Chapter 6: Applications of Derivatives.

Exercise 6. Miscellaneous

1. Using differentials, find the approximate value of each of the following

$$(a)\left(\frac{17}{81}\right)^{\frac{1}{4}} \qquad (b)(33)^{\frac{1}{5}}$$

Solution:

$$(a)\ y = x^{\frac{1}{4}}$$

$$x = \frac{16}{81}$$

$$\Delta x = \frac{1}{81}$$

$$\Delta y = (x+\Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x)$$

$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) = \frac{27}{4\times 8}\times\frac{1}{81} = \frac{1}{32\times 3} = \frac{1}{96} = 0.010$$

Approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3}+0.010=0.667+0.010$

$=0.677$

$(b)\ y=x^{\frac{1}{5}}$

$x=32$

$\Delta x=1$

$$\Delta y=(x+\Delta x)^{\frac{1}{5}}-x^{\frac{1}{5}}=(33)^{\frac{1}{5}}-(32)^{\frac{1}{5}}=(33)^{-\frac{1}{5}}-\frac{1}{2}$$

$$\therefore (33)^{\frac{1}{5}}=\frac{1}{2}+\Delta y$$

$$dy=\left(\frac{dy}{dx}\right)=(\Delta x)=\frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x)$$

$$=\frac{1}{5(2)^{6}}(1)=\frac{1}{320}=-0.003$$

Approximate value of $(33)^{\frac{1}{5}}$ is $\frac{1}{2}+(-0.003)=0.5-0.003=0.497$

2. Show that the function given by $f(x)=\frac{\log x}{x}$ has maximum at $x=e$

Solution:

$$f(x)=\frac{\log x}{x}$$

$$f'(x)=\frac{x\left(\frac{1}{x}\right)-\log x}{x^2}=\frac{1-\log x}{x^2}$$

$$f'(x)=0$$

$$\Rightarrow 1-\log x=0$$

$\Rightarrow \log x = 1$

$\Rightarrow \log x = \log e$

$\Rightarrow x = e$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1 - \log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

$$f''(e) = \frac{-3 + 2\log e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3} < 0$$

f is the maximum at x = e

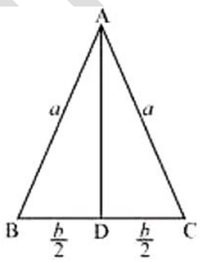
3. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3cm per second. How fast is the area decreasing when the two equal sides area equal to the base?

Solution:

Let ΔABC be isosceles where BC is the base of fixed length b

Let the length of the two equal sides of ΔABC be a

Draw $AD \perp BC$



$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

Area of triangle $=\frac{1}{2}b\sqrt{a^2-\frac{b^2}{4}}$

$$\frac{dA}{dt}=\frac{1}{2}b.\frac{2a}{2\sqrt{a^2-\frac{b^2}{4}}}\frac{da}{dt}=\frac{ab}{\sqrt{4a^2-b^2}}\frac{da}{dt}$$

$$\frac{da}{dt}=-3cm/s$$

$$\therefore \frac{da}{dt}=\frac{-3ab}{\sqrt{4a^2-b^2}}$$

When $a=b,$

$$\frac{dA}{dt}=\frac{-3b^2}{\sqrt{4b^2-b^2}}=\frac{-3b^2}{\sqrt{3b^2}}=-\sqrt{3}b$$

4. Find the equation of the normal to $y^2=4x$ curve at the point (1, 2)

Solution:

$$2y\frac{dy}{dx}=4$$

$$\Rightarrow \frac{dy}{dx}=\frac{4}{2y}=\frac{2}{y}$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,2)}=\frac{2}{2}=1$$

Slope of the normal at (1, 2) is $\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}}=\frac{-1}{1}=-1$

Equation of normal at (1, 2) is $y-2=-1(x-1)$

$$y-2=-x+1$$

$$x+y-3=0$$

5. Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta, y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin

Solution:

$$x = a\cos\theta + a\theta\sin\theta$$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$y = a\sin\theta - a\theta\cos\theta$$

$$\therefore \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

$$\therefore \frac{dy}{d\theta} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Slope of normal at any point θ is $\frac{1}{\tan\theta}$

Equation of normal at a given point (x, y) is given by,

$$y - a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}(x - a\cos\theta - a\theta\sin\theta)$$

