Chapter 5: Continuity and differentiability.

1.   Differentiate the following with respect to x.

$\cos x.\cos 2x.\cos 3x$

Solution:

Let $y = \cos x.\cos 2x.\cos 3x$

Taking logarithm or both the side, we obtain

$\log y = \log\left(\cos x.\cos 2x.\cos 3x\right)$

$\Rightarrow \log y = \log\left(\cos x\right) + \log\left(\cos 2x\right) + \log\left(\cos 3x\right)$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{\cos x}\dfrac{d}{dx}\left(\cos x\right) + \dfrac{1}{\cos 2x}\dfrac{d}{dx}\left(\cos 2x\right) + \dfrac{1}{\cos 3x}\dfrac{d}{dx}\left(\cos 3x\right)$

$\Rightarrow \dfrac{dy}{dx} = y\left[-\dfrac{\sin x}{\cos x} - \dfrac{\sin 2x}{\cos 2x}\dfrac{d}{dx}\left(2x\right) - \dfrac{\sin 3x}{\cos 3x}\dfrac{d}{dx}\left(3x\right)\right]$

$\therefore \dfrac{dy}{dx} = -\cos x.\cos 2x.\cos 3x\left[\tan x + 2\tan 2x + 3\tan 3x\right]$

2.   Differentiate the functions with respect to x.

$\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution:

Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm or both the side, we obtain

$\log y = \log\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

$$\Rightarrow \log y = \frac{1}{2} \log \left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right]$$

$$\Rightarrow \log y = \frac{1}{2} \Big[\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\} \Big]$$

$$\Rightarrow \log y = \frac{1}{2} \Big[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5) \Big]$$

Differentiating both sides with respect to, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}\frac{d}{dx}(x-1) + \frac{1}{x-2}\frac{d}{dx}(x-2) - \frac{1}{x-3}\frac{d}{dx}(x-3) - \frac{1}{x-4}\frac{d}{dx}(x-4) - \frac{1}{x-5}\frac{d}{dx}(x-5) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right]$$

3. Differentiate the functions with respect to x.

$$(\log x)^{\cos x}$$

Solution:

Let $y = (\log x)^{\cos x}$

Taking logarithm or both the side, we obtain

$$\log y = \cos x.\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\cos x) \times \log(\log x) + \cos x \times \frac{d}{dx}\big[\log(\log x)\big]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\sin x \log(\log x) + \cos x \times \frac{1}{\log x}\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\sin x \log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x} \right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\cos x} \left[\frac{\cos x}{x \log x} - \sin \times \log (\log x) \right]$$

4. Differentiate the functions with respect to x.

$$x^x - 2^{\sin x}$$

Solution:

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u$ and $2^{\sin x} = v$

$\therefore y = u - v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$u = x^x$

Taking logarithm on both sides, we obtain

$\log u = x \log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x) \right]$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \times \log x + x \times \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^x (\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x (1 + \log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to x, we obtain

$\log v = \sin x . \log 2$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \log 2 \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v\log 2\cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x}\cos x\log 2$$

$$\therefore \frac{dv}{dx} = x^2(1+\log x) - 2^{\sin x}\cos x\log 2$$

5. Differentiate the functions with respect to x.

$$(x+3)^2.(x+4)^3.(x+5)^4$$

Solution:

Let $y = (x+3)^2(x+4)^3(x+5)^4$

Taking logarithm on both sides, we obtain

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$

$$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 2.\frac{1}{x+3}\frac{d}{dx}(x+3) + 3.\frac{1}{x+4}\frac{d}{dx}(x+4) + 4.\frac{1}{x+5}\frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3.\left[2(x^2+9x+20)+3(x^2+9x+15)+4(x^2+7x+12)\right]$$

$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

6. Differentiate the function with respect to x.

$$\left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$$

Solution:

Let $y = \left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Also, let $u = \left(x+\frac{1}{x}\right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{..........(1)}$$

Then, $u = \left(x+\frac{1}{x}\right)^x$

Taking log on both sides

$$\Rightarrow \log u = \log\left(x+\frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right) + x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = 1\times\log\left(x+\frac{1}{x}\right) + x\times\frac{1}{\left(x+\frac{1}{x}\right)}\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log\left(x + \frac{1}{x}\right) + \frac{x}{\left(x + \frac{1}{x}\right)} \times \left(x + \frac{1}{x^2}\right)\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\log\left(x + \frac{1}{x}\right) + \frac{\left(x - \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\log\left(x + \frac{1}{x}\right) + \frac{x^2 - 1}{x^2 + 1}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)\left[\frac{x^2 - 1}{x^2 + 1} + \log\left(x + \frac{1}{x}\right)\right] \quad \dots\dots\dots\dots(2)$$

