Infinity Learn by Sri Chaitanya Educational Institutions

Chapter 5: Continuity and differentiability.

Exercise 5.6

1. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$, $x=2at^2, y=at^4$

Solution:

The given equation are $x=2at^2$ and $y=at^4$

Then,

$$\frac{dx}{dt}=\frac{d}{dt}\left(2at^2\right)=2a.\frac{d}{dt}\left(t^2\right)=2a.2t=4at$$

$$\frac{dy}{dx}=\frac{d}{dt}\left(at^4\right)a.\frac{d}{dt}\left(t^4\right)=a.4.t^3=4at^3$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{4at^3}{4at}=t^2$$

2. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=a\cos\theta, y=b\cos\theta$

Solution:

The given equations are $x=a\cos\theta$ and $y=b\cos\theta$

Then, $\frac{dx}{d\theta}=\frac{d}{d\theta}\left(a\cos\theta\right)=a\left(-\sin\theta\right)=-a\sin\theta$

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left(b\cos\theta\right)=b\left(-\sin\theta\right)=-b\sin\theta$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{-b\sin\theta}{-a\sin\theta}=\frac{b}{a}$$

3. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=\sin t, y=\cos 2t$

Solution:

The given equations are $x=\sin t$ and $y=\cos 2t$

Then, $\frac{dx}{dt}=\frac{d}{dt}(\sin t)=\cos t$

$$\frac{dy}{dt}=\frac{d}{dt}(\cos 2t)=-\sin 2t.\frac{d}{dt}(2t)=-2\sin 2t$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-2\sin 2t}{\cos t}=\frac{-2.2\sin t\cos t}{\cos t}=-4\sin t$$

4. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=4t, y=\frac{4}{t}$

Solution:

The equation are $x=4t$ and $y=\frac{4}{t}$

$$\frac{dx}{dt}=\frac{d}{dt}(4t)=4$$

$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{4}{t}\right)=4.\frac{d}{dt}\left(\frac{1}{t}\right)=4.\left(\frac{-1}{t^2}\right)=\frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-4}{t^2}\right)}{4}=\frac{-1}{t^2}$$

5. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=\cos\theta-\cos 2\theta, y=\sin\theta-\sin 2\theta$

Solution:

The given equations are $x=\cos\theta-\cos 2\theta$ and $y=\sin\theta-\sin 2\theta$

Then, $\frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos 2\theta)=\frac{d}{d\theta}(\cos\theta)-\frac{d}{d\theta}(\cos 2\theta)$

$=-\sin\theta(-2\sin 2\theta)=2\sin 2\theta-\sin\theta$

$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin 2\theta)=\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\sin 2\theta)$

$=\cos\theta-2\cos\theta$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\cos\theta-2\cos\theta}{2\sin 2\theta-\sin\theta}$$

6. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=a(\theta-\sin\theta), y=a(1+\cos\theta)$

Solution:

The given equations are $x=a(\theta-\sin\theta)$ and $y=a(1+\cos\theta)$

Then, $\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}(\theta)-\frac{d}{d\theta}(\sin\theta)\right]=a(1-\cos\theta)$

$$\frac{dy}{d\theta}=a\left[\frac{d}{d\theta}(1)+\frac{d}{d\theta}(\cos\theta)\right]=a\left[0+(-\sin\theta)\right]=-a\sin\theta$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=\frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=-\cot\frac{\theta}{2}$$

7. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x=\frac{\sin^3 t}{\sqrt{\cos x2t}},\ y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Solution:

The given equations are $x=\frac{\sin^3 t}{\sqrt{\cos x2t}}$ and $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$=\frac{\sqrt{\cos 2t}-\frac{d}{dt}\left(\sin^3 t\right)-\sin^3 t-\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\sin^2 t.\frac{d}{dt}(\sin t)-\sin^3 tx\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}.\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t\sqrt{\cos 2t}}$$

$$=\frac{3\cos 2t\sin^2 t\cot t+\sin^2 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right)-\cos^3 t.\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t.\frac{d}{dt}\left(\cos t\right)-\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}\left(\cos 2t\right)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}\cos^2 t.\left(-\sin t\right)-\cos^3 t.\frac{1}{\sqrt{\cos 2t}}\left(-2\sin 2t\right)}{\cos 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{\cos 2t.\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx}\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-3\cos 2t.\cos^2 t+\cos^3 t\sin 2t}{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}$$

$$=\frac{3\cos 2t.\cos^2 t\sin t+\cos^3 t\left(2\sin t\cos t\right)}{3\cos 2\text{tsin}^2 t.\cos t+\sin^3 t\left(2\sin t\cos t\right)}$$

$$=\frac{\sin t\cos t\left[-3\cos 2t.\cos t+2\cos^3 t\right]}{\sin t\cos t\left[3\cos 2t\sin t+2\sin^3 t\right]}$$

