Infinity Learn by Sri Chaitanya Educational Institutions

Chapter 5: Continuity and differentiability.

Exercise 5.8

1. Differentiate the following with respect to *x*.
$\cos x.\cos 2x.\cos 3x$

Solution-1

Let $y=\cos x.\cos 2x.\cos 3x$

Taking logarithm or both the side, we obtain

$$\log y=\log(\cos x.\cos 2x.\cos 3x)$$

$$\Rightarrow \log y=\log(\cos x)+\log(\cos 2x)+\log(\cos 3x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\cos x}.\frac{d}{dx}(\cos x)+\frac{1}{\cos 2x}.\frac{d}{dx}(\cos 2x)+\frac{1}{\cos 3x}.\frac{d}{dx}(\cos 3x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{\sin x}{\cos x}-\frac{\sin 2x}{\cos 2x}.\frac{d}{dx}(2x)-\frac{\sin 3x}{\cos 3x}.\frac{d}{dx}(3x)\right]$$

$$\therefore \frac{dy}{dx}=-\cos x.\cos 2x.\cos 3x[\tan x+2\tan 2x+3\tan 3x]$$

2. Differentiate the function with respect to *x*. $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution-2

Let $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm or both the side, we obtain

$$\log y=\log^{\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}}$$

$$\Rightarrow \log y=\frac{1}{2}\log\left[\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\begin{array}{c}\frac{1}{x-1}.\frac{d}{dx}(x-1)+\frac{1}{x-2}.\frac{d}{dx}(x-2)-\frac{1}{x-3}.\frac{d}{dx}(x-3)\\ -\frac{1}{1-4}.\frac{d}{dx}(x-4)-\frac{1}{x-5}.\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{1-4}-\frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{1-4}-\frac{1}{x-5}\right]$$

3. Differentiate the function with respect to *x*. $(\log x)^{\cos x}$

Solution-3

Let $y=(\log x)^{\cos x}$

Taking logarithm or both the side, we obtain

$$\log y=\cos x.\log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{\log x}-\sin x\log(\log x)\right]$$

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4. Differentiate the function with respect to *x*. $x^x-2^{\sin x}$

Solution-4

Let $y=x^x-2^{\sin x}$

Also, let $x^x=u$ and $2^{\sin x}=v$

$\therefore y = u - v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$$u = x^x$$

Taking logarithm on both sides, we obtain

$$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log x + x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to *x*, we obtain

$\log v = \sin x.\log 2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}.\frac{dv}{dx} = \log 2.\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^2(1 + \log x) - 2^{\sin x} \cos x \log 2$$

5. Differentiate the function with respect to *x*. $(x+3)^2.(x+4)^3.(x+5)^4$

Solution-5

Let $y = (x+3)^2.(x+4)^3.(x+5)^4$

Taking logarithm on both sides, we obtain.

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$

$$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Differentiating both sides with respect to *x*, we obtain

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3.\left[2\left(x^2+9x+20\right) + 3\left(x^2+9x+15\right) + 4\left(x^2+7x+12\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4.\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3.\left[2(x^2+9x+20)+3(x^2+9x+15)+4(x^2+7x+12)\right]$$

$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

6. Differentiate the function with respect to x. $\left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Solutoin-6

Let $y = \left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Also, let $u = \left(x+\frac{1}{x}\right)^x$ and $v = x^{\left(1+\frac{1}{x}\right)}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \qquad \text{................ (1)}$$

Then, $u = \left(x+\frac{1}{x}\right)^x$

Taking log on both sides

$$\Rightarrow \log u = \log\left(x+\frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times(\log)\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=1\times(\log)\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}.\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow\frac{du}{dx}=u\left[(\log)\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}x\left(x+\frac{1}{x^2}\right)\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[(\log)\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)\left[\frac{x^2-1}{x^2+1}+(\log)\left(x+\frac{1}{x}\right)\right]\quad\text{............(2)}$$

$$v=x^{\left(1+\frac{1}{x}\right)}$$

Taking log on both sides, we obtain

$$\log v=\log x^{\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow\log v=\left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right).\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(x+\frac{1}{x}\right).\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right]\quad\text{.......... (3)}$$

Therefore from (1) , (2) and (3), we obtain

$$\frac{dy}{dx}=\left(1+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2}+\log x\left(x+\frac{1}{x}\right)\right]+x^{\left(x+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

7. Differentiating both sides with respect to *x*. $(\log x)^x + x^{\log x}$

Solution-7

Let $y=(\log x)^x + x^{\log x}$

Also, let $u=(\log x)^x$ and $v=x^{\log x}$

$\therefore y=u+v$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ....(1)