$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta + y\sin\theta - a\left(\sin^2\theta + \cos^2\theta\right) = 0$$

$$\Rightarrow x\cos\theta + y\sin\theta - a = 0$$

Perpendicular distance of normal from origin is

$$\frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = |-a|, \text{ which is independent of } \theta$$

Perpendicular distance of normal from origin is constant

6. Find the intervals in which the function f given by $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is

(i) Increasing (ii) Decreasing

Solution:

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

$$\therefore f'(x)=\frac{(2+\cos x)(4\cos x-2-\cos x+x\sin x)-(4\sin x-2x-x\cos x)(-\sin x)}{(2+\cos x)^2}$$

$$=\frac{(2+\cos x)(3\cos x-2+x\sin x)+\sin x(4\sin x-2x-x\cos x)}{(2+\cos x)^2}$$

$$=\frac{6\cos x-4+2x\sin x+3\cos^2 x-2\cos x+x\sin x\cos x+4\sin^2 x-2\sin^2 x-2x\sin x-x\sin x\cos x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-4+3\cos^2 x+4\sin^2 x}{(2+\cos x)^2}$$

$$=\frac{4\cos x-\cos^2 x}{(2+\cos x)^2}=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

$$f'(x)=0$$

$$\Rightarrow \cos x=0, \cos x=4$$

$$\cos x\neq 4$$

$$\cos x=0$$

$$\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$$

In $\left(0,\frac{\pi}{2}\right)$ *and* $\left(\frac{3\pi}{2},2\pi\right)$, $f'(x)>0$

$f(x)$ is increasing for $0<x<\frac{x}{2}$ *and* $\frac{3\pi}{2}<x<2\pi$

In $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $f'(x)<0$

$f(x)$ is decreasing for $\frac{\pi}{2} < x < \frac{3\pi}{2}$

7. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is

(i) Increasing (ii) Decreasing

Solution:

$$f(x) = x^3 + \frac{1}{x^3}$$

$$\therefore f'(x) = 3x^2 + \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

$$f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = x \pm 1$$

In $(-\infty, 1)$ and $(1, \infty)$ i.e., when $x < -1$ and $x > 1$, $f'(x) > 0$

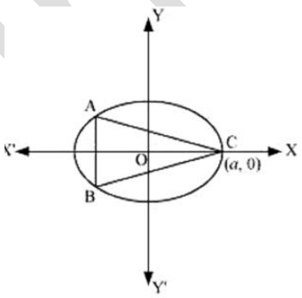
when $x < -1$ and $x > 1$, f is increasing

In $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) = 0$

Thus, when $-1 < x < 1$, f is decreasing

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis

Solution:



ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let ABC, be the triangle inscribed in the ellipse where vertex C is at (a, 0)

Since the ellipse is symmetrical with x – axis and y – axis

$$y_1 = \pm\frac{b}{a}\sqrt{a^2 - x_1^2}$$

Coordinates of A are $\left(-x_1, \frac{b}{a}\sqrt{a^2 - x_1^2}\right)$ and coordinates of B are $\left(x_1, -\frac{b}{a}\sqrt{a^2 - x_1^2}\right)$

As the point $(-x_1, y_1)$ lies on the ellipse, the area of triangle ABC is

$$A = \frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2 - x_1^2}\right) + (-x_1)\left(-\frac{b}{a}\sqrt{a^2 - x_1^2}\right) + (-x_1)\left(-\frac{b}{a}\sqrt{a^2 - x_1^2}\right)\right|$$

$$\Rightarrow A = ba\sqrt{a^2 - x_1^2} + x_1\frac{b}{a}\sqrt{a^2 - x_1^2}$$

$$\therefore \frac{dA}{dx_1} = \frac{-2xb}{2\sqrt{a^2 - x_1^2}} + \frac{b}{a}\sqrt{a^2 - x_1^2} - \frac{2bx_1^2}{a^2\sqrt{a^2 - x_1^2}}$$

$$= \frac{b}{2\sqrt{a^2 - x_1^2}}\left[-x_1 a + \left(a^2 - x_1^2\right) - x_1^2\right]$$

$$= \frac{b\left(-2x_1^2 - x_1^2 + a^2\right)}{a\sqrt{a^2 - x_1^2}}$$

$$\frac{dA}{dx_1} = 0$$

$$\Rightarrow -2x_1^2 - x_1 a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)\left(a^2\right)}}{2(-2)}$$

$$= \frac{a \pm \sqrt{9a^2}}{-4}$$

$$= \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

x_1 cannot be equal to a

$$\therefore x_1 = \frac{a}{2} \Rightarrow y_1 = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a}\sqrt{3} = \frac{\sqrt{3}b}{2}$$

