Chapter 6: Applications of Derivatives.

Exercise 6.2

1. Show, that the function given by $f(x)=3x+17$ is strictly increasing on R.

Solution:

Let x_1 *and* x_2, be any two numbers in R.

$x_1<x_2 \Rightarrow 3x_1+17<3x_2+17=f(x_1)<f(x_2)$

Thus, f is strictly increasing on R.

Alternate Method

$f'(x)=3>0$ on R.

Thus, f is strictly increasing on R.

2. Show, that the function given by $f(x)=e^{2x}$ is strictly increasing on R.

Solution:

Let x_1 *and* x_2 be any two numbers in R.

$x_1<x_2 \Rightarrow 2x_1<2x_2 \Rightarrow e^{2x_1}<e^{2x_2} \Rightarrow f(x_1)<f(x_2)$

Thus, f is strictly increasing on R.

3. Show that the function given by $f(x)=\sin x$ is

(A) Strictly increasing in $\left(0,\frac{\pi}{2}\right)$ (B) Strictly decreasing $\left(\frac{\pi}{2},\pi\right)$

(C) Neither increasing nor decreasing in $(0,\pi)$

Solution:

$f(x)=\sin x \Rightarrow f'(x)=\cos x$

(A) $x\in\left(0,\frac{\pi}{2}\right) \Rightarrow \cos x>0 \Rightarrow f'(x)>0$

Thus, f is strictly increasing in $\left(0,\frac{\pi}{2}\right)$

(B) $x\in\left(\frac{\pi}{2},\pi\right)\Rightarrow \cos x<0\Rightarrow f'(x)<0$

Thus, f is strictly decreasing in $\left(\frac{\pi}{2},\pi\right)$

(C) The results obtained in (A) and (B) are sufficient to state that f is neither increasing nor decreasing in $(0,\pi)$

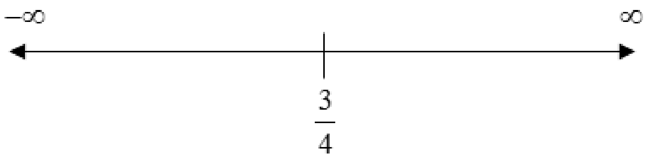
4. Find the intervals in which the function f given by $f(x)=2x^2-3x$ is

(A) Strictly increasing (B) Strictly decreasing

Solution:

$f(x)=2x^2-3x\Rightarrow f'(x)=4x-3$

$\therefore f'(x)=0\Rightarrow x=\frac{3}{4}$



In $\left(-\infty,\frac{3}{4}\right)$, $f'(x)=4x-3<0$

Hence, f is strictly decreasing in $\left(-\infty,\frac{3}{4}\right)$

In $\left(\frac{3}{4},\infty\right)$, $f'(x)=4x-3>0$

Hence, f is strictly increasing in $\left(\frac{3}{4},\infty\right)$

5. Find the intervals in which the function f given $f(x)=2x^2-3x^2-36x+7$ is

(A) Strictly increasing (B) Strictly decreasing

Solution:

$$f(x) = 2x^2 - 3x^2 - 36x + 7$$

$$f'(x) = 6x^2 - 6x + 36 = 6(x^2 - x - 6) = 6(x+2)(x-3)$$

$$\therefore f'(x) = 0 \Rightarrow x = -2, 3$$

In $(-\infty, -2)$ *and* $(3, \infty)$, $f'(x) > 0$

In $(-2, 3)$, $f'(x) < 0$

Hence, f is strictly increasing $(-\infty, -2)$ *and* $(3, \infty)$ and strictly decreasing in $(-2, 3)$

6. Find the intervals in which the following functions are strictly increasing or decreasing

(a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$ (c) $-2x^3 - 9x^2 - 12x + 1$

(d) $6 - 9x - x^2$ (e) $(x+1)^3 (x-3)^3$

Solution:

$$f(x) = x^2 + 2x - 5 \Rightarrow f'(x) = 2x + 2 \Rightarrow f'(x) = 0 \Rightarrow x = 1$$

$x = -1$ divides the number line into intervals $(-\infty, -1)$ *and* $(-1, \infty)$

