Chapter 5: Continuity and differentiability.

Exercise 5.2

1. Differentiate the function with respect to x. $\sin\left(x^2+5\right)$

Solution:

Let $f(x)=\sin\left(x^2+5\right), u(x)=x^2+5, \text{ and } v(t)=\sin t$

Then, $(vou)(x)=v\left(u(x)\right)=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, f is a composite of two functions

Put $t=u(x)=x^2+5$

Then, we obtain

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$$

Therefore, by chain rule, $\frac{df}{dx}=\frac{dv}{dt}\frac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$

Alternate method

$$\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]=\cos\left(x^2+5\right)\frac{d}{dx}\left(x^2+5\right)$$

$$=\cos\left(x^2+5\right)\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]$$

$$=\cos\left(x^2+5\right)[2x+0]$$

$$=2x\cos\left(x^2+5\right)$$

2. Differentiate the functions with respect of x. $\cos(\sin x)$

Solution:

Let $f(x)=\cos(\sin x), u(x)=\sin x, \text{ and } v(t)=\cos t$

Then, $(vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$

Thus, f is a composite function of two functions

Put $t=u(x)=\sin x$

$$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$

$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

By chain rule, $\frac{df}{dx}, \frac{dv}{dt}.\frac{dt}{dx}=-\sin(\sin x)\cos x=-\cos x\sin(\sin x)$

Alternate method

$$\frac{d}{dx}[\cos(\sin x)]=-\sin(\sin x)\frac{d}{dx}(\sin x)=-\sin(\sin x)-\cos x=-\cos x\sin(\sin x)$$

3. Differentiate the functions with respect of x. $\sin(ax+b)$

Solution:

Let $f(x)=\sin(ax+b), u(x)=ax+b, \text{ and } v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$

Thus, f is a composite function of two functions u and v

Put $t=u(x)=ax+b$

Therefore, $\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx}=\frac{dv}{dt}\frac{dt}{dx}=\cos(ax+b)a=a\cos(ax+b)$$

Alternate method

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b)\frac{d}{dx}(ax+b)$$

$$=\cos(ax+b)\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$

$$=\cos(ax+b)(a+0)$$

$$=a\cos(ax+b)$$

4. Differentiate the functions with respect of x. $\sec\left(\tan\left(\sqrt{x}\right)\right)$

Solution:

Let $f(x)=\sec\left(\tan\left(\sqrt{x}\right)\right), u(x)=\sqrt{x}, v(t)=\tan t, \text{ and } w(s)=\sec s$

Then, $(wovou)(x)=w\left[v\left(u(x)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, f is a composite function of three functions, u, v and w

Put $s=v(t)=\tan t \text{ and } t=u(x)=\sqrt{x}$

Then, $\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t)\tan(\tan t) \quad [s=\tan t]$

$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right) \quad \left[t=\sqrt{x}\right]$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}.x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\frac{ds}{dt}\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right).\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

5. Differentiate the functions with respect of X. $\frac{\sin(ax+b)}{\cos(cx+d)}$

Solution:

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where $g(x)=\sin(ax+b)$ and

$h(x)=\cos(cx+d)$

$\therefore h(x)=\cos(cx+d)$

$$\therefore f=\frac{g'h-gh'}{h^2}$$

Consider $g(x)=\sin(ax+b)$

Let $u(x)=ax+b, v(t)=\sin t$

Then $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

$\therefore$ g is a composite function of two functions, u and v

Put $t=u(x)=ax+b$

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Therefore, by chain rule, we obtain

$$g'=\frac{dg}{dx}=\frac{dv}{dt}\frac{dt}{dx}=\cos(ax+b).a=a\cos(ax+b)$$

Consider $h(x)=\cos(cx+d)$

Let $p(x)=cx+d, q(y)=\cos y$

Then, $(qop)(x)=q(p(x))=q(cx+d)=\cos(cx+d)=h(x)$

$\therefore$ h is a composite function of two functions, p and q

Put $y=p(x)=cx+d$

$$\frac{dq}{dy}=\frac{d}{dy}(\cos y)=-\sin y=-\sin(cx+d)$$

$$\frac{dy}{dx}=\frac{d}{dx}(cx+d)=\frac{d}{dx}(cx)+\frac{d}{dx}(d)=c$$

Therefore, by chain rule, we obtain

$$h'=\frac{dh}{dx}=\frac{dq}{dy}\frac{dy}{dx}=-\sin(cx+d)\times c=-c\sin(cx+d)$$

$$\therefore f' = \frac{a\cos(ax+b)\cos(cx+d) - \sin(ax+b)\{-c\sin cx+d\}}{[\cos(cx+d)]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\frac{\sin(cx+d)}{\cos(cx+d)} \times \frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+b)\sec(cx+d)$$

