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Chapter 6: Applications of Derivatives.

Exercise 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when

(a) r =3 cm (b) r = 4 cm

Solution:

We know that $A = \pi r^2$

$$\therefore \frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$$

(a) When r = 3 cm,

$$\frac{dA}{dr} = 2\pi(3) = 6\pi$$

The area is changing at $6\,cm^2/s$ when radius is 3 cm

(b) When r = 4 cm,

$$\frac{dA}{dr} = 2\pi(4) = 8\pi$$

The area is changing at $8\,cm^2/s$ when radius is 4 cm

2. The volume of a cube is increasing at the rate of $8\,cm^3/s$. How fast is the surface area increasing when the length of its edge is 12 cm?

Solution:

Let the side length, volume and surface area respectively be equal to x, V and S

$$V = x^3$$

$$S = 6x^2$$

$$\frac{dV}{dt} = 8cm^3/s$$

$$\therefore 8 = \frac{dV}{dt} = \frac{d}{dt}\left(x^3\right) = \frac{d}{dx}\left(x^3\right)\frac{dx}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$$

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$$\frac{ds}{dt}=\frac{d}{dt}\left(6x^2\right)=\frac{d}{dx}\left(6x^2\right)\frac{dx}{dt}=12x\frac{dx}{dt}=12x\left(\frac{8}{3x^2}\right)=\frac{32}{x}$$

So, when $x=12cm, \frac{ds}{dt}=\frac{32}{12}cm^2/s=\frac{8}{3}cm^2/s$

3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm/s

Solution:

We know that $A=\pi r^2$

$$\therefore \frac{dA}{dt}=\frac{d}{dr}\left(\pi r^2\right)\frac{dr}{dt}=2\pi r\frac{dr}{dt}$$

$$\frac{dr}{dt}=3cm/s$$

$$\therefore \frac{dA}{dt}=2\pi r(3)=6\pi r$$

So, when r = 10 cm,

$$\frac{dA}{dt}=6\pi(10)=60\pi\ cm^2/s$$

4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

Solution:

Let the length and the volume of the cube respectively be x and V

$$V=x^3$$

$$\therefore \frac{dV}{dt}=\frac{d}{dt}\left(x^3\right)=\frac{d}{dx}\left(x^3\right)\frac{dx}{dt}=3x^2\frac{dx}{dt}$$

$$\frac{dx}{dt}=3cm/s$$

$$\therefore \frac{dV}{dt}=3x^2(3)=9x^2$$

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So, when x = 10 cm,

$$\frac{dV}{dt} = 9(10)^2 = 900\ cm^3/s$$

5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Solution:

We know that $A = \pi r^2$

$$\frac{dA}{dt} = \frac{d}{dt}\left(\pi r^2\right) = \frac{d}{dr}\left(\pi r^2\right)\frac{dr}{dt} = 2\pi r\frac{dr}{dt}$$

$$\frac{dr}{dt} = 5\,cm/s$$

So, when r = 8 cm,

$$\frac{dA}{dt} = 2\pi(8)(5) = 80\pi\ cm^2/s$$

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Solution:

We know that $C = 2\pi r$

$$\therefore \frac{dC}{dt} = \frac{dC}{dr}\frac{dr}{dt} = \frac{d}{dr}(2\pi r)\frac{dr}{dt} = 2\pi\frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.7\,cm/s$$

$$\therefore \frac{dC}{dt} = 2\pi(0.7) = 1.4\pi\ cm/s$$

7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x= 8 cm and y = 6cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

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Solution:

It is given that $\frac{dx}{dt} = -5cm/\min, \frac{dx}{dt} = 4cm/\min, x = 8\,cm\ and\ \ y = 6cm,$

(a) The perimeter of a rectangle is given by $P = 2(x+y)$

$$\therefore \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4) = -2cm/\min$$

(b) The area of rectangle is given by $A = xy$

$$\therefore \frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} = -5y + 4x$$

When x = 8cm and y = 6 cm, $\frac{dA}{dt} = (-5\times6+4\times8)cm^2/\min = 2cm^2/\min$

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm

Solution:

We know that $V = \frac{4}{3}\pi r^3$

$$\therefore \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^2\right)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = 900\ cm^2/s$$

$$\therefore 900 = 4\pi r^2\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

So, when radius = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}cm/s$$

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is 10 cm

Solution:

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We know that $V = \frac{4}{3}\pi r^2$

$$\therefore \frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2$$

So, when radius = 10 cm, $\frac{dV}{dr} = 4\pi(10)^2 = 400\pi$

Thus, the volume of the balloon is increasing at the rate of $400\pi\ cm^3/s$

10. A ladder 5m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

Solution:

Let the height of the wall at which the ladder is touching it be y m and the distance of its foot from the wall on the ground be x m

$$\therefore x^2 + y^2 = 5^2 = 25 \Rightarrow y = \sqrt{25 - x^2}$$

$$\therefore \frac{dy}{dt} = \frac{d}{dt}\left(\sqrt{25 - x^2}\right) = \frac{d}{dx}\left(\sqrt{25 - x^2}\right)\frac{dx}{dt} = \frac{-x}{\sqrt{25 - x^2}}\frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\ cm/s$$

$$\therefore \frac{dy}{dt} = \frac{-2x}{\sqrt{25 - x^2}}$$

So, when x = 4 m,

$$\frac{dy}{dt} = \frac{-2 \times 4}{\sqrt{25 - 16}} = -\frac{8}{3}$$

11. A particle is moving along the curve $6y = x^3 + 2$. Find the points on the curve at which the Y coordinate is changing 8 times as fast as the X coordinate