In $(-\infty, -1)$, $f'(x) = 2x + 2 < 0$

$\therefore f$ is strictly decreasing in $(-\infty, -1)$

In $(-\infty, -1)$, $f'(x) = 2x + 2 > 0$, $\therefore f'(x) = 2x + 2 > 0$

$\therefore f$ is strictly decreasing in $(-1, \infty)$

(b) $f(x)=10-6x-2x^2 \Rightarrow f'(x)=-6-4x \Rightarrow f'(x)=0 \Rightarrow x=-\frac{3}{2}$

$x=-\frac{3}{2}$ divides the number line into two intervals $\left(-\infty,-\frac{3}{2}\right)$ *and* $\left(-\frac{3}{2},\infty\right)$

In $\left(-\infty,-\frac{3}{2}\right)$, $f'(x)=-6-4x<0$

$\therefore$ f is strictly increasing for $x<-\frac{3}{2}$

In $\left(-\frac{3}{2},\infty\right)$, $f'(x)=-6-4x>0$

$\therefore$ f is strictly increasing for $x>-\frac{3}{2}$

(c) $f(x)=-2x^3-9x^2-12x+1$

$\therefore f'(x)=-6x^2-18x-12=-6\left(x^2+3x+2\right)=-6(x+1)(x+2)$

$\therefore f'(x)=0 \Rightarrow x=-1,2$

$x=-1$ *and* $x=-2$ divide the number line into intervals $(-\infty,-2),(-2,-1)$ *and* $(-1,\infty)$

In $(-\infty,-2)$ *and* $(-1,\infty)$, $f'(x)=-6(x+1)(x+2)<0$

$\therefore$ f is strictly decreasing for $x<-2$ *and* $x>-1$

In $(-2,-1)$, $f'(x)=-6(x+1)(x+2)>0$

$\therefore$ f is strictly increasing for $-2<x<-1$

(d) $f(x)=6-9x-x^2 \Rightarrow f'(x)=-9-2x$

$f'(x)=0 \Rightarrow x=\frac{-9}{2}$

In $\left(-\infty,-\frac{9}{2}\right)$, $f'(x)>0$

$\therefore f$ is strictly increasing for $x < -\frac{9}{2}$

In $\left(-\frac{9}{2}, \infty\right)$, $f'(x) < 0$

$\therefore f$ is strictly decreasing for $x > -\frac{9}{2}$

(e) $f(x) = (x+1)^3 (x-3)^3$

$f'(x) = 3(x+1)^2 (x-3)^3 + 3(x-3)^2 (x+1)^3$

$= 3(x+1)^2 (x-3)^2 [x-3+x+1]$

$= 3(x+1)^2 (x-3)^2 (2x-2)$

$= 6(x+1)^2 (x-3)^2 (x-1)$

$f'(x) = 0 \Rightarrow x = -1, 3, 1$

$x = -1, 3, 1$ divides the number line into four intervals $(-\infty, -1), (-1, 1)(1, 3)$ *and* $(3, \infty)$

In $(-\infty, -1)$ *and* $(-1, 1)$, $f'(x) = 6(x+1)^2 (x-3)^2 (x-1) < 0$

$\therefore f$ is strictly decreasing in $(-\infty, -1)$ *and* $(-1, 1)$

In $(1, 3)$ *and* $(3, \infty)$, $f'(x) = 6(x+1)^2 (x-3)^2 (x-1) > 0$

$\therefore f$ is strictly increasing in $(1, 3)$ *and* $(3, \infty)$

7. Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$, is an increasing function throughout its domain

Solution:

$$y = \log(1+x) - \frac{2x}{2+x}$$

$$\therefore \frac{dy}{dx}=\frac{1}{1+x}-\frac{(2+x)(2)-2x(1)}{(2+x)^2}=\frac{1}{1+x}-\frac{4}{(2+x)^2}=\frac{x^2}{(1+x)(2+x)^2}$$

$$\frac{dy}{dx}=0$$

$$\Rightarrow \frac{x^2}{(2+x)^2}=0$$

$$\Rightarrow x^2=0$$

$$\Rightarrow x=0$$

Because $x>-1, x=0$ divides domain $(-1,\infty)$ in two intervals $-1<x<0$ *and* $x>0$

When $-1<x<0$,

$x<0 \Rightarrow x^2>0$

$x>-1 \Rightarrow (2+x)>0 \Rightarrow (2+x)^2>0$

$$\therefore y'=\frac{x^2}{(2+x)^2}>0$$

When x > 0,

$$\therefore y'=\frac{x^2}{(2+x)^2}>0$$

Hence, f is increasing throughout the domain.