$$v = x^{\left(x + \frac{1}{x}\right)}$$

Taking log on both sides, we obtain

$$v = x^{\left(x + \frac{1}{x}\right)}$$

$$\Rightarrow \log v = \left(1 + \frac{1}{x}\right)\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \left[\frac{d}{dx}\left(1 + \frac{1}{x}\right)\right] \times \log x + \left(1 + \frac{1}{x}\right)\frac{d}{dx}\log x$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left(-\frac{1}{x^2}\right)\log x + \left(1 + \frac{1}{x}\right)\frac{1}{x}$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{-\log x + x + 1}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = x^{\left(x+\frac{1}{x}\right)} \left(\frac{x+1-\log x}{x^2}\right) \quad \dots\dots\dots(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x \left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] + x^{\left(x+\frac{1}{x}\right)} \left(\frac{x+1-\log x}{x^2}\right)$$

7. Differentiate the functions with respect to x.

$$\left(\log x\right)^x + x^{\log x}$$

Solution:

Let $y = \left(\log x\right)^x + x^{\log x}$

Also, let $u = \left(\log x\right)^x \text{ and } v = x^{\log x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots\dots\dots(1)$$

$u = \left(\log x\right)^x$

$$\Rightarrow \log u = \log\left[\left(\log x\right)^x\right]$$

$$\Rightarrow \log u = x\log\left(\log x\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)x\log\left(\log x\right) + x.\frac{d}{dx}\left[\log\left(\log x\right)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1x\log\left(\log x\right) + x.\frac{1}{\log x}\frac{d}{dx}\left(\log x\right)\right]$$

$$\Rightarrow \frac{du}{dx} = \left(\log x\right)^x \left[\log\left(\log x\right) + \frac{x}{\log x}\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x} \right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x \left[\frac{\log(\log x).\log x + 1}{\log x} \right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x-1} \left[1 + \log x.\log(\log x) \right] \qquad ....(2)$$

$$v = x^{\log x}$$

$$\Rightarrow \log v = \log \left(x^{\log x} \right)$$

$$\Rightarrow \log v = \log x \log x = (\log x)^2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \frac{dv}{dx} = \frac{d}{dx} \left[(\log x)^2 \right]$$

$$\Rightarrow \frac{1}{v} \frac{dv}{dx} = 2(\log x) \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x} \frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x - 1} \log x \qquad ....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx} = (\log x)^{x-1} \left[1 + \log x.\log(\log x) \right] + 2x^{\log x - 1} \log x$$

8.   Differentiate the functions with respect to x

$$(\sin x)^x + \sin^{-1} \sqrt{x}$$

Solution:

Let $y = (\sin x)^x + \sin^{-1} \sqrt{x}$

Also, let $u = (\sin x)^x \ and \ v = \sin^{-1} \sqrt{x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad ....(1)$$

$$u = (\sin x)^x$$

$$\Rightarrow \log u = \log (\sin x)^x$$

$$\Rightarrow \log u = x \log (\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\sin x) + x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1.\log(\sin x) + x.\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x}.\cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left(x\cot x + \log \sin x\right) \qquad ....(2)$$

$$v = \sin^{-1}\sqrt{x}$$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1 - \left(\sqrt{x}\right)^2}}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \qquad ....(3)$$

Therefore, from (1), (2) and (3) we obtain

$$\frac{dy}{dx} = (\sin x)^2\left(x\cot x + \log \sin x\right) + \frac{1}{2\sqrt{x-x^2}}$$

9. Differentiate the function with respect to x.

$$x^{\sin x} + (\sin x)^{\cos x}$$

Solution:

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$\therefore y = u + v$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} \qquad ....(1)$

$u = x^{\sin x}$

$\Rightarrow \log u = \log \left(x^{\sin x} \right)$

$\Rightarrow \log u = \sin x \log x$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{u}\dfrac{du}{dx} = \dfrac{d}{dx}\left(\sin x \right).\log x + \sin x \dfrac{d}{dx}\left(\log x \right)$

$\Rightarrow \dfrac{du}{dx} = u = \left[\cos x \log x + \sin x.\dfrac{1}{x} \right]$

$\Rightarrow \dfrac{du}{dx} = x^{\sin x}\left[\cos x \log x + \dfrac{\sin x}{x} \right] \qquad ....(2)$

