If $c=1$, then $f(c)=f(1)=2\times 1=2$

The left hand limit of f at $x=1$ is ,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(2x)=2\times 1=2$

The right hand limit of f at $x=1$ is ,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+} 2=2$

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$\lim_{x\to 1} f(x) = f(c)$

Therefore, f is continuous at $x = 2$

Case –V :

If $c > 1, f(c) = 2\, and \lim_{x\to 2} f(x) = \lim_{x\to 2}(2) = 2$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore , f is continuous at all points , x , such that $x > 1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real time.

17. Find the relationship be *a* and *b* so that the function *f* defined by *f (x)* = $\begin{cases} ax+1, if\ x \le 3 \\ bx+3, if\ x > 3 \end{cases}$ is continuous at *x=3*.

Solution: The given function *f* is *f (x)* = $\begin{cases} ax+1, if\ x \le 3 \\ bx+3, if\ x > 3 \end{cases}$ If *f* is continuous at *x=3*, then

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} f(x) = f(3)$

Also,

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} f(ax+1) = 3a+1$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} f(ax+1) = 3b+3$

$f(3) = 3a+1$

therefore, from (1), we obtain

$3a+1 = 3b+3 = 3a+1$

$\Rightarrow 3a+1 = 3b+3$

$\Rightarrow 3a=3b+2$

$\Rightarrow a=b+\frac{2}{3}$

Therefore, the required relationship is given by, $a=b+\frac{2}{3}$

18. For what value of λ is the function defined by *f (x)* = $\begin{cases} \lambda(x^2-2x), if\ x\le 0 \\ 4x+1, \quad if\ \ x>0 \end{cases}$ continuous at $x = 0$? What about continuity at $x = 1$?

Solution: The given function *f* is $f(x) = \begin{cases} \lambda(x^2-2x), if\ x \le 0 \\ 4x+1, \quad if \quad x>0 \end{cases}$

If *f* is continuous at $x = 0$, then

$$\lim_{x\to 0^-} f(x) \lim_{x\to 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} \lambda(x^2-2x) = \lim_{x\to 0^+}(4x+1) = \lambda(0^2-2\times 0)$$

$$\Rightarrow \lambda(02-2\times 0) = 4\times 0+1 = 0$$

$$\Rightarrow 0=1=0, \textit{which is not possible}$$

Therefore, there is no value of λ for which *f* is continuous at $x = 0$

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$$\lim_{x\to 1}(4x+1) = 4\times 1+1 = 5$$

19. Show that the function defined by $g(x) = x-[x]$ is discontinuous at all integral point. Here $[x]$ denoted the greatest integer less than or equal to *x*.

Solution:

The given function is $g(x) = x-[x]$

It is evident that *g* is defined at all integral points.

Let *n* be a integer

Then,

$$g(n) = n-[n] = n-n = 0$$

The left hand limit of *f* at $x = n$ is

$$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}[x-[x]] = \lim_{x\to n^-} x - \lim_{x\to n^-}[x] = n-(n-1) = 1$$

The right hand limit of *f* at $x = n$ is

$$\lim_{x\to n^-} g(x) = \lim_{x\to n^-}[x-[x]] = \lim_{x\to n^-} x - \lim_{x\to n^-}[x] = n-n = 0$$

It is observed that the left and right hand limits of *f* at $x = n$ do not coincide.

Therefore, *f* is not continuous at $x = n$

Hence, *g* is discontinuous at all integral points

20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

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Solution:

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at $x = \pi$

At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} f\left(x^2 - \sin x + 5\right)$