8. Find the values of x for which $y=[x(x-2)]^2$ is an increasing function

Solution:

$$y=[x(x-2)]^2=[x^2-2x]^2$$

$$\therefore \frac{dy}{dx}=y'=2(x^2-2x)(2x-2)=4x(x-2)(x-1)$$

$$\therefore \frac{dy}{dx}=0 \Rightarrow x=0, x=2, x=1$$

$x=0, x=1$ *and* $x=2$ divide the number line into intervals $(-\infty,0),(0,1),(1,2)$ *and* $(2,\infty)$

In $(-\infty,0)$ *and* $(1,2), \frac{dy}{dx}<0$

$\therefore$ y is strictly decreasing in intervals $(-\infty,0)$ *and* $(1,2)$

In intervals $(0,1)$ *and* $(2,\infty), \frac{dy}{dx}>0$

$\therefore$ y is strictly increasing in intervals $(0,1)$ *and* $(2,\infty)$

9. Prove that $y=\frac{4\sin\theta}{(2+\cos\theta)}-\theta$ is an increasing function of θ in $\left[0,\frac{\pi}{2}\right]$

Solution:

$$y=\frac{4\sin\theta}{(2+\cos\theta)}-\theta$$

$$\therefore \frac{dy}{d\theta}=\frac{(2+\cos\theta)(4\cos\theta)-4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4\cos^2\theta+4\sin^2\theta}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4}{(2+\cos\theta)}-1$$

$$\frac{dy}{d\theta}=0$$

$$\Rightarrow \frac{8\cos\theta+4}{(2+\cos\theta)^2}=1$$

$$\Rightarrow 8\cos\theta+4=4+\cos^2\theta+4\cos\theta$$

$$\Rightarrow \cos^2\theta-4\cos\theta=0$$

$$\Rightarrow \cos\theta(\cos\theta-4)=0$$

$$\Rightarrow \cos\theta=0 \text{ or } \cos\theta=4$$

Because $\cos\theta\neq 4, \cos\theta=0$

$$\cos\theta=0\Rightarrow \theta=\frac{\pi}{2}$$

$$\frac{dy}{d\theta}=\frac{8\cos\theta+4-\left(4+\cos^2\theta+4\cos\theta\right)}{\left(2+\cos\theta\right)^2}=\frac{4\cos\theta-\cos^2\theta}{\left(2+\cos\theta\right)^2}=\frac{\cos\theta\left(4-\cos\theta\right)}{\left(2+\cos\theta\right)^2}$$

In $\left[0,\frac{\pi}{2}\right], \cos\theta>0,$

$4>\cos\theta \Rightarrow 4-\cos\theta>0$

$\therefore \cos\theta\left(4-\cos\theta\right)>0$

$\left(2+\cos\theta\right)^2>0$

$\Rightarrow \frac{\cos\theta\left(4-\cos\theta\right)}{\left(2+\cos\theta\right)^2}>0$

$\Rightarrow \frac{dy}{d\theta}>0$

So, y is strictly increasing in $\left(0,\frac{\pi}{2}\right)$

The function is continuous at $x=0$ *and* $x=\frac{\pi}{2}$

So, y is increasing in $\left[0,\frac{\pi}{2}\right]$

10. Prove that the logarithmic function is strictly increasing on $\left(0,\infty\right)$

Solution:

$f\left(x\right)=\log x$

$\therefore f'\left(x\right)=\frac{1}{x}$

For $x>0, f'\left(x\right)=\frac{1}{x}>0$

Thus, the logarithmic function is strictly increasing in interval $\left(0,\infty\right)$

11. Prove that the function f given by $f\left(x\right)=x^2-x+1$ is neither strictly increasing nor stricty decreasing on $\left(-1,1\right)$

Solution:

$f(x) = x^2 - x + 1$

$\therefore f'(x) = 2x - 1$

$\therefore f'(x) = 0 \Rightarrow x = \frac{1}{2}$

$x = \frac{1}{2}$ divides $(-1,1)$ into $\left(-1, \frac{1}{2}\right)$ *and* $\left(\frac{1}{2}, 1\right)$