6. Differentiate the function with respect to x. $\cos x^3.\sin^2(x^5)$

Solution:

$$\cos x^3.\sin^2(x^5)$$

$$\frac{d}{dx}\left[\cos x^3.\sin^2(x^5)\right] = \sin^2(x^5) \times \frac{d}{dx}(\cos x^3) + \cos^3 \times \frac{d}{dx}\left[\sin^2(x^5)\right]$$

$$= \sin^2(x^5) \times (-\sin x^3) \times \frac{d}{dx}(x^3) + \cos x^3 + 2\sin(x^5)\frac{d}{dx}\left[\sin x^5\right]$$

The given function is $= \sin x^3 \sin^2(x^5) \times 3x^2 + 2\sin x^5 \cos x^3.\cos x^5 \times \frac{d}{dx}(x^5)$

$$= 3x^2 \sin x^3 \sin^3(x^5) + 2\sin x^5 \cos x^5 \cos x^3 \times 5x^4$$

$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2(x^5)$$

7. Differentiate the functions with respect to x.

$$\sqrt[2]{\cot(x^2)}$$

Solution:

$$\frac{d}{dx}\left[\sqrt[2]{\cot(x^2)}\right]$$

$$= 2.\frac{1}{\sqrt[2]{\cot(x^2)}} \times \frac{d}{dx}\left[\cot(x^2)\right]$$

$$=\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times -\cos ec^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$

$$=\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{1}{\sin^2\left(x^2\right)}\times\left(2x\right)$$

$$=\frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2\sin x^2}}$$

$$=\frac{-2\sqrt{2}x}{\sqrt{2\sin x^2\cos x^2\sin x^2}}$$

$$=\frac{-2\sqrt{2}x}{\sin x^2\sqrt{\sin 2x^2}}$$

8. Differentiate the functions with respect to x

$\cos\left(\sqrt{x}\right)$

Solution:

Let $f\left(x\right)=\cos\left(\sqrt{x}\right)$

Also, let $u\left(x\right)=\sqrt{x}$

And, $v\left(t\right)=\cos t$

Then, $\left(vou\right)\left(x\right)=v\left(u\left(x\right)\right)$

$=v\left(\sqrt{x}\right)$

$=\cos\sqrt{x}$

$=f\left(x\right)$

Clearly, f is a composite function of two functions, u and v, such that $t=u\left(x\right)=\sqrt{x}$

Then, $\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

$\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

And, $\frac{dv}{dt}=\frac{d}{dt}(\cos t)=-\sin t=-\sin\sqrt{x}$

By using chain rule, we obtain

$$\frac{dt}{dx}=\frac{dv}{dt}\frac{dt}{dx}$$

$$=-\sin\left(\sqrt{x}\right)\frac{1}{2\sqrt{x}}$$

$$=-\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)$$

$$=-\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right]=-\sin\left(\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=-\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=-\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}$$

$$=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

9. Prove that the function f given by $f(x)=|x-1|, x\in R$ is not differentiable at x = 1

Solution:

The given function is $f(x)=|x-1|, x\in R$

It is known that a function f is differentiable at a point x = c in its domain if both

$\lim\limits_{k\to 0^-}\frac{f(c+h)-f(c)}{k}$ *and* $\lim\limits_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal

To check the differentiability of the given function at $x=1$,

Consider the left hand limit of f at $x=1$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{f|I+h-1||1-1|}{h}$$

$$\lim_{h\to0^-}\frac{|h|-0}{h}=\lim_{h\to0^-}\frac{-h}{h}\qquad (h<0\Rightarrow|h|=-h)$$

= - 1

Consider the right hand limit of f at $x=1$

$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{f|I+h-1|-|1-1|}{h}$$

$$=\lim_{h\to0^+}\frac{|h|-0}{h}=\lim_{h\to0^+}\frac{h}{h}\qquad (h>0\Rightarrow|h|=h)$$

= 1

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at $x=1$

10. Prove that the greatest integer function defined by $f(x)=[x],0<x<3$is not differentiable at $x=1\,and\,x=2$

Solution:

The given function f is $f(x)=[x],0<x<3$

It is known that a function f is differentiable at a point $x=c$in its domain if both $\lim_{h\to0^-}\frac{f(c+h)-f(c)}{h}$ *and* $\lim_{h\to0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal

To check the differentiable of the given function at x = 1, consider the left hand limit of f at x = 1

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to0^-}\frac{0-1}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at x = 1

$$= \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[1+h][1]}{h}$$

$$= \lim_{h \to 0^+} \frac{1-1}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right limits of f at x = 1 are not equal, f is not differentiable at x = 1

To check the differentiable of the given function at x=2, consider the left hand limit of f at x = 2

$$= \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^-} \frac{1-2}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

Consider the right hand limit of f at x= 1

$$= \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^+} \frac{1-2}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at x = 2 are not equal, f is not differentiable at $x = 2$