$u=(\log x)^x$

$\Rightarrow \log u=\log\left[(\log x)^x\right]$

$\Rightarrow \log u = x\log(\log x)$

Differentiating both sides with respect to *x*, we obtain

$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)x\log(\log x)+x.\frac{d}{dx}\left[\log(\log x)\right]$

$\Rightarrow \frac{du}{dx}=u\left[1x\log(\log x)+x.\frac{1}{\log x}.\frac{d}{dx}(\log x)\right]$

$\Rightarrow \frac{du}{dx}=(\log x)^x\left[\log(\log x)+\frac{x}{\log x}.\frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx}=(\log x)^x\left[\log(\log x)+\frac{1}{\log x}\right]$

$\Rightarrow \frac{du}{dx}=(\log x)^x\left[\frac{\log(\log x).\log x+1}{\log x}\right]$

$\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]$ ....(2)

$v=x^{\log x}$

$\Rightarrow \log v=\log\left(x^{\log x}\right)$

$\Rightarrow \log v=\log x\log x=(\log x)^2$

Differentiating both sides with respect to *x*, we obtain

$\frac{1}{v}.\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^2\right]$

$\Rightarrow \frac{1}{v}.\frac{dv}{dx}=2(\log x).\frac{d}{dx}(\log x)$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x - 1}.\log x \qquad \text{........(3)}$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dv}{dx} = (\log x)^{x-1}\left[1+\log x.\log(\log x)\right] + 2x^{\log x - 1}.\log x$$

8. Differentiating both sides with respect to *x*. $(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution-8

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad \text{............ (1)}$$

$u = (\sin x)^x$

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x\log(\sin x)$

Differentiating both sides with respect to *x*, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)x\log(\sin x) + x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1.\log(\sin x) + x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x}.\cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x(x\cot x + \log\sin x) \qquad \text{..........(2)}$$

$v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to *x*, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}}.\frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \qquad \text{...........(3)}$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx}=(\sin x)^2(x\cot x+\log\sin x)+\frac{1}{2\sqrt{x-x^2}}$$

9. Differentiate the function with respect to x. $x^{\sin x}+(\sin x)^{\cos x}$

Solution-9

Let $y=x^{\sin x}+(\sin x)^{\cos x}$

Also $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$

$\therefore y=u+v$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ....(1)

$u=x^{\sin x}$

$\Rightarrow \log u=\log\left(x^{\sin x}\right)$

$\Rightarrow \log u=\sin x\log x$

Differentiating both sides with respect to x, we obtain

$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x).\log x+\sin x.\frac{d}{dx}(\log x)$

$\Rightarrow \frac{du}{dx}=u=\left[\cos x\log x+\sin x.\frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx}=x^{\sin x}=\left[\cos x\log x+\frac{\sin x}{x}\right]$

$v=(\sin x)^{\cos x}$

$\Rightarrow \log v=\log(\sin x)^{\cos x}$

$\Rightarrow \log v=\cos x\log(\sin x)$

Differentiating both sided with respect to x, we obtain

$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}(\cos x)\times\log(\sin x)+\cos x\times\frac{d}{dx}\left[\log(\sin x)\right]$

$\Rightarrow \frac{dv}{dx}=v\left[-\sin x.\log(\sin x)+\cos x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$

$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\frac{\cos x}{\sin x}\cos x\right]$

$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\cot x\cos x\right]$

$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}\left[\cot x\cos x-\sin x\log\sin x\right]$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dv}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}\left[\cos x\cot x-\sin x\log\sin x\right]$$

10. Differentiate the function with respect to *x*. $x^{\cos x}+\dfrac{x^2+1}{x^2-1}$

Solution-10

Let $y=x^{x\cos x}+\dfrac{x^2+1}{x^2-1}$

Also, let $u=x^{x\cos x}$ and $v=\dfrac{x^2+1}{x^2-1}$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dy}{dx}$$

$$\therefore y=u+v$$

$$u=x^{x\cos x}$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x).\cos x\log x+x.\frac{d}{dx}(\cos x).\log x+x\cos x.\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1.\cos x\log x+x.(-\sin x).\log x+x\cos x.\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left(\cos x\log x-x\sin x.\log x+\cos x\right)$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]\qquad\text{.......(2)}$$

$$v=\frac{x^2+1}{x^2-1}$$

$$\Rightarrow\log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}=\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{2x\left(x^2-2\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{\left(x^2-1\right)^2} \qquad \text{..............(3)}$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dy}{dx} = x^{x\cos x}\left[\cos x\left(1+\log x\right) - x\sin x\log x\right] - \frac{4x}{\left(x^2-1\right)^2}$$

11. Differentiate the function with respect to x $\left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$

Solution-11

Let $y = \left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$

Also, let $u = \left(x\cos x\right)^x$ and $v = \left(x\sin x\right)^{\frac{1}{x}}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dy}{dx} \qquad \text{..............(1)}$$