$v = \left(\sin x \right)^{\cos x}$

$\Rightarrow \log v = \log \left(\sin x \right)^{\cos x}$

$\Rightarrow \log v = \cos x \log \left(\sin x \right)$

Differentiating both sides with respect to x, we obtain

$\dfrac{1}{v}\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\cos x \right) \times \log \left(\sin x \right) + \cos x \times \dfrac{d}{dx}\left[\log \left(\sin x \right) \right]$

$\Rightarrow \dfrac{dv}{dx} = v\left[-\sin x.\log \left(\sin x \right) + \cos x.\dfrac{1}{\sin x}\dfrac{d}{dx}\left(\sin x \right) \right]$

$\Rightarrow \dfrac{dv}{dx} = \left(\sin x \right)^{\cos x}\left[-\sin x.\log \sin x + \dfrac{\cos x}{\sin x}\cos x \right]$

$\Rightarrow \dfrac{dv}{dx} = \left(\sin x \right)^{\cos x}\left[-\sin x \log \sin x + \cot x \cos x \right]$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x} \left[\cot x \cos x - \sin x \log \sin x \right] \qquad ......(3)$$

Therefore, from (1), (2) and (3) we obtain

$$\frac{dy}{dx} = x^{\sin x} \left(\cos x \log x + \frac{\sin x}{x} \right) + (\sin x)^{\cos x} \left[\cos x \cot x - \sin x \log \sin x \right]$$

10. Differentiate the function with respect to x.

$$x^{x\cos x} + \frac{x^2 + 1}{x^2 - 1}$$

Solution:

Let $y = x^{x\cos x} + \dfrac{x^2 + 1}{x^2 - 1}$

Also, let $u = x^{x\cos x}$ and $v = \dfrac{x^2 + 1}{x^2 - 1}$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\therefore y = u + v$$

$$u = x^{x\cos x}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\cos x \log x + x \frac{d}{dx}(\cos x)\log x + x\cos x \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[1.\cos x.\log x + x.(-\sin x)\log x + x\cos x.\frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x} \left(\cos x \log x - x\sin x \log x + \cos x \right)$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x} \left[\cos x(1 + \log x) - x\sin x \log x \right] \qquad ....(2)$$

$$v = \frac{x^2 + 1}{x^2 - 1}$$

$$\Rightarrow \log v = \log\left(x^2 + 1\right) - \log\left(x^2 - 1\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} = \frac{dv}{dx} = \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{2x\left(x^2 - 1\right) - 2x\left(x^2 + 1\right)}{\left(x^2 + 1\right)\left(x^2 - 1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{x^2 + 1}{x^2 - 1} \times \left[\frac{-4x}{\left(x^2 + 1\right)\left(x^2 - 1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{\left(x^2 - 1\right)^2} \qquad \ldots\ldots\ldots(3)$$

Therefore, from (1), (2) and (3) we obtain

$$\frac{dy}{dx} = x^{x\cos x}\left[\cos x\left(1 + \log x\right) - x\sin x \log x\right] - \frac{4x}{\left(x^2 - 1\right)^2}$$

11. Differentiate the function with respect to x.

$$\left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$$

Solution:

Let $y = \left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$

Also, let $u = \left(x\cos x\right)^x$ and $v = \left(x\sin x\right)^{\frac{1}{x}}$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots\ldots(1)$$

$$u = \left(x\cos x\right)^2$$

$$\Rightarrow \log u = \log\left(x\cos x\right)^x$$

$$\Rightarrow \log u = x \log (x \cos x)$$

$$\Rightarrow \log u = x [\log x + \log \cos x]$$

$$\Rightarrow \log u = x \log x + x \log \cos x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \frac{du}{dx} = \frac{d}{dx} (x + \log x) + \frac{d}{dx} (x \log \cos x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\left\{ \log x \frac{d}{dx} (x) + x \frac{d}{dx} (\log x) \right\} + \left\{ \log \cos x \frac{d}{dx} (x) + x \frac{d}{dx} (\log \cos x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[\left(\log x.1 + x.\frac{1}{x} \right) + \left\{ \log \cos x.1 + x.\frac{1}{\cos x}.\frac{d}{dx} (\cos x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[\left(\log x.1 + x.\frac{1}{x} \right) + \left\{ \log \cos x.1 + x.\frac{1}{\cos x}.\frac{d}{dx} (\cos x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[(\log x + 1) + \left\{ \log \cos x + \frac{x}{\cos x} (-\sin x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[(1 + \log x) + (\log \cos x - x \tan x) \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[1 - x \tan x + (\log x + \log \cos x) \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[1 - x \tan x + \log (x \cos x) \right] \quad ....(2)$$

$$v = (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x} \log (x \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} (\log x + \log \sin x)$$