Now, $$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{\sqrt{a^2 - x_1^2}\,(-4x_1 - a) - \left(-2x_1^2 - x_1 a + a^2\right)\frac{(-2x_1)}{2\sqrt{a^2 - x_1^2}}}{a^2 - x_1^2}\right\}$$

$$= \frac{b}{a}\left\{\frac{\left(a^2 - x_1^2\right)(-4x_1 - a) + x_1\left(-2x_1^2 - x_1 a + a^2\right)}{\left(a^2 - x_1^2\right)^{\frac{2}{3}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{2x^3 - 3a^2x - a^3}{\left(a^2 - x_1^2\right)^{\frac{2}{3}}}\right\}$$

When $x_1 = \frac{a}{2}$,

$$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left[\frac{2\frac{a^3}{8} - 3\frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right] = \frac{b}{a}\left\{\frac{\frac{a^3}{4} - \frac{3}{2}a^3 - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$= \frac{b}{a}\left[\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right] < 0$$

Area is the maximum when $x_1 = \frac{a}{2}$

Maximum area of the triangle is

$$A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$

$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

9. A tank with rectangular base and rectangular sides, open at the top is to constructed so that its depth is 2m and volume is $8m^3$. If building of tank costs Rs 70 per sq meters for the base and Rs 45 per sq meters for sides. What is the cost of least expensive tank?

Solution:

Let l, b and h represent the length, breadth and height of the tank respectively

height $(h) = 2m$

Volume of the tank = $8m^3$

Volume of the tank = $l \times b \times h$

$8 = l \times b \times 2$

$$\Rightarrow lb = 4 \Rightarrow \frac{4}{l}$$

Area of the base $= lb = 4$

Area of 4 walls $(A) = 2h(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$

$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$

Now, $\frac{dA}{dl} = 0$

$$\Rightarrow l - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

Therefore, we have $l=4$

$$\therefore b=\frac{4}{l}=\frac{4}{2}=2$$

$$\frac{d^2A}{dl^2}=\frac{32}{l^3}$$

$$l=2, \frac{d^2A}{dl^2}=\frac{32}{8}=4>0$$

Area is the minimum when $l=2$

We have $l=b=h=2$

Cost of building base = $Rs70\times(lb)=Rs70(4)=Rs280$

Cost of building walls =

$$Rs\ 2h(l+h)\times 45=Rs\ 90(2)(2+21)=Rs\ 8(90)=Rs\ 720$$

Required total cost $=Rs(280+720)=Rs\ 1000$

10. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their area is least when the side of square is double the radius of the circle

Solution:

$2\pi r+4a=k$ (where k is constant)

$$\Rightarrow a=\frac{k-2\pi r}{4}$$

Sum of the areas of the circle and the square (A) is given by,

$$A=\pi r^2+a^2=\pi r^2+\frac{(k=2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr}=2\pi r+\frac{2(k-2\pi r)(2\pi)}{16}=2\pi r=-\frac{\pi(k-2\pi r)}{4}$$

Now, $\frac{dA}{dr}=0$

$$\Rightarrow 2\pi r=\frac{\pi(k-2\pi r)}{4}$$

$$8r=k-2\pi r$$

$$\Rightarrow (8+2\pi)r=k$$

$$\Rightarrow r = \frac{k}{8+2\pi} = \frac{k}{2(4+\pi)}$$

Now, $\frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$

$\therefore$ where $r = \frac{k}{2(4+\pi)}, \frac{d^2A}{dr^2} > 0$

Area is least when $r = \frac{k}{2(4+\pi)}$

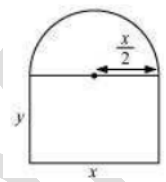
Where $r = \frac{k}{2(4+\pi)}, a = \frac{k - 2\pi\left[\frac{k}{2(4+\pi)}\right]}{4} = \frac{8k + 2\pi k - 2\pi k}{2(4+\pi)\times 4} = \frac{k}{4+\pi} = 2r$

11. A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening

Solution:

x and y be length and breadth of rectangular window

Radius of semicircular opening $= \frac{x}{2}$



$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1 + \frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1 + \frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi}{8} \times x^2$$

$$= 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{8} \times x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$\frac{d^2A}{dx^2} = -\left(1 - \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

$$\frac{dA}{dx} = 0$$

$$\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow x\left(1 + \frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

$$x = \frac{20}{\pi + 4}, \frac{d^2A}{dx^2} < 0$$

Area is maximum when length $x = \frac{20}{\pi + 4}$ m.