Put $x = \pi + h$

If $x \to \pi$, then it is evident that $h \to 0$

$$\therefore \quad \lim_{x\to\pi} f(x) = \lim_{x\to\pi}\left(x^2 - \sin x\right) + 5$$

$$= \lim_{h\to0}\left[(\pi + h)^2 - \sin(\pi + h) + 5\right]$$

$$= (\pi + 0)^2 - \lim_{h\to0}[\sin\pi \cosh + \cos\pi + \sinh] + 5$$

$$= \pi^2 - \lim_{h\to0}\sin\pi\cosh - \lim_{h\to0}\cos\pi + \sinh + 5$$

$$= \pi^2 - \sin\pi\cos0 - \cos\pi\sin0 + 5$$

$$= \pi^2 - 0\times1 - (-1)\times0 + 5$$

$$= \pi^2 + 5$$

$$\therefore \lim_{x\to x} f(x) = f(\pi)$$

Therefore, the given function f is continuous at $x = \pi$

21. Discuss the continuity of the following functions.

A) $f(x) = \sin x + \cos x$

B) $f(x) = \sin x - \cos x$

C) $f(x) = \sin x \times \cos x$

Solution:

It is known that if g and h are two continuous functions, then $g + h, g - h$ and $g.h$ are also continuous.

It has to proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

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Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$g(c) = \sin c$

$$\lim_{x\to c} g(x) = \lim_{x\to c} g \sin x$$

$$= \lim_{h\to 0} \sin(c+h)$$

$$= \lim_{h\to 0}[\sin c \cosh + \cos c \sinh]$$

$$= \lim_{h\to 0}(\sin c \cosh) + \lim_{h\to 0}(\cos c \sinh)$$

$$= \sin c \cos 0 + \cos c \sin 0$$

$$= \sin c + 0$$

$$= \sin c$$

$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\lim_{x\to c}(x) = \lim_{x\to c} \cos x$$

$$= \lim_{h\to 0} \cos(c+h)$$

$$= \lim_{h\to 0}[\cos c \cosh - \sin c \sinh]$$

$$= \lim_{h\to 0} \cos c \cosh - \lim_{h\to 0} \sin c \sinh$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c \times 1 - \sin c \times 0$$

$$= \cos c$$

$$\therefore \lim_{h\to 0}(x) = h(c)$$

Therefore, h is a continuous function.

Therefore, it can be concluded that

a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Solution:

It is known that if p and h are two continuous functions, then

i. $\frac{h(x)}{g(x)}.g(x) \neq 0$ is continuous

ii. $\frac{1}{g(x)}.g(x) \neq 0$ is continuous

iii. $\frac{1}{h(x)}.h(x) \neq 0$ is continuous

It has to be proved first that $g(x) - \sin x$ and $h(x) - \cos x$ are continuous functions.

Let $g(x) - \sin x$

It has evident that $g(x) - \sin x$ is defined foc every real number.

Let c be a real number. Put $x - c + h$

If $x \to C,$ then $h \to 0$

$g(c) - \sin x$

$\lim_{x+c} g(c) - \lim_{x+c} \sin x$

$- \lim_{x+c} \sin(c+h)$

$- \lim_{x+c} [\sin c \cosh + \cos c \sinh]$

$- \lim_{x+c} (\sin c \cosh) + \lim_{x+c} (\cos c \sinh)$

$- \sin c \cos 0 + \cos c \sin 0$

$- \sin c + 0$

$- \sin c$

$\therefore \lim_{0+c} g(x) - g(c)$

Therefore, g is a continuous function.

Let $h(x) - \cos x$

It is evident that $h(x) - \cos x$ is defined for every real number.

Let c be a real number. Put $x - c + h$

If $x \to c,$ then $h \to 0x$

$h(c) - \cos c$

$- \lim_{x \to 0} \cos(c+h)$

$- \lim_{x \to 0} [\cos c \cosh - \sin c \sinh]$

$- \lim_{x \to 0} \cos c \cosh - \lim_{x \to 0} \sin c \sinh$

$- \cos c \cos 0 - \sin c \sin 0$

$- \cos c\, x 1 - \sin c\, x 0$

$= \cos c$

$\therefore \lim_{x+c} h(x) - h(c)$

Therefore, $h(x) - \cos x$ is continuous function.