$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log \sin x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}(\log x)\right] + \left[\log(\sin x).\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{x}\right] + \left[\log(\sin x).\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1-\log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}.\cos x\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(x\sin x)^{\frac{1}{x}} + \left[\frac{1-\log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x - \log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x) + x\cot x}{x^2}\right] \quad .....(3)$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx} = (x\cos x)^2\left[1-x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

12. Find $\frac{dy}{dx}$ of function.

$$x^y + y^x = 1$$

Solution:

The given function is $x^y + y^x = 1$

Let $x^y = u \ and \ y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad .... (1)$$

$$u = x^y$$

$$\Rightarrow \log u = \log\left(x^y\right)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y.\frac{d}{dx}\left(\log x\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad .... (2)$$

$$v = y^x$$

$$\Rightarrow \log v = \log\left(y^x\right)$$

$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \log y \frac{d}{dx}\left(x\right) + x.\frac{d}{dx}\left(\log y\right)$$

$$\Rightarrow \frac{dv}{dx} = v\left(\log y.1 + x.\frac{1}{y}.\frac{d}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad .... (3)$$

Therefore, from (1), (2) and (3) we obtain

$$x^y \left(\log x \frac{dy}{dx} + + \frac{y}{x} \right) + y^x \left(\log y + \frac{x}{y} \frac{dy}{dx} \right) = 0$$

$$\Rightarrow \left(x^2 + \log x + xy^{y-1} \right) \frac{dy}{dx} = -\left(yx^{y-1} + y^x \log y \right)$$

$$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

13. Find $\frac{dy}{dx}$ of function $y^x = x^y$

Solution:

The given function is $y^x = x^y$

Taking logarithm on both sides, we obtain

$x \log y = y \log x$

Differentiating both sides with respect to x, we obtain

$$\log y . \frac{d}{dx}(x) + x . \frac{d}{dx}(\log y) = \log x . \frac{d}{dx}(y) + y . \frac{d}{dx}(\log x)$$

$$\Rightarrow \log y . 1 + x . \frac{1}{y} . \frac{dy}{dx} = \log x . \frac{dy}{dx} + y . \frac{1}{x}$$

$$\Rightarrow \log y + \frac{x}{y} \frac{dy}{dx} = \log x \frac{dy}{dx} + \frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y} - \log x \right) \frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \left(\frac{x - y \log x}{y} \right) \frac{dy}{dx} = \frac{y - x \log y}{x}$$

$$\Rightarrow \left(\frac{x - y \log x}{y} \right) \frac{dy}{dx} = \frac{y - x \log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \left(\frac{y - x \log y}{x - y \log x} \right)$$

14. Find $\dfrac{dy}{dx}$ of friction $(\cos x)^y = (\cos y)^x$

Solution:

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both sides, we obtain

$y = \log \cos x = x \log \cos y$

Differentiating both sides with respect to x, we obtain

$\log \cos x.\dfrac{dy}{dx} + y.\dfrac{d}{dx}(\log \cos x) = \log \cos y.\dfrac{d}{dx}(x) + x.\dfrac{d}{dx}(\log \cos y)$

$\Rightarrow \log \cos x.\dfrac{dy}{dx} + y.\dfrac{1}{\cos x}.\dfrac{d}{dx}(\cos x) = \log \cos y.1 + x.\dfrac{1}{\cos y}\dfrac{d}{dx}(\cos y)$

$\Rightarrow \log \cos x\dfrac{dy}{dx} + \dfrac{y}{\cos x}(-\sin x) = \log \cos y + \dfrac{x}{\cos y}(-\sin y)\dfrac{dy}{dx}$

$\Rightarrow \log \cos x\dfrac{dy}{dx} - y\tan x = \log \cos y - x\tan y\dfrac{dy}{dx}$

$\Rightarrow (\log \cos x + x\tan y)\dfrac{dy}{dx} = y\tan x + \log \cos y$

$\therefore \dfrac{dy}{dx} = \dfrac{y\tan x + \log \cos y}{x\tan y + \log \cos x}$

15. Find $\dfrac{dy}{dx}$ of function $xy = e^{(x-y)}$

Solution:

The given function is $xy = e^{(x-y)}$

Taking logarithm on both sides, we obtain

$\log(xy) = \log\left(e^{x-y}\right)$

$\Rightarrow \log x + \log y = (x - y)\log e$

$$\Rightarrow \log x + \log y = (x - y) \times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

16. Find the derivative of the function given by $f(x) = (1-x)(1+x^2)(1+x^4)(1+x^8)$ and

hence find $f'(1)$

Solution:

The given relationship is $f(x) = (1-x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both sides, we obtain

$$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)} \cdot \frac{d}{dx}[f(x)] = \frac{d}{dx}\log(1+x) + \frac{d}{dx}\log(1+x^2) + \frac{d}{dx}\log(1+x^4) + \frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)} \cdot f'(x) = \frac{1}{1+x} \cdot \frac{d}{dx}(1+x) + \frac{1}{1+x^2} \cdot \frac{d}{dx}(1+x^2) + \frac{1}{1+x^4} \cdot \frac{d}{dx}(1+x^4) + \frac{1}{1+x^8} \cdot \frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{1+x} + \frac{1}{1+x^2} \cdot 2x + \frac{1}{1+x^4} \cdot 4x^3 + \frac{1}{1+x^8} \cdot 8x^7\right]$$

$$\therefore f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Hence, $f'(1) = (1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1} + \frac{2\times1}{1+1^2} + \frac{4\times1^3}{1+1^4} + \frac{8\times1^7}{1+1^8}\right]$

$$= 2\times2\times2\times2\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$$

$$= 16\times\left(\frac{1+2+4+8}{2}\right)$$

$$= 16\times\frac{15}{2} = 120$$

17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three easy mentioned below

    i. By using product rule

    ii. By expanding the product to obtain a single polynomial

    iii. By logarithm differentiate

    Do they all given the same answer?

Solution:

Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

(i) Let $x = x^2 - 5x + 8$ and $u = x^3 + 7x + 9$

$\therefore y = uv$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dv}v + u\frac{dv}{dx} \qquad \text{(By using product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 - 5x + 8).(x^3 + 7x + 9) + (x^2 + 5x + 8).\frac{d}{dx}(x^3 + 7x + 9)$$

$$\Rightarrow \frac{dy}{dx} = (2x - 5)(x^3 + 7x + 9) + (x^2 - 5x + 8)(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + x^2(3x^2 + 7) - 5x(3x^2 + 7) - 8(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = \left(2x^4 + 14x^2 + 18x\right) - 5x^3 - 35x - 45 + \left(3x^2 + 7x^2\right) - 15x^3 - 35x + 24x^2 + 56$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(ii) $y = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)$

$$= x^2\left(x^3 + 7x + 9\right) - 5x\left(x^3 + 7x + 9\right) + 8\left(x^3 + 7x + 9\right)$$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

$$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72\right)$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^5\right) - 5\frac{d}{dx}\left(x^4\right) + 15\frac{d}{dx}\left(x^3\right) - 26\frac{d}{dx}\left(x^2\right) + 11\frac{d}{dx}\left(x\right) + \frac{d}{dx}\left(72\right)$$

$$= 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(iii) Taking logarithm on both sides, we obtain

$$\log y = \log\left(x^2 - 5x + 8\right) + \log\left(x^3 + 7x + 9\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\log\left(x^2 - 5x + 8\right) + \frac{d}{dx}\log\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}\cdot\frac{d}{dx}\left(x^2 - 5x + 8\right) + \frac{1}{x^3 + 7x + 9}\cdot\frac{d}{dx}\left(x^3 + 7x + 9\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x^2 - 5x + 8}\times\left(2x - 5\right) + \frac{1}{x^3 + 7x + 9}\times\left(3x^2 + 7\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{2x - 5}{x^3 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{2x - 5\left(x^3 + 7x + 9\right) + \left(3x^2 + 7\right)\left(x^2 - 5x + 8\right)}{\left(x^3 - 5x + 8\right) + \left(x^3 + 7x + 9\right)}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2x\left(x^3 + 7x + 9\right) - 5\left(x^3 + 7 + 9\right) + 3x^2\left(x^2 - 5x + 8\right) + 7\left(x^2 - 5x + 8\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(2x^5 + 14x^2 + 18x\right) - 5x^3 - 35x - 45 + \left(3x^4 - 15x^3 + 24x^2\right) + \left(7x^2 - 35x + 56\right)$$

$$\Rightarrow \frac{dy}{dx} = 5x^2 - 20x^3 + 45x^2 - 52x + 11$$

From the above three observations, it can be concluded that all the result of $\frac{dy}{dx}$ are same

18. If u, v and w are function of x, then show that $\frac{d}{dx}(u.v.w) = \frac{du}{dx}.v.w + u\frac{dv}{dx}.w + u.v\frac{dw}{dx}$.

In two ways – first by repeated application of product rule, second by logarithmic differentiation.

Solution:

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}(v.w)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v.w + u\left[\frac{dv}{dx}.w + v.\frac{dv}{dx}\right] \qquad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y = u.v.w,$ we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v\frac{dw}{dx}$$