Now, $y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4}m$

The required dimensions

$$\text{Length} = \frac{20}{\pi+4} m \text{ and breadth } = \frac{10}{\pi+4} m$$

12. A point of the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

Solution:

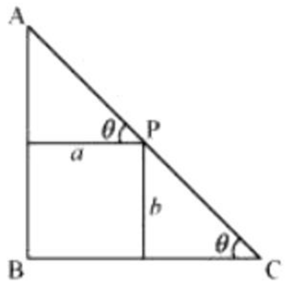
ΔABC right – angled at B

AB = x and BC = y

P be a point on hypotenuse such that P is at a distance of a and b from the sides AB and

BC respectively

$\angle c = \theta$



$AC = \sqrt{x^2 + y^2}$

$PC = b \cos ec\theta$

$AP = a \sec\theta$

$AC = AP + PC$

$AC = b \cos ec\theta + \sec\theta \quad \ldots\ldots\ldots\ldots\ldots(1)$

$$\therefore \frac{d(AC)}{d\theta} = -b \cos ec\theta \cot\theta + a \sec\theta \tan\theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0$$

$$\Rightarrow a\sec\theta\tan\theta = b\cos ec\theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\sin^3\theta = b\cos^3\theta$$

$$\Rightarrow (a)^{\frac{1}{3}}\sin\theta = (b)^{\frac{1}{3}}\cos\theta$$

$$\Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}} \; and \; \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}} \quad ......(2)$$

Clearly $\frac{d^2(AC)}{d\theta^2} < 0 \; when \; \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$

The length of the hypotenuse is the maximum when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$

Now, when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$

$$\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$AC = \frac{b\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{a^{\frac{1}{3}}}$$

$$= a\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}\left(b^{\frac{2}{3}}+a^{\frac{2}{3}}\right)$$

$$= \left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

Maximum length of the hypotenuse is $=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

13. Find the points at which the function f given by $f(x)=(x-2)^4(x+1)^3$ has

(i) local maxima (ii) local minima (iii) point of inflexion

Solution:

$$f(x)=(x-2)^4(x+1)^3$$

$$\therefore f'(x)=4(x-2)^3(x+1)^3+3(x+1)^2(x-2)^4$$

$$=(x-2)^3(x+1)^2\left[4(x+1)+3(x-2)\right]$$

$$=(x-2)^3(x+1)^2(7x-2)$$

$$f'(x)=0\Rightarrow x=-1 \; and \; x=\frac{2}{7} \; or \; x=2$$

For x close to $\frac{2}{7}$ and to left of $\frac{2}{7}$, $f'(x)>0$

For x close to $\frac{2}{7}$ and to right of $\frac{2}{7}$, $f'(x)>0$

$x=\frac{2}{7}$ is point of local minima

As the value of x varies $f'(x)$ does not changes its sing

$x=-1$ is point of inflexion

14. Find the absolute maximum and minimum values of the function f given by

$f(x)=\cos^2 x+\sin x, x\in[0,\pi]$

Solution:

$$f(x)=\cos^2 x+\sin x$$

$$f'(x)=2\cos x(-\sin x)+\cos x$$

$= -2\sin x\cos x + \cos x$

$f'(x) = 0$

$\Rightarrow 2\sin x\cos x = \cos x \Rightarrow \cos x(2\sin x - 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}$ *or* $\cos x = 0$

$\Rightarrow x = \frac{\pi}{6}, or\ \frac{\pi}{2}\ as\ x \in [0, \pi]$

$f\left(\frac{\pi}{6}\right) = \cos^2\frac{\pi}{6} + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$

$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$

$f(\pi) = \cos^2\pi + \sin\pi = (-1)^2 + 0 = 1$

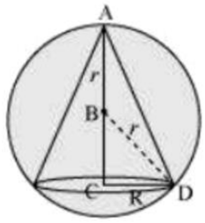
$f\left(\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$

Absolute maximum value of f is $\frac{5}{4}$ at $x = \frac{\pi}{6}$

Absolute minimum value of f is 1 at $x = 0, x = \frac{\pi}{2}, and\ \pi$

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$

Solution:



$V = \frac{1}{3}\pi R^2 h$

$BC=\sqrt{r^2-R^2}$

$h=r+\sqrt{r^2-R^2}$

$\therefore V=\frac{1}{3}\pi R^2\left(r+\sqrt{r^2-R^2}\right)=\frac{1}{3}\pi R^2 r+\frac{1}{3}\pi R^2\sqrt{r^2-R^2}$

$\therefore \frac{dV}{dR}=\frac{2}{3}\pi Rr+\frac{2\pi}{3}\pi R\sqrt{r^2-R^2}+\frac{R^2}{3}.\frac{(-2R)}{2\sqrt{r^2-R^2}}$

$=\frac{2}{3}\pi Rr+\frac{2\pi}{3}\pi R\sqrt{r^2-R^2}-\frac{R^2}{3\sqrt{r^2-R^2}}$

$=\frac{2}{3}\pi Rr+\frac{2\pi Rr\left(r^2-R^2\right)-\pi R^3}{3\sqrt{r^2-R^2}}$

$=\frac{2}{3}\pi Rr+\frac{2\pi Rr^2-3\pi Rr^3}{3\sqrt{r^2-R^2}}$

$=\frac{dV}{dR^2}=0$

$\Rightarrow \frac{2\pi rR}{3}=\frac{3\pi R^2-2\pi Rr^2}{3\sqrt{r^2-R^2}}$

$\Rightarrow 2r\sqrt{r^2-R^2}=\left(3R^2-2r^2\right)^2$

$\Rightarrow 4r^2\left(r^2-R^2\right)=\left(3R^2-2r^2\right)^2$

$\Rightarrow 14r^2-4r^2R^2=9R^4+4r^4-12R^2r^2$

$\Rightarrow 9R^4-8r^2R^2=0$

$\Rightarrow 9R^2=8r^2$

$\Rightarrow R^2=\frac{8r^2}{9}$

$$\frac{d^2V}{dR^2}=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)-\left(2\pi R^3-3\pi R^3\right)(-6)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

$$=\frac{2\pi r}{3}+\frac{3\sqrt{r^2-R^2}\left(2\pi r^2-9\pi R^2\right)-\left(2\pi R^3-3\pi R^3\right)(3R)\frac{1}{2\sqrt{r^2-R^2}}}{9\left(r^2-R^2\right)}$$

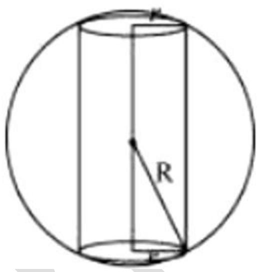
When $R^2=\frac{8r^2}{9},\frac{d^2V}{dR^2}<0$

Volume is the maximum when $R^2=\frac{8r^2}{9}$

$R^2=\frac{8r^2}{9}$, height of the cone $=r+\sqrt{r^2-\frac{8R^2}{9}}=r+\sqrt{\frac{r^2}{9}}=r+\frac{r}{3}=\frac{4r}{3}$

16. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$, also find the maximum volume

Solution:



$h=2\sqrt{R^2-r^2}$

$V=\pi r^2h=2\pi r^2\sqrt{R^2-r^2}$

$$\therefore\frac{dV}{dr}=4\pi r\sqrt{R^2-r^2}+\frac{2\pi r^2(-2r)}{2\sqrt{R^2-r^2}}$$

$$=4\pi r\sqrt{R^2-r^2}+\frac{2\pi r^3}{\sqrt{R^2-r^2}}$$

$$= \frac{4\pi r\left(R^2 - r^2\right) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi r R^2 - 6\pi r^3\right)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{\left(R^2 - r^2\right)}$$

$$= \frac{\left(R^2 - r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi r R^2 - 6\pi r^3\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0$$

Volume is maximum when $r^2 = \frac{2R^2}{3}$

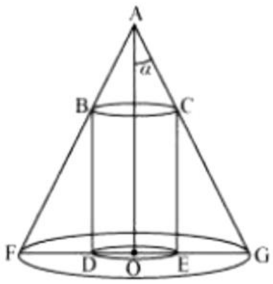
$$r^2 = \frac{2R^2}{3}$$

Height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$

Volume of the cylinder is maximum when height of cylinder is $\frac{2R}{\sqrt{3}}$

17. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi veritical angle a is one – third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^2 \tan^2 a$

Solution:



$r = h\tan a$

Since ΔAOG is similar to ΔCEG

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r-R}$$

$$\Rightarrow H = \frac{h}{r}(r-R) = \frac{h}{h\tan a}(h\tan a - R) = \frac{1}{\tan a}(h\tan a - R)$$

Volume of the cylinder is

$$V = \pi R^2 H = \frac{\pi R^2}{\tan a}(h\tan a - R) = \pi R^2 h - \frac{\pi R^3}{\tan a}$$

$$\therefore \frac{dV}{dR} = 2\pi Rh \frac{3\pi R^2}{\tan a}$$

$$\frac{dV}{dR} = 0$$

$$\Rightarrow 2\pi Rh = \frac{3\pi R^2}{\tan a}$$

$$\Rightarrow 2h\tan a = 3R$$

$$\Rightarrow R=\frac{2h}{3}\tan a$$

$$\frac{d^2V}{dR^2}=2\pi Rh-\frac{6\pi R}{\tan a}$$

And, for $R=\frac{2h}{3}\tan a$, we have

$$\frac{d^2V}{dR^2}=2\pi h-\frac{6\pi}{\tan a}\left(\frac{2h}{3}\tan a\right)=2\pi h-4\pi h=-2\pi h<0$$

Volume of the cylinder is greatest when $R=\frac{2h}{3}\tan a$

$$R=\frac{2h}{3}\tan a, H=\frac{1}{\tan a}\left(h\tan a-\frac{2h}{3}\tan a\right)=\frac{1}{\tan a}\left(\frac{h\tan a}{3}\right)=\frac{h}{3}$$

The maximum volume of cylinder can be obtained as

$$\pi\left(\frac{2h}{3}\tan a\right)^2\left(\frac{h}{3}\right)=\pi\left(\frac{4h^2}{9}\tan^2 a\right)\left(\frac{h}{3}\right)=\frac{4}{27}\pi h^3\tan^2 a$$

18. A cylindrical tank of radius 10m is being filled with wheat at the rate of 314 cubic mere per hour. Then the depth of the wheat is increasing at the rate of

$(A)\,1\,m/h$ $(B)\,0.1\,m/h$ $(C)\,1.1\,m/h$ $(D)\,0.5\,m/h$

Solution:

$$V=\pi(radius)^2\times height$$

$$=\pi(10)^2 h \qquad (radius=10m)$$

$$=100\pi h$$

$$\frac{dV}{dt}=100\pi\frac{dh}{dt}$$

Tank is being filled with wheat at rate of 314 cubic meters per hour

$$\frac{dV}{dt}=314\,m^3/h$$

$$314 = 100\pi \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

The depth of wheat is increasing at 1 m/h

The correct answer is A

19. The slope of the tangent to the curve $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is

$(A)\frac{22}{7}$ $(B)\frac{6}{7}$ $(C)\frac{7}{6}$ $(D)\frac{-6}{7}$

Solution:

$$\therefore \frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{4t-2}{2t+3}$$

Given point is $(2, -1)$

At x = 2

$$t^2 + 3t - 8 = 2$$

$$\Rightarrow t^2 + 3t - 10 = 0$$

$$\Rightarrow (t-2)(t+5) = 0$$

$$\Rightarrow t = 2 \text{ or } t = -5$$

At $y = -1$, we have

$$2t^2 - 2t - 5 = -1$$

$$\Rightarrow 2t^2 - 2t - 4 = 0$$

$$\Rightarrow 2(t^2 - t - 2) = 0$$

$\Rightarrow (t-2)(t+1)=0$

$\Rightarrow t=2 \text{ or } t=-1$

Common value of t is 2

Slope of tangent to given curve at point $(2,-1)$ is

$$\left.\frac{dy}{dx}\right]_{t=2}=\frac{4(2)-2}{2(2)+3}=\frac{8-2}{4+3}=\frac{6}{7}$$

The correct answer is B

20. The line $y=mx+1$ is tangent to the given curve $y^2=4x$ if the value on m is

$(A)1$ $(B)2$ $(C)3$ $(D)\frac{1}{2}$

Solution:

Equation of the tangent to curve is $y=mx+1$

Substituting $y=mx+1$ in $y^2=4x$

$\Rightarrow (mx+1)^2=4x$

$\Rightarrow m^2x^2+1+2mx-4x=0$

$\Rightarrow m^2x^2+x(2m-4)+1=0 \quad ....(i)$

$(2m-4)^2-4(m^2)(1)=0$

$\Rightarrow 4m^2+16-16m-4m^2=0$

$\Rightarrow 16-16m=0$

$\Rightarrow m=1$

The required value of m is 1

The correct answer is A.