It can be conclned that,

$\cos ex - \frac{1}{\sin x}, \sin x \neq 0$ is continuous

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$\Rightarrow \cos ec\, x, x \neq nx(n \in Z)$ is continuous

Therefore, secant is continuous except at $X - np, nIZ$

$\sec x = \frac{1}{\cos x}, \cos x \neq 0$ is continuous

$\Rightarrow \sec x, x \neq (2n+1)\frac{\pi}{2}(n \in Z)$ is continuous

Therefore, secant is continuous except at $x - (2n+1)\frac{\pi}{2}(n \in Z)$

$\cot x = \frac{\cos x}{\sin x}, \sin x \neq 0$ is continuous

$\Rightarrow \cot x, x \neq n\pi(n \in Z)$ is continuous

Therefore, cotangent is continuous except at $x - np, nIZ$

23. Find the points of discontinuity of f, where $f(x) = \begin{cases} \frac{\sin x}{x}, & if \quad x < 0 \\ x+1, & if \quad x \geq 0 \end{cases}$

Solution:

The given function f is $f(x) = \begin{cases} \frac{\sin x}{x}, & if \quad x < 0 \\ x+1, & if \quad x \geq 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number

Case I:

If $c < 0$, then $f(c) = \frac{\sin c}{c}$ and $\lim_{x \to c} f(x) = \lim_{x \to c}\left(\frac{\sin x}{x}\right) = \frac{\sin c}{c}$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, then $f(c) = c+1$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(x+1) = c+1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $f(c) = f(0) = 0+1 = 1$

The left hand limit of f at $x = 0$ is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$

The right hand limit of f at $x = 0$ is,

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(x+1) = 1$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be conducted that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

24. Determine if f defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & if \neq 0 \\ 0, & if \neq 0 \end{cases}$ is a continuous function?

Solution:

The given function f is $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & if \neq 0 \\ 0, & if \neq 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin\frac{1}{c}$

$$\lim_{x \to c} f(x) = \lim_{x \to c}\left(x^2 \sin\frac{1}{x}\right) = \left(\lim_{x \to c} x^2\right)\left(\lim_{x \to c} \sin\frac{1}{x}\right) = c^2 \sin\frac{1}{c}$$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all p[oints $x \neq 0$

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{2}\right)$$

It is known that, $-1 \leq \sin\frac{1}{x} \leq 1, \ x \neq 0$

$$\Rightarrow -x^2 \leq \sin\frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x \to 0}\left(-x^2\right) \leq \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \leq \lim_{x \to 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \leq 0$$

$$\Rightarrow \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$\therefore \lim_{x \to 0} f(x) = 0$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x) = 0$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus f is a continuous function.

25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & if \quad x \neq 0 \\ 1 \quad , & if \quad x = 0 \end{cases}$$

Solution:

The given function f is $f(x) = \begin{cases} \sin x - \cos x, & if \quad x \neq 0 \\ 1 \quad , & if \quad x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$\lim_{x\to c} f(x) = \lim_{x\to c}(\sin x - \cos x) = \sin c - \cos c$

$\therefore \lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x \neq 0$

Case II:

If $c = 0$, then $f(0) = -1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

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26. Find the value of k so that the function f is continuous at the indicated point.

$$f(x)\begin{cases}\dfrac{k\cos x}{\pi-2x}, & if \quad x\neq\dfrac{\pi}{2}\\ 3, & if \quad x=\dfrac{\pi}{2} \quad at x=\dfrac{\pi}{2}\end{cases}$$

Solution:

The given function f is $f(x)\begin{cases}\dfrac{k\cos x}{\pi-2x}, & if \quad x\neq\dfrac{\pi}{2}\\ 3, & if \quad x=\dfrac{\pi}{2}\end{cases}$

The given function f is continuous at $x=\frac{\pi}{2}$, it is defined at $x=\frac{\pi}{2}$ and if the value of the f at $x=\frac{\pi}{2}$ equals the limit of f at $x=\frac{\pi}{2}$.

