Chapter 6: Applications of Derivatives.

Exercise 6.5

1. Find the maximum and minimum values, if any, of the following given by

$(i)\ f(x)=(2x-1)^2+3 \quad (ii)\ f(x)=9x^2+12x+2 \quad (iii)\ f(x)=-(x-1)^2+10$

$(iv)\ g(x)=x^3+1$

Solution:

$(i)\ f(x)=(2x-1)^2+3$

$(2x-1)^2 \geq 0$ for every $x \in R$

$f(x)=(2x-1)^2+3 \geq 3\ for\ x \in R$

The minimum value of f occurs when $2x-1=0$

$2x-1=0, x=\frac{1}{2}$

Min value of $f\left(\frac{1}{2}\right)=\left(2.\frac{1}{2}-1\right)^2+3=3$

The function f does not have a maximum value

$(ii)\ f(x)=9x^2+12x+2=(3x^2+2)^2-2$

$(3x^2+2)^2 \geq 0\ for\ x \in R$

$f(x)=(3x^2+2)^2-2 \geq -2\ for\ x \in R$

Minimum value of f is when $3x+2=0$

$3x+2=0=0, x=\frac{-2}{3}$

Minimum value of $f\left(\frac{-2}{3}\right)=\left(3\left(\frac{-2}{3}\right)+2\right)^2-2=-2$

f does not have a maximum value

$(iii)\ f(x)=-(x-1)^2+10$

$(x-1)^2 \geq 0\ for\ x \in R$

$f(x)=-(x-1)^2+10 \leq 0\ for\ x \in R$

maximum value of f is when $(x-1)=0$

$(x-1)=0, x=0$

Maximum value of $f = f(1)=-(1-1)^2+10=10$

F does not have a minimum value

$(iv)\ g(x)=x^3+1$

g neither has a maximum value nor a minimum value.

2. Find the maximum and minimum values, if any, of the following functions given by

$(i) f(x)=|x+2|-1$ $(ii) g(x)=-|x+1|+3$ $(iii)\ h(x)=\sin(2x)+5$

$(iv)\ f(x)=|\sin 4x+3|$ $(v)\ h(x)=x+4, x \in (-1,1)$

Solution:

$(i)\ f(x)=|x+2|-1$

$|x+2| \geq 0\ for\ x \in R$

$f(x)=|x+2|-1 \geq -1\ for\ x \in R$

minimum value of f is when $|x+2|=0$

$|x+2|=0$

$\Rightarrow x=-2$

minimum value of $f = f(-2) = |-2+2| - 1 = 1$

f does not have a maximum value

$(ii)\ g(x) = -|x+1| + 3$

$-|x+1| \le 0\ for\ x \in R$

$g(x) = -|x+1| + 3 \le 3\ for\ x \in R$

maximum value of g is when $|x+1| = 0$

$|x+1| = 0$

$\Rightarrow x = -1$

Maximum value of $g = g(-1) = -|1+1| + 3 = 3$

g does not have a minimum value

$(iii)\ h(x) = \sin 2x + 5$

$-1 \le \sin 2\mathrm{x} \le 1$

$-1+5 \le \sin 2\mathrm{x}+5 \le 1+5$

$4 \le \sin 2\mathrm{x}+5 \le 6$

maximum and minimum values of h are 6 and 4 respectively

$(iv)\ f(x) = |\sin 4x + 3|$

$-1 \le \sin 4\mathrm{x} \le 1$

$2 \le \sin 4\mathrm{x}+3 \le 4$

$2 \le |\sin 4\mathrm{x}+3| \le 4$

maximum and minimum values of f are 4 and 2 respectively

$(v)\ h(x) = x + 4, x \in (-1,1)$

Here, if a point x_0 is closest to – 1, then we find $\frac{x_0}{2}+1 < x_0+1$ for $x_0 \in (-1,1)$

Also if x_1 is closet to – 1, then we find $x_1+1 < \frac{x_1+1}{2}+1$ for all $x_0 \in (-1,1)$