21. The normal at the point (1, 1) on the curve $2y+x^2=3$ is

$(A)\, x+y=0 \quad (B)\, x-y=0 \quad (C)\, x+y+1=0 \quad (D)\, x-y=1$

Solution:

$$\frac{2dy}{dx}+2x=0$$

$$\Rightarrow \frac{dy}{dx}=-x$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,1)}=-1$$

Slope of normal to curve at point (1, 1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}}=1$$

Equation of normal to given curve at (1, 1) is

$$\Rightarrow y-1=1(x-1)$$

$$\Rightarrow y-1=x-1$$

$$\Rightarrow x-y=0$$

The correct answer is B

22. The normal to the curve $x^2=4y$ passing (1, 2) is

$(A)\, x+y=3 \quad (B)\, x-y=3 \quad (C)\, x+y=1 \quad (D)\, x-y=1$

Solution:

$$2x=4\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{2}$$

Slope of normal to curve at point (h,k) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(h,k)}} = -\frac{2}{h}$$

Equation of normal at point (h,k) is

$$y-k=\frac{-2}{h}(x-h)$$

Normal passes through the point $(1,2)$

$$2-k=\frac{-2}{h}(1-h) \text{ or } k=2+\frac{2}{h}(1-h) \quad ......(i)$$

(h,k) lies on the curves $x^2=4y$, we have $h^2=4k$

$$\Rightarrow k=\frac{h^2}{4}$$

$$\frac{h^2}{4}=2+\frac{2}{h}(1-h)$$

$$\Rightarrow \frac{h^3}{4}=2h+2-2h=2$$

$$\Rightarrow h^3=8$$

$$\Rightarrow h=2$$

$$\therefore k=\frac{h^2}{4} \Rightarrow k=1$$

Equation of normal is

$$\Rightarrow y-1=\frac{-2}{2}(x-2)$$

$$\Rightarrow y-1=(x-2)$$

$$\Rightarrow x+y=3$$

The correct answer is A

23. The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are

$(A)\left(4,\pm\frac{8}{3}\right)$ $(B)\left(4,\frac{-8}{3}\right)$ $(C)\left(4,\pm\frac{3}{8}\right)$ $(D)\left(\pm4,\frac{8}{3}\right)$

Solution:

$$9(2y)\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

Slope of normal to given curve at point (x_1, y_1) is

$$\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_1,y_1)}} = -\frac{6y_1}{x_1^2}$$

Equation of normal to curve at (x_1, y_1) is

$$(y - y_1) = \frac{-6y_1}{x_1^2}(x - x_1)$$

$$\Rightarrow x_1^2 y + x_1^2 y_1 = 6xy_1 + 6x_1y_1$$

$$\Rightarrow 6x_1y_1 + x_1^2 y = 6x_1y_1 + x_1^2 y_1$$

$$\Rightarrow \frac{6xy_1}{6x_1y_1 + x_1^2 y_1} = \frac{x^2 y}{6x_1y_1 + x^2 y} = 1$$

$$\Rightarrow \frac{x}{\frac{x_1(6+x_1)}{6}} + \frac{y}{\frac{y_1(6+x_1)}{x_1}} = 1$$

Normal makes equal intercepts with axes

$$\therefore \frac{x_1(6+x_1)}{6} + \frac{y_1(6+x_1)}{x_1}$$

$$\Rightarrow \frac{x_1}{6} = \frac{y_1}{x_1}$$

$$\Rightarrow x_1^2 = 6y_1$$

(x_1, y_1) lies on the curve, so

$$9y_1^2 = x_1^3$$

$$9\left(\frac{x_1^2}{6}\right)^2 = x_1^3 \Rightarrow \frac{x_1^4}{4} = x_1^3 \Rightarrow x_1 = 4$$

$$9y_1^2 = (4)^3 = 64$$

$$\Rightarrow y_1^2 = \frac{64}{9}$$

$$\Rightarrow y_1 = \pm\frac{8}{3}$$

Required points are $\left(4, \pm\frac{8}{3}\right)$

The correct answer is A