It is evident that f is defined at $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=3$

$$\lim_{x\to}\frac{\pi}{2}f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

Put $x=\frac{\pi}{2}+h$

Then, $x\to\frac{\pi}{2}\Rightarrow h\to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}}f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{x\to 0}\frac{-\sinh}{-2h}=\frac{k}{2}\lim_{x\to 0}\frac{\sinh}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}}f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of k is 6.

27. Find the value of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx^2, & if\ x\le 2 \\ 3, & if\ x>2 \end{cases} \text{ at } x=2$$

Solution:

The given function is $f(x)=\begin{cases} kx^2, & if\ x\le 2 \\ 3, & if\ x>2 \end{cases}$

The given function f is continuous at $x=2$, if f is defined at $x=2$ and if the value of f at $x=2$ equals the limit of f at $x=2$

It is evident that f is defined at $x=2$ and $f(2)=k(2)^2=4k$

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\Rightarrow \lim_{x\to 2^-}(kx)^2=\lim_{x\to 2^+}(3)=4k$

$\Rightarrow k\times 2^2=3=4k$

$\Rightarrow 4k=3=4k$

$\Rightarrow 4k=3$

$\Rightarrow k=\frac{3}{4}$

Therefore, the required value of k is $\frac{3}{4}$.

28. Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & if\ x\le \pi \\ \cos x, & if\ x>\pi \end{cases} at\ x=\pi$$

Solution:

The given function is $f(x)=\begin{cases} kx+1, & if\ x\le \pi \\ \cos x, & if\ x>\pi \end{cases}$

The given function f is continuous at $x=\pi$ and, if f is defined at $x=\pi$ and if the value of f at $x=\pi$ equals the limit of f at $x=\pi$

It is evident that f is defined at $x=\pi$ and $f(\pi)=k\pi+1$

$\lim_{x\to \pi^-} f(x)=\lim_{x\to \pi^+} f(x)=f(\pi)$

$\Rightarrow \lim_{x\to \pi^-}(kx+1)=\lim_{x\to \pi^+}\cos x=k\pi+1$

$\Rightarrow k\pi+1=\cos\pi=k\pi+1$

$\Rightarrow k\pi+1=-1=k\pi+1$

$\Rightarrow k\pi+1=-1=k\pi+1$

$\Rightarrow k=-\frac{2}{\pi}$

Therefore, the required value of k is $-\frac{2}{\pi}$.

29. Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & if \quad x\le 5 \\ 3x-5, & if \quad x>5 \end{cases}$$

Solution:

The given function of f is $f(x)=\begin{cases} kx+1, & if \quad x\le 5 \\ 3x-5, & if \quad x>5 \end{cases}$

The given function f is continuous at $x=5$, if f is defined at $x=5$ and if the value of f at $x=5$ equals the limit of f at $x=5$

It is evident that f is defined at $x=5$ and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$

$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$

$$\Rightarrow 5k+1=15-5=5k+1$$

$$\Rightarrow 5k+1=10$$

$$\Rightarrow 5k=9$$

$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$

30. Find the vales of a and b such that the function defined

$f(x)=\begin{cases} 5, & if \quad x\le 2 \\ ax+b, & if \quad 2<x<10 \\ 21 & if \quad x\ge 10 \end{cases}$ is continuous function.

Solution:

The given function f is $f(x)=\begin{cases} 5, & if \quad x\le 2 \\ ax+b, & if \quad 2<x<10 \\ 21 & if \quad x\ge 10 \end{cases}$

It is evident that the given function *f* is defined at all points of the real line.

If *f* is a continuous function, then *f* is a continuous at all real numbers.