$u = \left(x\cos x\right)^2$

$\Rightarrow \log u = \log\left(x\cos x\right)^x$

$\Rightarrow \log u = x\log\left(x\cos x\right)^x$

$\Rightarrow \log u = x\left[\log x + \log\cos x\right]$

$\Rightarrow \log u = x\log x + x\log\cos x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}\left(x + \log x\right) + \frac{d}{dx}\left(x\log\cos x\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\left\{\log x.\frac{d}{dx}\left(x\right) + x.\frac{d}{dx}\left(\log x\right)\right\} + \left\{\log\cos x.\frac{d}{dx}\left(x\right) + x.\frac{d}{dx}\left(\log\cos x\right)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x\cos x\right)^x\left[\left(\log x.1 + x.\frac{1}{x}\right) + \left\{\log\cos x.1 + x.\frac{1}{\cos x}.\frac{d}{dx}\left(\cos x\right)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x\cos x\right)^x\left[\left(\log x + 1\right) + \left\{\log\cos x + \frac{x}{\cos x}.\left(-\sin x\right)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x\cos x\right)^x\left[\left(1+\log x\right) + \left(1 + \log\cos x - x\tan x\right)\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x\cos x\right)^x\left[1 - x\tan x + \left(\log x + \log\cos x\right)\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[1 - x\tan x + +\log(x\cos x)\right] \qquad \text{..............(2)}$$

$$v = (x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}(\log x + \log\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log\sin x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left(\frac{1}{x}\log(\sin x)\right)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}(\log x)\right] + \left[\log(\sin x).\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{x}\right] + \left[\log(\sin x).\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1 - \log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}.\cos x\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(x\sin x)^{\frac{1}{x}} + \left[\frac{1 - \log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1 - \log x - \log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1 - \log(x\sin x) + x\cot x}{x^2}\right] \qquad \text{..............(3)}$$

Therefore, from (1), (2) and (3), we obtain

$$\frac{dv}{dx} = (x\cos x)^2\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

12. Find $\frac{dy}{dx}$ of function. $x^y + y^x = 1$

Solution-12

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \text{.........(1)}$$

$$u = x^y$$

$$\Rightarrow \log u = \log\left(x^y\right)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y.\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y.\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad \text{...........(2)}$$

$$v = y^x$$

$$\Rightarrow \log v = \log\left(y^x\right)$$

$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \log y \frac{d}{dx}(x) + x.\frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v\left[\log y.1 + x.\frac{1}{y}.\frac{dy}{dx}\right]$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad \text{...........(3)}$$

Therefore, from (1), (2) and (3), we obtain

$$x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) + y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \left(x^2 + \log x + xy^{y-1}\right)\frac{dy}{dx} = -\left(yx^{y-1} + y^x \log y\right)$$

$$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

13. Find $\frac{dy}{dx}$ of function $y^x = x^y$

Solution:13

The given function $y^x = x^y$

Taking logarithm on both sides, we obtain

$x\log y = y\log x$

Differentiating both sides with respect to x, we obtain

$$\log y.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log y)=\log x.\frac{d}{dx}(y).\frac{d}{dx}(\log x)$$

$$\Rightarrow \log y.1+x.\frac{1}{y}.\frac{dy}{dx}=\log x.\frac{dy}{dx}+y.\frac{1}{x}$$

$$\Rightarrow \log y+\frac{x}{y}.\frac{dy}{dx}=\log x.\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow \left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-\log y}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right).$$

14. Find $\frac{dy}{dx}$ of function $(\cos x)^y = (\cos y)^x$

Solution:14

The given function $(\cos x)^y = (\cos y)^x$

Taking logarithm on both sides, we obtain

$y=\log\cos x = x\log\cos y$

Differentiating both sides with respect to x, we obtain

$$\log\cos x.\frac{d}{dx}+y.\frac{d}{dx}(\log\cos x)=\log\cos y.\frac{d}{dx}(x).+x\frac{d}{dx}(\log\cos y)$$

$$\Rightarrow \log\cos x.\frac{dy}{dx}+y\frac{1}{\cos x}.\frac{d}{dx}(\cos x)=\log\cos y.1+x.\frac{1}{\cos y}.\frac{d}{dx}(\cos y)$$

$$\Rightarrow \log\cos x.\frac{dy}{dx}+\frac{y}{\cos x}.(-\sin x)=\log\cos y+\frac{x}{\cos y}(-\sin y)\frac{dy}{dx}$$

$$\Rightarrow \log\cos x.\frac{dy}{dx} - y\tan x = \log\cos y - x\tan y\frac{dy}{dx}$$

$$\Rightarrow (\log\cos x + x\tan y)\frac{dy}{dx} = y\tan x + \log\cos y$$

$$\therefore \frac{dy}{dx} = \frac{y\tan x + \log\cos y}{x\tan y + \log\cos x}$$

15. Find $\frac{dy}{dx}$ of function $xy = e^{(x-y)}$

Solution-15

The given function is $xy = e^{(x-y)}$

Taking logarithm on both sides, we obtain.