In $\left(-1, \frac{1}{2}\right)$, $f'(x) = 2x - 1 < 0$

So, f is strictly decreasing in $\left(-1, \frac{1}{2}\right)$

In $\left(\frac{1}{2}, 1\right)$, $f'(x) = 2x - 1 > 0$

So, f is strictly increasing in interval $\left(\frac{1}{2}, 1\right)$

Thus, f is neither strictly increasing nor strictly decreasing in interval $(-1,1)$

12. Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

$(A)\cos x$ $(B)\cos 2x$ $(C)\cos 3x$ $(D)\tan x$

Solution:

$(A)\ f_1(x) = \cos x$

$\therefore f_1'(x) = -\sin x$

In $\left(0, \frac{\pi}{2}\right)$, $f_1'(x) = -\sin x < 0$

$\therefore f_1'(x) = \cos x$ is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$

$(B)\ f_2(x) = \cos 2x$

$\therefore f_2'(x) = -2\sin 2x$

$0 < x < \frac{\pi}{2} \Rightarrow 0 < 2x < \pi \Rightarrow \sin 2x > 0 \Rightarrow -2\sin 2x < 0$

$\therefore f_2'(x) = -2\sin 2x < 0$ in $\left(0, \frac{\pi}{2}\right)$

$\therefore f_2(x) = \cos 2x$ is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$

$(C)\ f_3(x) = \cos 3x$

$\therefore f_3'(x) = -3\sin 3x$

$f_3'(x) = 0$

$\Rightarrow \sin 3x = 0 \Rightarrow 3x = \pi, as\ x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow x = \frac{\pi}{3}$

$x = \frac{\pi}{3}$ *divides* $\left(0, \frac{\pi}{2}\right)$ into $\left(0, \frac{\pi}{3}\right)$ *and* $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

In $\left(0, \frac{\pi}{3}\right)$, $f_3(x) = -3\sin 3x < 0$ $\quad \left[0 < x < \frac{\pi}{3} \Rightarrow 0 < 3x < \pi\right]$

$\therefore f_3$ is strictly decreasing in $\left(0, \frac{\pi}{3}\right)$

In $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, $f_3(x) = -3\sin 3x > 0$ $\quad \left[\frac{\pi}{3} < x < \frac{\pi}{2} \Rightarrow \pi < 3x < \frac{3\pi}{2}\right]$

$\therefore f_3$ is strictly increasing in $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

So, f_3 is neither increasing nor decreasing in interval $\left(0, \frac{\pi}{2}\right)$

$(D)\ f_4(x) = \tan x$

$\therefore\ f_4'(x) = \sec^2 x$

In $\left(0, \frac{\pi}{2}\right)$, $f_4'(x) = \sec^2 x > 0$

$\therefore\ f_4$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

So, the correct answer are A and B

13. On which of the following intervals is the function f is given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?

$A. (0,1)$ $B. \left(\frac{\pi}{2}, \pi\right)$ $C. \left(0, \frac{\pi}{2}\right)$ D. None of these

Solution:

$f(x) = x^{100} + \sin x - 1$

$\therefore\ f'(x) = 100x^{99} + \cos x$

In $(0,1)$, $\cos x > 0$ *and* $100x^{99} > 0$

$\therefore\ f'(x) > 0$

So, f is strictly increasing in (0, 1)

In $\left(\frac{\pi}{2}, x\right)$, $\cos x < 0$ *and* $100x^{99} > 0$

$100x^{99} > \text{cosx}$

$\therefore\ f'(x) > 0\ in \left(\frac{\pi}{2}, x\right)$

So, f is strictly increasing in interval $\left(\frac{\pi}{2}, x\right)$

In interval $\left(0, \frac{\pi}{2}\right)$, $\cos x > 0$ *and* $100x^{99} > 0$

$\therefore 100x^{99} + \cos x > 0$

$\Rightarrow f'(x) > 0$ *on* $\left(0, \frac{\pi}{2}\right)$

$\therefore f$ is strictly increasing in interval $\left(0, \frac{\pi}{2}\right)$

Hence, f is strictly decreasing in none of the intervals.

The correct answer is D.