In a particular, *f* is continuous at $x=2$ and $x=10$

Since *f* is continuous at $x=2$, we obtain

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-}(5) = \lim_{x\to 2^+}(ax+b) = 5$$

$$\Rightarrow 5 = 2a+b = 5$$

$$\Rightarrow 2a+b=5 \qquad \text{.......... (1)}$$

Since *f* is continuous at $x=10$, we obtain

$$\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$$

$$\Rightarrow \lim_{x\to 10^-}(ax+b) = \lim_{x\to 2^+}(21) = 21$$

$$\Rightarrow 10a+b-21=21$$

$$\Rightarrow 10a+b=21 \qquad \text{.......... (2)}$$

On subtracting equation (1) from equation (2), we obtain

$$8a=16$$

$$\Rightarrow a=2$$

By putting $a=2$ in equation (1), we obtain

$$2\times 2+b=5$$

$$\Rightarrow 4+b=5$$

$$\Rightarrow b=1$$

Therefore, the values of *a* and *b* for which *f* is a continuous function are 2 and 1 respectively.

31. Show that the function defined by $f(x)=\cos(x^2)$ is a continuous function.

Solution:

The given function is $f(x)=\cos(x^2)$

This function *f* is defined for every real number and f can be written as the composition of two functions as,

$f = goh$, where $g(x)=\cos x$ and $h(x)=x^2$

$\left[\because (goh)(x)=g(h(x))=g(x^2)=\cos(x^2)=f(x)\right]$

It has to be first proves that $g(x)=\cos x$ and $h(x)=x^2$ are continuous functions.

It is evident that *g* is defined foe every real number.

Let *c* be a real number.

Then, $g(c)=\cos c$

Put $x=c+h$

If $x \to c$, then $h \to 0$

$\lim_{x\to c} g(x)=\lim_{x\to c}\cos x$

$=\lim_{h\to 0}\cos(c+h)$

$=\lim_{h\to 0}[\cos c\cosh-\sin c\sinh]$

$=\lim_{h\to 0}\cos c\cosh-\lim_{h\to 0}\sin c\sinh$

$=\cos c\cos 0-\sin c\sin 0$

$=\cos c\times 1-\sin c\times 0$

$=\cos c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, $g(x)=\cos x$ is a continuous function

$h(x)=x^2$

Clearly, *h* is defined for every real number.

Let *k* be a real number, then $h(k)=k^2$

$\lim_{x\to k} h(x)=\lim_{x\to k} x^2=k^2$

$\therefore \lim_{x\to k} h(x)=h(k)$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*, such that (goh) is defined at *c*, it *g* is continuous at *c* and it *f* is continuous at *c*.

Therefore, $f(x)=(goh)(x)=\cos(x^3)$ is a continuous function.

32. Show that the function defined by $f(x)=|\cos x|$ is a continuous function

Solution:

The given function is $f(x)=|\cos x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f=goh$, where $g(x)=|x|$ and $h(x)=\cos x$

$\left[\because (goh)(x)=g(h(x))=g(\cos x)=|\cos x|=f(x)\right]$

It has to be first proves that $g(x)=|x|$ and $h(x)=\cos x$ are continuous functions.

$g(x)=|x|$, can be written as

$$g(x)=\begin{cases}-x & if \quad x<0\\ x & if \quad x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that $x<0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim_{x\to c} g(x)=\lim_{x\to c}(-x)=c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that $x>0$

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Case III:

If $c=0$, then $g(c)=g(0)=0$ and $\lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$

$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$

$$\therefore \lim_{x\to c^-} g(x)=\lim_{x\to c^+} g(x)=g(0)$$

Therefore, g is continuous at $x=0$

From the above three observations, it can be concluded that g is continuous at all points.

$$h(x)=\cos x$$

It is evident that $h(x)=\cos x$ is defined for every real number.

Let c be a real number. Put $x=c+h$

If $x\to c$, then $h\to 0$

$$h(c)=\cos c$$

$$\lim_{x\to c} h(x)=\lim_{x\to c}\cos x$$

$$=\lim_{h\to 0}\cos(c+h)$$

$$=\lim_{h\to 0}[\cos c\cosh-\sin c\sinh]$$

$$=\lim_{h\to 0}\cos c\cosh-\lim_{h\to 0}\sin c\sinh$$

$$=\cos c\cos 0-\sin c\sin 0$$

$$=\cos c\times 1-\sin c\times 0$$

$$=\cos c$$

$$=\lim_{h\to c} h(x)=h(c)$$

Therefore, $h(x)=\cos x$ is continuous function/

It is known that for real values functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\cos x)=|\cos x|$ is a continuous function.