$$=\frac{\left[-3\left(2\cos^2 t-1\right)\cos t+2\cos^3 t\right]}{\left[3\left(1-2\sin^2 t\right)\sin t+2\sin^3 t\right]} \qquad \left[\begin{array}{l}\cos 2t=\left(2\cos^2 t-1\right)\\ \cos 2t=\left(1-2\sin^2 t\right)\end{array}\right]$$

$$=\frac{-4\cos^3 t+3\cos t}{3\sin t-4\sin^3 t} \qquad \left[\begin{array}{l}\cos 3t=4\cos^3 t-3\cos t\\ \sin 3t=3\sin t-4\sin^2 t\end{array}\right]$$

$$=\frac{-\cos 3t}{\sin 3t}$$

$$=-\cot 3t$$

8. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$$x=a\left(\cos t+\log\frac{t}{2}\right),y=a\sin t$$

Solution:

The given equations are $x=a\left(\cos t+\log\frac{t}{2}\right)$ and $y=a\sin t$

Then, $\frac{d}{dx}=a\left[\frac{d}{dt}(\cos t)+\frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$

$$=a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}.\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\cot\frac{t}{2}.\sec^2\frac{t}{2}.\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\frac{\cot\frac{t}{2}}{\sin\frac{t}{2}}\times\frac{1}{\cos^2\frac{t}{2}}\times\frac{1}{2}\right]$$

$$=a\left[-\sin t+\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$=a\left(-\sin t+\frac{1}{\sin t}\right)$$

$$=a\left(\frac{-\sin^2 t+1}{\sin t}\right)$$

$$=a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt}=a\frac{d}{dt}(\sin t)=a\cos t$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)}=\frac{\sin t}{\cos t}=\tan t$$

9. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=a\sec, y=b\tan\theta$

Solution:

The given equations are $x=a\sec$ and $y=b\tan\theta$

Then, $\frac{dx}{d\theta}=a.\frac{d}{d\theta}(\sec\theta)=a\sec\theta\tan\theta$

$$\frac{dy}{d\theta}=b.\frac{d}{d\theta}(\tan\theta)=b\sec^2\theta$$

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b}{a}\sec\theta\cot\theta=\frac{b\cos\theta}{a\cos\theta\sin\theta}=\frac{b}{a}\times\frac{1}{\sin\theta}=\frac{b}{a}\cos ec\theta$$

10. If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

$x=a(\cos\theta+\theta\sin\theta), y=a(\sin\theta-\theta\cos\theta)$

Solution:

The given equations are $x=a(\cos\theta+\theta\sin\theta)$ and $y=a(\sin\theta-\theta\cos\theta)$

Then, $\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}\cos\theta+\frac{d}{d\theta}(\theta\sin\theta)\right]=a\left[-\sin\theta+\theta\frac{d}{d\theta}(\sin\theta)+\sin\theta\frac{d}{d\theta}(\theta)\right]$

$$=a[-\sin\theta+\theta\cos\theta+\sin\theta]=a\theta\cos\theta$$

$$\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\theta\cos\theta)\right]=a\left[\cos\theta-\left\{\theta\frac{d}{d\theta}(\cos\theta)+\cos\theta.\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a\left[\cos\theta + \theta\sin\theta - \cos\theta\right]$$

$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\sin\theta} = \tan\theta$$

11. If $x = \sqrt{a^{\sin-1t}}, y = \sqrt{a^{\cos-1t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Solution:

The given equation are $x = \sqrt{a^{\sin-1t}}$ *and* $y = \sqrt{a^{\cos-1t}}$

$$x = \sqrt{a^{\sin-1t}} \text{ and } y = \sqrt{a^{\cos-1t}}$$

$$\Rightarrow x = \left(a^{\sin-1t}\right) \text{ and } y = \left(a^{\cos-1t}\right)^{\frac{1}{2}}$$

$$\Rightarrow x = a^{\frac{1}{2}\sin-1t} \text{ and } y = a^{\frac{1}{2}\cos-1t}$$

Consider $x = a^{\frac{1}{2}\sin-1t}$

Taking logarithm on both sides, we obtain

$$\log x = \frac{1}{2}\sin^{-1}t\log a$$

$$\therefore \frac{1}{x}\frac{dx}{dt} = \frac{1}{2}\log a.\frac{d}{dt}\left(\sin^{-1}t\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x}{2}\log a.\frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{x\log a}{2\sqrt{1-t^2}}$$

Then, consider

$y=a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both sides, we obtain

$$\log y=\frac{1}{2}\cos^{-1}t\log a$$

$$\therefore \frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\log a.\frac{d}{dt}\left(\cos^{-1}t\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{y\log a}{2}\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dy}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\frac{x\log a}{2\sqrt{1-t^2}}}=-\frac{y}{x}$$

Hence proved