$$\log(xy) = \log\left(e^{x-y}\right)$$

$$\Rightarrow \log x + \log y = (x-y)\log e$$

$$\Rightarrow \log x + \log y = (x-y)\times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y-1)}$$

16. Find the derivative of the function given by

$f(x) = (1-x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$

Solution-16

The given relationship is $f(x) = (1-x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both sides, we obtain

$$\log f(x)=\log(1-x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to *x*, we obtain.

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1-x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1-x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\cdot\frac{d}{dx}\log(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}\log(1+x^8$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}+\frac{1}{1+x^8}.8x^7\right]$$

$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Hence, $f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+x^4}+\frac{8\times1^7}{1+x^8}\right]$

$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\left[\frac{1+2+4+8}{2}\right]$$

$$=16\times\frac{15}{2}=100$$

17. Differentiate $(x^2-5x+8)(x^3+7x+9)$ in three ways mentioned below

i. By using product rule

ii. By expanding the product to obtain a single polynomial

iii. By

iii. By logarithm Differentiate

Do they all given the same answer?

Solution-17

Let $y=(x^2-5x+8)(x^3+7x+9)$

(i) Let $x=x^2-5x+8$ and $u=x^3+7x+9$

$\therefore$ $y=uv$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dv}v+u\frac{dv}{dx}$$ (By using product rule)

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(x^2-5x+8).(x^3+7x+9)+(x^2-5x+8).\frac{d}{dx}(x^3+7x+9)$$

$$\Rightarrow \frac{dy}{dx} = (2x-5)(x^3+7x+9)+(x^2-5x+8)(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3+7x+9)-5(x^3+7x+9)+x^2(3x^2+7)-5x(3x^2+7)-8(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx} = (2x^4+14x^2+18x)-5x^3-35x-45+(3x^4+7x^2)-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx} = 5x^4-20x^3+45x^2-52x+11$$

(ii) $y=(x^2-5x+8)(x^3+7x+9)$

$$=x^2(x^3+7x+9)-5x(x^3+7x+9)+8(x^3+7x+9)$$

$$=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$

$$=x^5-5x^4+15x^3-26x^2+11x+72$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}=(x^5-5x^4+15x^3-26x^2+11x+72)$$

$$=\frac{d}{dx}=(x^5)-5\frac{d}{dx}(x^4)+15\frac{d}{dx}(x^3)-26\frac{d}{dx}(x^2)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)$$

$$=5x^4-5\times 4x^3+15\times 3x^2-26\times 2x+11\times 1+0$$

$$=5x^4-20x^3+45x^2-52x+11$$

(iii) Taking logarithm on both sides, we obtain.

$$\log y=\log(x^2+5x+8)+\log(x^3+7x+9)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log(x^2-5x+8)+\frac{d}{dx}\log(x^3+7x+9)$$

Differentiating both sides with respect to *x*, we obtain

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log(x^2-5x+8).\frac{d}{dx}\log(x^3+7x+9)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}.\frac{d}{dx}(x^2-5x+8)+\frac{1}{x^3+7x+9}.\frac{d}{dx}\log(x^3+7x+9)$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times(2x-5)+\frac{1}{x^3+7x+9}\times(3x^2+7)\right]$$

$$\Rightarrow \frac{dy}{dx}=(x^2-5x+8)(x^3+7x+9)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{(2x-5)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)+\left(x^3+7x+9\right)}\right]$$

$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$

$$\Rightarrow \frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$

$$\Rightarrow \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the result of $\frac{dy}{dx}$ are same

18. If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u\frac{dv}{dx}.w+u.v\frac{dw}{dx}$$

In two ways-first by repeated application of product rule, second by logarithmic differentiation.

Solution – 18

Let $y=u.v.w=u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}.(v.w)+u.\frac{d}{dx}.(v.w)$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v.w+u\left[\frac{dv}{dx}.w+v.\frac{dv}{dx}\right] \quad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y=u.v.w$, we obtain

$$\log y=\log u+\log v+\log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=\frac{d}{dx}(\log u)+\frac{d}{dx}(\log v)+\frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{1}{u}\frac{d}{dx}+\frac{1}{v}\frac{d}{dx}+\frac{1}{w}\frac{d}{dx}$$

$$\Rightarrow \frac{dy}{dx}=y\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx}u.w + u\frac{dv}{dx}w + u.v\frac{dw}{dx}$$