33. Examine that $\sin|x|$ is a continuous function

Solution:

Let $f(x)=\sin|x|$

This function *f* is defined for every real number and *f* can be written as the composition of two functions as,

$f=goh$, where $g(x)=|x|$ and $h(x)=\sin x$

$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$

It has to be prove first that $g(x)=|x|$ and $h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x & if \quad x<0\\ x & if \quad x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, that $x<0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that $x>0$

Case III:

If $c=0$, then $g(c)=g(0)=0$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$

$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$

$$\therefore \lim_{x\to c^-} g(x)=\lim_{x\to c^+} g(x)=g(0)$$

Therefore, g is continuous at $x=0$

From the above three observations, it can be concluded that g is continuous at all points.

$$h(x)=\sin x$$

It is evident that $h(x)=\sin x$ is defined for every real number.

Let c be a real number. Put $x=c+k$

If $x\to c$, then $k\to 0$

$$h(c)=\sin c$$

$$\lim_{x\to c} h(x)=\lim_{x\to c}\sin x$$

$$=\lim_{k\to 0}\sin(c+k)$$

$$=\lim_{k\to 0}[\sin c\cos k+\cos c\sin k]$$

$$=\lim_{k\to 0}(\sin c\cos k)-\lim_{k\to 0}(\cos c\sin k)$$

$$=\sin c\cos 0+\cos c\sin 0$$

$$=\sin c+0$$

$$=\sin c$$

$$=\lim_{x\to c} h(x)=g(c)$$

Therefore, h is continuous function.

It is known that for real values functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (foh) is continuous at c.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\sin x)=|\sin x|$ is a continuous function.

34. Find all the points of discontinuity of *f* defined by $f(x)=|x|-|x+1|$.

Solution:

The given function is $f(x)=|x|-|x+1|$

The two functions, *g* and *h*, are defined as

$g(x)=|x|$ and $h(x)=|x+1|$

Then $f=g-h$

The continuous of *g* and *h* is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases} -x & if \quad x<0 \\ x & if \quad x\geq 0 \end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=g(0)=-c$ and $\lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, that $x<0$

Case II:

If $c>0$, then $g(c)=c$ and $\lim_{x\to c} g(x)=\lim_{x\to c} x=c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that $x>0$

Case III:

If $c=0$, then $g(c)=g(0)=0$

$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(x) = 0$$

$$\therefore \lim_{x\to c^-} g(x) = \lim_{x\to c^+} g(x) = g(0)$$

Therefore, *g* is continuous at $x = 0$

From the above three observations, it can be concluded that *g* is continuous at all points.

$$h(x) = |x+1|$$

$$h(x) = \begin{cases} -(x+1), & if, \quad x < -1 \\ x+1, & if, \quad x \geq -1 \end{cases}$$

Clearly, *h* is defined for every real number.

Let c be a real number

Case I:

If $c < -1$, then $h(c) = -(c+1)$ and $\lim_{x\to c} h(x) = \lim_{x\to c}[-(x+1)] = -(c+1)$

$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, *h* is continuous at all points *x*, such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c+1$ and $\lim_{x\to c} h(x) = \lim_{x\to c} x+1 = (c+1)$

$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, *g* is continuous at all points *x*, such that $x > -1$

Case III:

If $c = -1$, then $h(c) = h(-1) = -1+1 = 0$

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-}[-(x+1)] = -(-1+1)0$$

$$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+}(x+1) = (-1+1) = 0$$

$$\therefore \lim_{x\to 1^-} h = \lim_{x\to 1^+} h(x) = h(-1)$$

Therefore, *h* is continuous at $x = -1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.