Function has neither maximum nor minimum value in $(-1,1)$

3. Find the local maxima and minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be

$(i)\ f(x)=x^2 \quad (ii)\ g(x)=x^3-3x \quad (iii)\ h(x)=\sin x+\cos .0<x<\frac{\pi}{2}$

$(iv)\ f(x)=\sin x-\cos x, 0<x<2\pi \quad (v)\ f(x)=x^3-6x^2+9x+15 \quad (vi)\ g(x)=\frac{x}{2}+\frac{2}{x}, x>0$

$(vii)\ g(x)=\frac{1}{x^2+2} \quad (vii)\ f(x)=x\sqrt{1-x}, x>0$

Solution:

$(i)\ f(x)=x^2$

$\therefore f'(x)=2x$

$f'(x)=0 \Rightarrow x=0$

We have $f'(0)=2,$

by second derivative test, $x=0$ is a point of local minima and local minimum value of f

at x= 0 is $f(0)=0$

$(ii)\ g(x)=x^3-3x$

$\therefore g'(x)=3x^2-3$

$\therefore g'(x)=0 \Rightarrow 3x^2=3 \Rightarrow x=\pm 1$

$g'(x)=6x$

$g'(1)=6>0$

$g'(-1)=-6>0$

By second derivative test, x = 1 is a point of local minima and local minimum value of g

At $x=1$ is $g(1)=1^3-3=1-3=-2$

x= -1 is a point of local maxima and local maximum value of g at

x = -1 is $g(1)=(-1)^3-3(-1)=-1+3=-2$

$(iii)\ h(x)=\sin x+\cos .0<x<\frac{\pi}{2}$

$\therefore h'(x)=\cos x+\sin x$

$\therefore h'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}\in\left(0,\frac{\pi}{2}\right)$

$h'(x)=\sin x-\cos x=-(\sin x+\cos x)$

$h'\left(\frac{\pi}{4}\right)=-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)=-\frac{2}{\sqrt{2}}=-\sqrt{2}<0$

Therefore, by second derivative test, $x=\frac{\pi}{4}$ is a point of local maxima and the local

Maximum value of h at $x=\frac{\pi}{4}$ is $h\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$

$(iv)\ f(x)=\sin x-\cos x, 0<x<2\pi$

$\therefore f'(x)=\cos x+\sin x$

$\therefore f'(x)=0 \Rightarrow \cos x=-\sin x \Rightarrow \tan x=-1 \Rightarrow x=\frac{3\pi}{4},\frac{7\pi}{4}\in(0,2\pi)$

$f''(x)=-\sin x+\cos x$

$$f''\left(\frac{3\pi}{4}\right)=-\sin\frac{3\pi}{4}+\cos\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\sqrt{2}>0$$

$$f''\left(\frac{7\pi}{4}\right)=-\sin\frac{7\pi}{4}+\cos\frac{7\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}>0$$

Therefore, by second derivative test, $x=\frac{3\pi}{4}$ is a point of local maxima and the local maximum value of f at $x=\frac{3\pi}{4}$ is

$$f\left(\frac{3\pi}{4}\right)=\sin\frac{3\pi}{4}+\cos\frac{3\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$$

$x=\frac{7\pi}{4}$ is a point of local minima and the local minimum value of f at $x=\frac{7\pi}{4}$ is

$$f\left(\frac{7\pi}{4}\right)=\sin\frac{7\pi}{4}-\cos\frac{7\pi}{4}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\sqrt{2}$$

$(v)\ f(x)=x^3-6x+9x+15$

$\therefore\ f'(x)=3x^2-12x+9$

$f(x)=0\Rightarrow 3\left(x^2-4x+3\right)=0$

$\Rightarrow 3(x-1)(x-3)=0$

$\Rightarrow x=1,3$

$f''(x)=6x-12=6(x-2)$

$f''(1)=6(1-2)=-6<0$

$f''(3)=6(3-2)=6<0$

By second derivative test, x = 1 is a point of local maxima and the local maximum value of f at x= 1 is $f(1)=1-6+9+15=19$

x =3 is a point of local minima and the local minimum value of f at x = 3 is $f(3)=27-54+27+15=15$