Solution:

The equation of the curve is $6y = x^3 + 2$

Differentiating with respect to time, we have,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} \Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

According to the question, $\left(\frac{dy}{dt}=8\frac{dx}{dt}\right)$

$$\therefore 2\left(8\frac{dx}{dt}\right)=x^2\frac{dx}{dt}\Rightarrow 16\frac{dx}{dt}=x^2\frac{dx}{dt}\Rightarrow\left(x^2-16\right)\frac{dx}{dt}=0\Rightarrow x^2=16\Rightarrow x=\pm 4$$

When $x=4, y=\frac{4^3+2}{6}=\frac{66}{6}=11$

When $x=-4, y=\frac{\left(-4^3\right)+2}{6}=\frac{62}{6}=\frac{31}{3}$

Thus, the points on the curve are (4, 11) and $\left(-4,\frac{-31}{3}\right)$

12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}cm/s$. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Solution:

Assuming that the air bubble is a sphere,

$$V=\frac{4}{3}\pi r^2$$

$$\therefore \frac{dV}{dt}=\frac{d}{dt}\left(\frac{4\pi}{3}r^3\right)=\frac{d}{dr}\left(\frac{4\pi}{3}r^3\right)\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt}=\frac{1}{2}cm/s$$

So, when r = 1 cm, $\frac{dV}{dt}=4\pi(1)^2\left(\frac{1}{2}\right)=2\pi\ cm^3/s$

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x

Solution:

We know that $V=\frac{4}{3}\pi r^3$

$$d=\frac{3}{2}(2x+1)\Rightarrow r=\frac{3}{4}(2x+1)$$

$$\therefore V=\frac{4}{3}\pi\left(\frac{3}{4}\right)^3(2x+1)^3=\frac{9}{16}\pi(2x+1)^3$$

$$\therefore \frac{dV}{dx}=\frac{9}{116}\pi\frac{d}{dx}(2x+1)^3=\frac{27}{8}\pi(2x+1)^3$$

14. Sand is pouring from a pipe at the rate of $12\ cm^3/s$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one – sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Solution:

We know that $V=\frac{1}{3}\pi r^2h$

$$h=\frac{1}{6}r\Rightarrow r=6h$$

$$\therefore V=\frac{1}{3}\pi(6h)^2h=12\pi h^3$$

$$\therefore \frac{dV}{dt}=12\pi\frac{d}{dt}\left(h^3\right)\frac{dh}{dt}=12\pi\left(3h^2\right)\frac{dh}{dt}=36\pi h^2\frac{dh}{dt}$$

$$\frac{dV}{dt}=12\ cm^2/s$$

So, when h = 4 cm,

$$12=36\pi(4)^2\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt}=\frac{12}{36\pi(16)}=\frac{1}{48\pi}cm/s$$

15. The total cost C(x) in Rupees associated with the production of x units of an item is given by $C(x)=0.007x^3-0.003x^2+15x+4000$. Find the marginal cost when 17 units are produced

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Solution:

Marginal cost is the rate of change of the total cost with respect to the output.

$\therefore$ Marginal cost $MC = \frac{dC}{dx} = 0.007\left(3x^2\right) - 0.003(2x) + 15 = 0.021x^2 - 0.006x + 15$

When $x = 17, MC = 0.021(17)^2 - 0.006(17) + 15$

$= 0.021(289) - 0.006(17) + 15$

$= 6.069 - 0.102 + 15$

$= 20.967$

So, when 17 units are produced, the marginal cost is Rs. 20.967.

16. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when x = 7

Solution:

Marginal revenue is the rate of change of the total revenue with respect to the number of

units sold.

$\therefore$ Marginal Revenue $MR = \frac{dR}{dx} = 13(2x) + 26 = 26x + 26$

When $x = 7, MR = 26(7) + 26 = 182 + 26 = 208$

Thus, the marginal revenue is Rs 208

17. The rate of change of the area of a circle with respect to its radius r at r = 6 cm is

$(A) 10\pi$ $(B) 12\pi$ $(C) 8\pi$ $(D) 11\pi$

Solution:

We know that $A = \pi r^2$

$\therefore \frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$

So, when $r = 6\,cm, \dfrac{dA}{dr} = 2\pi \times 6 = 12\pi\ cm^2 / s$

Thus, the rate of change of the area of the circle is $12\pi\ cm^2 / s$

The correct answer is option B

18. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when x = 15 is

(A) 116 (B) 96 (C) 90 (D) 126

Solution:

Marginal revenue is the rate of change of the total revenue with respect to the number of

units sold.

$\therefore$ Marginal Revenue $MR = \dfrac{dR}{dx} = 3(2x) + 36 = 6x + 36$

So, when $x = 15, MR = 6(15) + 36 = 90 + 36 = 126$

Hence, the marginal revenue is Rs 126.

The correct answer is option D.

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