Chapter 5: Continuity and differentiability.

Exercise 5.7

1. Find the second order derivatives of the function x^2+3x+2

Solution:

Let $y=x^2+3x+2$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(3x)+\frac{d}{dx}(2)=2x+3+0=2x+3$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)=\frac{d}{dx}(2x)+\frac{d}{dx}(3)=2+0=2$$

2. Find the second order derivative of the function x^{20}

Solution:

Let $y=x^{20}$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{20}\right)=20x^{19}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(20x^{19}\right)=20\frac{d}{dx}\left(x^{19}\right)=20.19.x^{18}=380x^{18}$$

3. Find the second order derivatives of the function $x.\cos x$

Solution:

Let $y=x.\cos x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x.\cos x)=\cos x.\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x.1+x(-\sin x)=\cos x-x\sin x$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}[\cos x-\sin x]=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)$$

$$=-\sin x-\left[\sin x\frac{d}{dx}(x)+x\frac{d}{dx}(\sin x)\right]$$

$$=-\sin x-(\sin x+\cos x)$$

$$=-(x\cos x+2\sin x)$$

4. Find the second order derivatives of the function $\log x$

Solution:

Let $y=\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)=\frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

5. Find the second order derivatives of the function $x^3\log x$

Solution:

Let $y=x^3\log x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^3\log x\right]=\log x.\frac{d}{dx}\left(x^3\right)+x^3\frac{d}{dx}(\log x)$$

$$=\log x.3x^2+x^3.\frac{1}{x}=\log x.3x^2+x^2$$

$$=x^2(1+3\log x)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[x^2(1+3\log x)\right]$$

$$=(1+3\log x).\frac{d}{dx}(x^2)+x^2\frac{d}{dx}(1+3\log x)$$

$$=(1+3\log x).2x+x^3.\frac{3}{x}$$

$$=2x+6\log x+3x$$

$$=5x+6x\log x$$

$$=x(5+6\log x)$$

6. Find the second order derivative of the function $e^x \sin 5x$

Solution:

Let $y=e^x \sin 5x$

$$\frac{dy}{dx}=\frac{d}{dx}(e^x \sin 5x)=\sin 5x\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(\sin 5x)$$

$$=\sin 5x.e^x+e^x.\cos 5x.\frac{d}{dx}(5x)=e^x\sin 5x+e^x\cos 5x.5$$

$$=e^x(\sin 5x+5\cos 5x)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x(\sin 5x+5\cos 5x)\right]$$

$$(\sin 5x+5\cos 5x)\frac{d}{dx}(e^x)+e^x.\frac{d}{dx}(\sin 5x+5\cos 5x)$$

$$(\sin 5x+5\cos 5x)e^x+e^x\left[\cos 5x.\frac{d}{dx}(5x)+5(-\sin 5x).\frac{d}{dx}(5x)\right]$$

$$=e^x(\sin 5x+5\cos 5x)+e^x(5\cos 5x-25\sin 5x)$$

Thus, $e^x(10\cos 5x-24\sin 5x)=2e^x(5\cos 5x-12\sin 5x)$

7. Find the second order derivatives of the function $e^{6x}\cos 3x$

Solution:

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x}\cos 3x\right) = \cos 3x.\frac{d}{dx}\left(e^{6x}\right) + e^{6x}\frac{d}{dx}\left(\cos 3x\right)$$

$$= \cos 3x.e^{6x}\frac{d}{dx}(6x) + e^{6x}(-\sin 3x)\frac{d}{dx}(3x)$$

$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x.........(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6.\frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3.\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$= 6.\left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3\left[\sin 3x.\frac{d}{dx}\left(e^{6x}\right) + e^{6x}.\frac{d}{dx}(\sin 3x)\right] \qquad [\text{using }(1)]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x.e^{6x}.6 + e^{6x}.\cos 3x - 3\right]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$

$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

8. Find the second order derivatives of the function $\tan^{-1}x$

Solution:

Let $y = \tan^{-1}x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}\left(1+x^2\right)^{-1} = (-1)\left(1+x^2\right)^{-2}\frac{d}{dx}\left(1+x^2\right) - \frac{1}{\left(1+x^2\right)^2}\times 2x = -\frac{2x}{\left(1+x^2\right)^2}$$

9. Find the second order derivative of the function $\log(\log x)$

Solution:

Let $y=\log(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\log x)]=\frac{1}{\log x}.\frac{d}{dx}(\log x)=\frac{1}{\log x}=(x\log x)^{-1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]=(-1)(x\log x)^{-2}\frac{d}{dx}(x\log x)$$

$$=\frac{-1}{(x\log x)^2}.\left[\log x.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log x)\right]$$

$$=\frac{-1}{(x\log x)^2}.\left[\log x.1x+\frac{1}{x}\right]=\frac{-1(1+\log x)}{(x\log x)^2}$$

10. Find the second order derivatives of the function $\sin(\log x)$

Solution:

Let $y=\sin(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}[\sin(\log x)]=\cos(\log x).\frac{d}{dx}(\log x)=\frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$=\frac{x.\frac{d}{dx}[\cos(\log x)]-\cos(\log x).\frac{d}{dx}(x)}{x^2}$$

$$=\frac{x\left[-\sin(\log x).\frac{d}{dx}(\log x)\right]-\cos(\log x).1}{x^2}$$

$$= \frac{-x\sin(\log x).\frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}$$

11. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Solution:

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$

$$= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[-(5\sin x + 3\cos x)]$$

$$= -\left[5.\frac{d}{dx}(\sin x) + 3.\frac{d}{dx}(\cos x)\right]$$

$$= [5\cos x + 3(-\sin x)]$$

$$= -[5\cos x - 3\sin x]$$

$$= -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence, proved

12. If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone

Solution:

It is given that, $y=\cos^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}=-\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$=\left(-\frac{1}{2}\right).\left(1-x^2\right)^{\frac{-3}{2}}.\frac{d}{dx}\left(1-x^2\right)$$

$$=\frac{1}{\sqrt{\left(1-x^2\right)^3}}\times(-2x)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{\left(1-x^2\right)^3}}.........(i)$$

$$y=\cos^{-1}x\Rightarrow x=\cos y$$

Putting $x=\cos y$ in equation (1), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\cot y.\cos ec^2 y$$

13. If $y=3\cos(\log x)+4\sin(\log x)$, show that $x^2y_2+xy_1+y=0$

Solution:

It is given that $y=3\cos(\log x)+4\sin(\log x)$ and $x^2y_2+xy_1+y=0$

Then,

$$y_1=3.\frac{d}{dx}\left[\cos(\log x)\right]+4.\frac{d}{dx}\left[\sin(\log x)\right]$$

$$=3.\left[-\sin(\log x).\frac{d}{dx}(\log x)\right]+4.\left[\cos(\log x).\frac{d}{dx}(\log x)\right]$$

$$\therefore y_1=\frac{-3\sin(\log x)}{x}+\frac{4\cos(\log x)}{x}=\frac{4\cos(\log x)-3\sin(\log x)}{x}$$

$$\therefore y_2=\frac{d}{dx}\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)$$

$$=x\frac{\left[4\{\cos(\log x)\}-\{-3\sin(\log x)\}'-\{4\cos(\log x)\}-3\sin(\log x)\right].1}{x^2}$$

$$=x\frac{\left[-4\sin(\log x).(\log x)'-3\cos(\log x)(\log x)'\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=x\frac{\left[-4\sin(\log x)\frac{1}{x}-3\cos(\log x)\frac{1}{x}\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{4\sin(\log x)-3\cos(\log x)-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{-\sin(\log x)-7\cos(\log x)}{x^2}$$

$$\therefore x^2y_2+xy_1+y$$

$$=x^2\left(\frac{-\sin(\log x)-7\cos(\log x)}{x^2}\right)+x\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)+3\cos(\log x)+4\sin(\log x)$$

$= \sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$

=0

Hence, proved

14. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Solution:

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A.\frac{d}{dx}(e^{mx}) + B.\frac{d}{dx}(e^{nx}) = A.e^{mx}.\frac{d}{dx}(mx) + B.e^{nx}\frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(Ame^{mx} + Bne^{nx}) = Am.\frac{d}{dx}(e^{mx}) + Bn.\frac{d}{dx}(e^{nx})$$

$$= Am.e^{mx}.\frac{d}{dx}(mx) + bn.e^{nx}.\frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n).(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{mx})$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2ex^{mx} - Bmne^{mx} - Bn^2e^{mx} + Amne^{mx} + Bmne^{nx}$$

=0

Hence, proved

15. If $y = 500e^{7x} + 600e^{-7x}$,show that $\frac{d^2y}{dx^2} = 49y$

Solution:

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx}=500.\frac{d}{dx}\left(e^{7x}\right)+600.\frac{d}{dx}\left(e^{-7x}\right)$$

$$=500.e^{7x}.\frac{d}{dx}(7x)+600.e^{-7x}.\frac{d}{dx}(-7x)$$

$$=3500e^{7x}-4200e^{-7x}$$

$$\therefore \frac{d^2y}{dx^2}=3500.\frac{d}{dx}\left(e^{7x}\right)-4200\frac{d}{dx}\left(e^{-7x}\right)$$

$$3500.e^{7x}\frac{d}{dx}(7x)-4200.e^{-7x}.\frac{d}{dx}(-7x)$$

$$=7\times 3500.e^{7x}+7\times 4200e^{-7x}$$

$$=49\times 500e^{7x}+49\times 600e^{-7x}$$

$$=49\left(500e^{7x}+600e^{-7x}\right)$$

$$=49y$$

Hence, proved

16. If $e^y(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

Solution:

The given relationship is $e^y(x+1)=1$

$$e^y(x+1)=1$$

$$\Rightarrow e^y=\frac{1}{x+1}$$

Taking logarithm on both sides, we obtain

$$y=\log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to x we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{(x+1)}\right)=(x+1)\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved

17. If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2y_2+2x\left(x^2+1\right)y_1=2$

Solution:

The given relationship is $y=\left(\tan^{-1}x\right)^2$

Then,

$$y_1=2\tan^{-1}x\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\Rightarrow \left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating with respect to x on both sides, we obtain

$$\left(1+x^2\right)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)y_2+2x\left(1+x^2\right)=2$$

Hence proved