$(vi)\ g(x)=\frac{x}{2}+\frac{2}{x}, x>0$

$\therefore g'(x)=\frac{1}{2}-\frac{2}{x^2}$

$\therefore g'(x)=0 \Rightarrow \frac{2}{x^2}=\frac{1}{2} \Rightarrow x^3=4 \Rightarrow x=\pm$

$x>0, x=2$

$g''(x)=\frac{4}{x^3}$

$g''(2)=\frac{4}{2^3}=\frac{1}{2}>0$

By second derivative test, x = 2 is a point of local minima and the local minimum value of g at x = 2 is $g(2)=\frac{2}{2}+\frac{2}{2}=1+1=2$

$(vii)\ g(x)=\frac{1}{x^2+2}$

$\therefore g'(x)=\frac{-(2x)}{(x^3+2)^2}$

$g'(x)=0 \Rightarrow \frac{-2x}{(x^3+2)^2}=0 \Rightarrow x=0$

For value close to x = 0 and left of 0, $g^1(x)>0$

For value close to x= 0 and to right of 0 $g^1(x)<0$

By first derivative test x = 0 is a point of local maxima and the local maximum value of g(0) is $\frac{1}{0+2}=\frac{1}{2}$

(viii) $f(x)=x\sqrt{1-x},x>0$

$$\therefore f'(x)=x\sqrt{1-x}+x.\frac{1}{2\sqrt{1-x}}(-1)=\sqrt{1-x}-\frac{x}{2\sqrt{1-x}}$$

$$=\frac{2(1-x)-x}{2\sqrt{1-x}}-\frac{2-3x}{2\sqrt{1-x}}$$

$$f'(x)=0\Rightarrow\frac{2-3x}{2\sqrt{1-x}}=0\Rightarrow 2-3x=0\Rightarrow x=\frac{2}{3}$$

$$f''(x)=\frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$

$$=\frac{\sqrt{1-x}(-3)+2(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$=\frac{-6(1-x)+2(2-3x)}{4(1-x)^{\frac{3}{2}}}$$

$$=\frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f'\left(\frac{2}{3}\right)=\frac{3\left(\frac{2}{3}\right)-4}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}}=\frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}}=\frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}}<0$$

By second derivative test, $x=\frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x=\frac{2}{3}$ is

$$f\left(\frac{2}{3}\right)=\frac{2}{3}\sqrt{1-\frac{2}{3}}=\frac{2}{3}\sqrt{\frac{1}{3}}=\frac{2}{3\sqrt{3}}=\frac{2\sqrt{3}}{9}$$

4. Prove that the following functions do not have maxima or minima

$(i)\ f(x)=e^x \quad (ii)\ g(x)=\log x \quad (iii)\ h(x)=x^3+x^2+x+1$

Solution:

$(i)\ f(x)=e^x$

$\therefore f'(x)=e^x$

If $f'(x)=0, e^x=0$. But exponential function can never be 0 for any value of x

There is no $c\in R$ such that $f'(c)=0$

f does not have maxima or minima

(ii) We have, $g(x)=\log x$

$\therefore g'(x)=\frac{1}{x}$

$\log x$ is defined for positive x, $g'(x)>0$ for any x

there does not exist $c\in R$ such that $g'(c)=0$

function g does not have maxima or minima

(iii) We have, $h(x)=x^3+x^2+x+1$

$\therefore h'(x)=3x^2+2x+1$

there does not exist $c\in R$ such that $h'(c)=0$

function h does not have maxima or minima

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals

$(i)\ f(x)=x^3, x\in[-2,2] \qquad (ii)\ f(x)=\sin x+\cos x, x\in[0,\pi]$

$(iii)\ f(x)=4x-\frac{1}{2}x^2, x\in\left[-2,\frac{9}{2}\right] \qquad (iv)\ f(x)=(x-1)^2+3, x\in[-3,1]$

Solution:

$(i)\ f(x)=x^3$

$\therefore f'(x)=3x^2$

$f'(x)=0 \Rightarrow x=0$

$f(0)=0$

$f(-2)=(-2)^3=-8$

$f(2)=(2)^3=8$

Hence, the absolute maximum of f on $[-2,2]$ is 8 at $x=-2$

The absolute minimum of f on $[-2,2]$ is -8 at $x=-2$

$(ii)\ f(x)=\sin x+\cos x$

$\therefore f'(x)=\cos x-\sin x$

$f'(x)=0 \Rightarrow \sin x=\cos x \Rightarrow \tan x=1 \Rightarrow x=\frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$

$f(0)=\sin 0+\cos 0=0+1=1$

$f(\pi)=\sin\pi+\cos\pi=0-1=1$

The absolute maximum of f on $[0,\pi]$ is $\sqrt{2}$ at $x=\frac{\pi}{4}$

The absolute minimum of f on $[0,\pi]$ is -1 at $x=\pi$

$(iii)\ f(x)=4x-\frac{1}{2}x^2$

$\therefore f'(x)=4x-\frac{1}{2}(2x)=4-x$

$\therefore f'(x)=0 \Rightarrow x=4$

$f(4)=16-\frac{1}{2}(16)=16-8=8$

$f(-2)=-8-\frac{1}{2}(4)=-8-2=-10$

$f\left(\frac{9}{2}\right)=4\left(\frac{9}{2}\right)-\frac{1}{2}\left(\frac{9}{2}\right)^2=18-\frac{81}{8}=18-10.125=7.875$

The absolute maximum of f on $\left[-2,\frac{9}{2}\right]$ is 8 at x = 4

The absolute minimum of f on $\left[-2,\frac{9}{2}\right]$ is -10 at x = - 2

$(iv)\ f(x)=(x-1)^2+3$

$\therefore f'(x)=2(x-1)$

$f'(x)=0 \Rightarrow 2(x-1)=0, x=1$

$f(1)=(1-1)^2+3=0+3=3$

$f(-3)=(-3-1)^2+3=16+3=19$

Absolute maximum value of f on $[-3,1]$ is 19 at x= - 3

Minimum value of f on $[-3,1]$ is at x = 1

6. Find the maximum profit that a company can make, if the profit function is given by $p(x)=41-24x-18x^2$

Solution:

$p(x)=41-24x-18x^2$

$\therefore p'(x)=-24-36x$

$p''(x) = -36$

$p'(x) = 0 \Rightarrow \frac{-24}{36} = -\frac{2}{3}$

$p'\left(\frac{-2}{3}\right) = -36 < 0$

By second derivative test, $x = -\frac{2}{3}$ is the point of local maximum of p

$\therefore$ Maximum profit $= p = \left(-\frac{2}{3}\right)$

$= 41 - 24\left(-\frac{2}{3}\right) - 18\left(-\frac{2}{3}\right)^2$

$= 41 + 16 - 8$

$= 49$

7. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is

(i) Increasing (ii) Decreasing

Solution:

$f(x) = x^3 + \frac{1}{x^3}$

$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$

$f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = 1 \Rightarrow x \pm 1$

In $(-\infty, -1)$ and $(1, \infty)$, $f'(x) > 0$

when $x < -1$ *and* $x > 1$, f is increasing

In $(-1, 1)$, $f(x) < 0$

when $-1 < x < 1$, f is decreasing

8. At which point in the interval $\left[0, 2\pi\right]$, does the function $\sin 2x$ attain, its maximum value?

Solution:

$$\mathrm{f}\left(x\right) = \sin 2x$$

$$\therefore\ f\,'\left(x\right) = 2\cos 2x$$

$$f\,'\left(x\right) = 0 \Rightarrow \cos 2x = 0$$

$$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1, \mathrm{f}\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = -1$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{2\pi}{2} = 1, \mathrm{f}\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{2} = -1$$

$$f\left(0\right) = \sin 0 = 0,\ f\left(2\pi\right) = \sin 2\pi = 0$$

Absolute maximum value of $f\left[0, 2\pi\right]$ is at $x = \frac{\pi}{4}$ *and* $x = \frac{5\pi}{4}$