14. Find the least value of a such that the function f given $f(x) = x^2 + ax + 1$ is strictly increasing on (1, 2)

Solution:

$f(x) = x^2 + ax + 1$

$\therefore f'(x) = 2x + a$

$\therefore f'(x) > 0$ *in* $(1, 2)$

$\Rightarrow 2x + a > 0$

$\Rightarrow 2x > -a$

$\Rightarrow x > \frac{-a}{2}$

So, we need to find the smallest value of a such that

$x > \frac{-a}{2}$, when $x \in (1, 2)$

$\Rightarrow x > \frac{-a}{2} (when\, 1 < x < 2)$

$\frac{-a}{2} = 1 \Rightarrow a = -2$

Hence, the required value of a is – 2

15. Let I be any interval disjoint from (-1, 1), prove that the function f given by

$f(x) = x + \frac{1}{x}$ is strictly increasing on I

Solution:

$$f(x) = x + \frac{1}{x}$$

$$\therefore f'(x) = 1 - \frac{1}{x^2}$$

$$\therefore f'(x) = 0 \Rightarrow \frac{1}{x^2} \Rightarrow x = \pm 1$$

x = 1 and x = - 1 divide the real line in intervals $(-\infty, 1), (-1, 1)$ *and* $(1, \infty)$

In $(-1, 1)$,

$$-1 < x < 1$$

$$\Rightarrow x^2 < 1$$

$$\Rightarrow 1 < \frac{1}{x^2}, x \neq 0$$

$$\Rightarrow 1 - \frac{1}{x^2} < 0, x \neq 0$$

$$\therefore f'(x) = 1 - \frac{1}{x^2} < 0 \text{ on } (-1, 1) \sim \{0\}$$

$\therefore f$ is strictly decreasing on $(-1, 1) \sim \{0\}$

In $(-\infty, -1)$ *and* $(1, \infty)$

$$x < -1 \text{ or } 1 < x$$

$$\Rightarrow x^2 > 1$$

$\Rightarrow 1 > \frac{1}{x^2}$

$\Rightarrow 1 - \frac{1}{x^2} > 0$

$\therefore f'(x) = 1 - \frac{1}{x^2} > 0 \; on \; (-\infty, -1) \; and \; (1, \infty)$

$\therefore f$ is strictly increasing on $(-\infty, 1) \; and \; (1, \infty)$

Hence, f is strictly increasing in $I - (-1, 1)$

16. Prove that the function f given by $f(x) = \log \sin x$ strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$

Solution:

$f(x) = \log \sin x$

$f'(x) = \frac{1}{\sin x} \cos x = \cot x$

In $\left(0, \frac{\pi}{2}\right)$, $f'(x) = \cot x > 0$

$\therefore f$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

In $\left(\frac{\pi}{2}, \pi\right)$, $f'(x) = \cot x < 0$

$\therefore f$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$

17. Prove that the function f given by $f(x) = \log \cos x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$

Solution:

$f(x) = \log \cos x$

$\therefore f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

In $\left(0, \frac{\pi}{2}\right)$, $\tan x > 0 \Rightarrow -\tan x < 0$

$\therefore f'(x) =< 0 \ on \left(0, \frac{\pi}{2}\right)$

$\therefore f$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

In $\left(\frac{\pi}{2}, \pi\right)$, $\tan x < 0 \Rightarrow -\tan x > 0$

$\therefore f'(x) > 0 \ on \left(\frac{\pi}{2}, \pi\right)$

$\therefore f$ is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x = 100$ is increasing in R

Solution:

$f(x) = x^3 - 3x^2 + 3x = 100$

$f'(x) = 3x^2 - 6x + 3$

$= 3(x^2 - 2x + 1)$

$= 3(x-1)^2$

For $x \in R (x-1)^2 \geq 0$

So $f'(x)$ is always positive in R

So, the f is increasing in R

19. The interval in which $y = x^2e^{-x}$ is increasing is

$A.(-\infty,\infty)$ $B.(-2,0)$ $C.(2,\infty)$ $D.(0,2)$

Solution:

$$y = x^2e^{-x}$$

$$\therefore \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$$

$$\frac{dy}{dx} = 0$$

$$\Rightarrow x = 0 \text{ and } x = 2$$

In $(-\infty,0)$ *and* $(2,\infty)$, $f'(x) < 0$ *as* e^{-x} is always positive

$\therefore f$ is decreasing on $(-\infty,0)$ *and* $(2,\infty)$

In $(0,2)$, $f'(x) > 0$

$\therefore f$ is strictly increasing on (0, 2)

So, f is strictly increasing in (0, 2)

The correct answer is D.