9. What is the maximum value of the function $\sin x + \cos x$?

Solution:

$$\mathrm{f}\left(x\right) = \sin x + \cos x$$

$$\therefore\ f\,'\left(x\right) = \cos x - \sin x$$

$$f\,'\left(x\right) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}......,$$

$$f\,'\left(x\right) = -\sin x - \cos x = -\left(\sin x + \cos x\right)$$

$f\,''\left(x\right)$ will be negative when $\left(\sin x + \cos x\right)$ is positive

We know that sin x and cos x are positive in the first quad5rant

$f''(x)$ will be negative when $x \in \left(0, \frac{\pi}{2}\right)$

Consider $x = \frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = -\left(\frac{2}{\sqrt{2}}\right) = -\sqrt{2} < 0$$

By second derivative test, f will be that maximum at $x = \frac{\pi}{4}$ and the maximum value of f is

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in $[-3, -1]$

Solution:

$$f(x) = 2x^3 - 24x + 107$$

$$f'(x) = 6x^2 - 24 = 6(x^2 - 4)$$

$$f'(x) = 0 \Rightarrow 6(x^2 - 4) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

Consider $[1, 3]$

$$f(2) = 2(8) - 24(2) + 107 = 16 - 48 + 107 = 75$$

$$f(1) = 2(1) - 24(1) + 107 = 2 - 24 + 107 = 85$$

$$f(3) = 2(27) - 24(3) + 107 = 54 - 72 + 107 = 89$$

Absolute maximum of $f(x)$ in the $[1, 3]$ is 89 at x = 3

Consider $[-3, -1]$

$$f(-3) = 2(-27) - 24(-3) + 107 = 54 + 72 + 107 = 125$$

$$f(-1) = 2(-1) - 24(-1) + 107 = 2 + 24 + 107 = 129$$

$$f(-2) = 2(-8) - 24(-2) + 107 = -16 + 48 + 107 = 139$$

Absolute maximum of $f(x)$ in $[-3, -1]$ is 139 at x= - 2

11. It is given that at x=1, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of a

Solution:

$$f(x) = x^4 - 62x^2 + ax + 9$$

$$\therefore f'(x) = 4x^2 - 124x + a$$

$$\therefore f'(1) = 0$$

$$\Rightarrow 4 - 124 + a = 0$$

$$\Rightarrow a = 120$$

The value of a is 120

12. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$

Solution:

$$f(x) = x + \sin 2x$$

$$\therefore f'(x) = 1 + 2\cos 2x$$

$$f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$2x = 2\pi \pm \frac{2\pi}{3} n \in Z$$

$$\Rightarrow x = \pi \pm \frac{\pi}{3}, n \in Z$$

$$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}+\sin\frac{4\pi}{3}=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right)=\frac{4\pi}{3}+\sin\frac{8\pi}{3}=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{3}\right)=\frac{5\pi}{3}+\sin\frac{10\pi}{3}=\frac{5\pi}{3}-\frac{\sqrt{3}}{2}$$

$$f(0)=0+\sin 0=0$$

$$f(2\pi)=2\pi+\sin 4\pi=2\pi+0=2\pi$$

Absolute maximum value of $f(x)$ in $[0,2\pi]$ is at $x=2\pi$

Absolute minimum value of $f(x)$ in $[0,2\pi]$ is at x = 0

13. Find two numbers whose sum is 24 and whose product is as large as possible

Solution:

Let number be x

The other number is $(24-x)$

$p(x)$ denote the product of the two numbers

$$P(x)=x(24-x)=24x-x^2$$

$$\therefore P'(x)=24-2x$$

$$P''(x)=-2$$

$$P'(x)=0\Rightarrow x=12$$

$$P''(12)=-2<0$$

x =12 is point of local maxima of P

Product of the numbers is the maximum when numbers are 12 and $24-12=12$

14. Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum

Solution:

Numbers are x and y such that $x+y=60$

$y=60-x$

$f(x)=xy^3$

$\Rightarrow f(x)=x(60-x)^3$

$\therefore f'(x)=(60+x)^3-3x(60-x)^2$

$=(60+x)^3[60-x-3x]$

$=(60+x)^3[60-4x]$

$f''(x)=-2(60-x)(60-4x)-4(60-x)^2$

$=-2(60-x)[60-4x+2(60-x)]$

$=-2(60-x)(180-6x)$

$=-12(60-x)(30-x)$

$f'(x)=0 \Rightarrow x=60\ or\ x=15$

$x=60,\ f''(x)=0$

$x=15, f''(x)=-12(60-15)(30-15)=12\times 45\times 15<0$

x =15 is a point of local maxima of f

function xy^3 is maximum when $x=15\ and\ y=60-15=45$

required numbers are 15 and 45

15. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum

Solution:

One number be x

Other number is $y=(35-x)$

$p(x)=x^2y^5$

$P(x)=x^2(35-x)^5$

$\therefore P'(x)=2x(35-x)^5-5x^2(35-x)^4$

$=x(35-x)^4\left[2(35-x)-5x\right]$

$=x(35-x)^4(70-7x)$

$=7x(35-x)^4(10-x)$

And $P''(x)=7(35-x)^4(10-x)+7x\left[-(35-5)^4-4(35-x)^3(10-x)\right]$

$=7(35-x)^4(10-x)-7x(35-x)^4-28x(35-x)^3(10-x)$

$=7(35-x)^3\left[(35-x)(10-x)-x(35-x)-4x(10-x)\right]$

$=7(35-x)^3\left[350-45x+x^2-35x+x^2-40x+4x^2\right]$

$=7(35-x)^3(6x^2-120x+350)$

$P'(x)=0\Rightarrow x=0, x=35, x=10$

$x=35, f'(x)=f(x)=0$ *and* $y=35-35=0$

$x=0, y=35-0=34$ and product x^2y^2 will be 0

$x=0$ *and* $x=35$ cannot be the possible values of x

$x = 10,$

$$P''(x) = 7(35-10)^3(6\times100-120\times10+350)$$

$$= 7(25)^3(-250) < 0$$

$P(x)$ will be the maximum when $x = 10$ *and* $y = 35-10 = 25$

The numbers are 10 and 25

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum

Solution:

One number be x

The other number is $(16-x)$

Sum of cubes of these numbers be denote by $S(x)$

$$\therefore S'(x) = 3x^2 - 3(16-x)^2, S''(x) = 6x + 6(16-x)$$

$$S'(x) = 0 \Rightarrow 3x^2 - 3(16-x^2) = 0$$

$$\Rightarrow x^2 - (16-x)^2 = 0$$

$$\Rightarrow x^2 - 256 - x^2 + 32x = 0$$

$$\Rightarrow x = \frac{256}{32} = 8$$

$$S''(8) = 6(8) + 6(16-8) = 48 + 48 = 96 > 0$$

By second derivative test, x = 8 is point of local minima of S.

Sum of the cubes of the numbers is minimum when the numbers are 8 and 16 – 8 = 8

17. A square piece of tin od side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution:

Side of the square to be cut off be x cm

Length and breath of the box will be $(18-2x)cm$ each and the height of the box is x cm.

$$V(x)=x(18-2x)^2$$

$$\therefore V'(x)=(18-2x)^2-4x(18-2x)$$

$$=(18-2x)[18-2x-4x]$$

$$=(18-2x)(18-6x)$$

$$=6\times 2(9-x)(3-x)$$

$$=12(9-x)(3-x)$$

$$V''(x)=12[-(9-x)-(3-x)]$$

$$=-12(9-x+3-x)$$

$$=-12(12-2x)$$

$$=-24(6-x)$$

$v'(x)=0 \Rightarrow x=9 \; or \; x=3$

$x=9,$ then the length and the breadth will become 0

$\therefore x\neq 9$

$\Rightarrow x=3$

$V''(3) = -24(6-3) = -72 < 0$

$\therefore$ By second derivative test, x = 3 is the point of maxima of V

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Solution:

Side of the square to be cut be x cm

Height of the box is x, the length is $45-2x$,

Breadth is $24-2x$

$V(x) = x(45-2x)(24-2x)$

$= x(1080 - 90x - 48x + 4x^2)$

$= 4x^3 - 138x^2 + 1080x$

$\therefore V'(x) = 12x^2 - 276x + 1080$

$= 12(x^2 - 23x + 90)$

$= 12(x-18)(x-5)$

$V''(x) = 24x - 276 = 12(2x-23)$

$V'(x) = 0 \Rightarrow x = 18 \text{ and } x = 5$

Not possible to cut a square of side 18cm from each corner of rectangular sheet, x cannot be equal to 18

$x = 15$

$V''(5) = 12(10-23) = 12(-13) = -156 < 0$

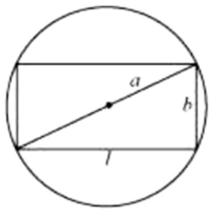
x = 5 is the point of maxima

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area

Solution:

A rectangle of length *l* and breadth b be inscribed in the given circle of radius a.

The diagonal passes through the centre and is of length 2a cm



$$(2a)^2 = l^2 + b^2$$

$$\Rightarrow b^2 = 4a^2 - l^2$$

$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

$$A = l\sqrt{4a^2 - l^2}$$

$$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l) = \sqrt{4a^2 - l^2} - \frac{1}{\sqrt{4a^2 - l^2}}$$

$$= \frac{4a^2 - l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)}$$

$$= \frac{(4a^2 - l^2)(-4l) + 1(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-12a^2l + 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}} = \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$\frac{dA}{dl}=0 \text{ gives } 4a^2=2l^2 \Rightarrow l=\sqrt{2}a$$

$$\Rightarrow b=\sqrt{4a^2-2a^2}=\sqrt{2a^2}=\sqrt{2}a$$

when $l=\sqrt{2}a$,

$$\frac{d^2A}{dl^2}=\frac{-2\left(\sqrt{2}a\right)\left(6a^2-2a^2\right)}{2\sqrt{2a^3}}=\frac{-8\sqrt{2a^3}}{2\sqrt{2a^3}}=-4<0$$

when $l=\sqrt{2}a$, then area of rectangle is maximum

Since $l=b=\sqrt{2}a$, rectangle is a square

20. Show that the right circular cylinder of given surface and maximum volume is such that is heights is equal to the diameter of the base

Solution:

$$S=2\pi r^2+2\pi rh$$

$$\Rightarrow h=\frac{S-2\pi r^2}{2\pi r}$$

$$=\frac{S}{2\pi}\left(\frac{1}{r}\right)-r$$

$$V=\pi r^2h=\pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right)-r\right]=\frac{Sr}{2}=-\pi r^3$$

$$\frac{dV}{dr}=\frac{S}{2}-3\pi r^2,\frac{d^2V}{dr^2}=-6\pi r$$

$$\frac{dV}{dr}=0\Rightarrow\frac{S}{2}=3\pi r^2\Rightarrow r^2=\frac{S}{6\pi}$$

$$r^2=\frac{S}{6\pi},\frac{d^2V}{dr^2}=-6\pi\left(\sqrt{\frac{S}{6\pi}}\right)<0$$

Volume is maximum when $r^2=\frac{S}{6\pi}$

When $r^2 = \dfrac{S}{6\pi}$, then $h = \dfrac{6\pi r^2}{2\pi}\left(\dfrac{1}{r}\right) - r = 3r - r = 2r$

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area?

Solution:

$$V = \pi r^2 h = 100$$

$$\therefore h = \frac{100}{\pi r^2}$$

$$S = 2\pi^2 + 2\pi r h = 2\pi r^2 + \frac{200}{r}$$

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

When $r = \left(\dfrac{50}{\pi}\right)^{\frac{1}{3}}, \dfrac{d^2S}{dr^2} > 0$

The surface area is the minimum when the radius of the cylinder is $\left(\dfrac{50}{\pi}\right)^{\frac{1}{3}} cm$

$$r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, h = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm$$

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the circle is minimum?

Solution:

Piece of length l be cut from wire to make square

Other piece of wire to be made into circle is $(28-l)m$

Side of square = $\frac{l}{4}$

$$2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28-l)$$

$$A = \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28-l)\right]^2$$

$$= \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$

$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28-l)$$

$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

$$\frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{l}{2\pi}(28-l) = 0$$

$$\Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0$$

$$(\pi+4)l - 112 = 0$$

$$\Rightarrow l = \frac{112}{\pi+4}$$

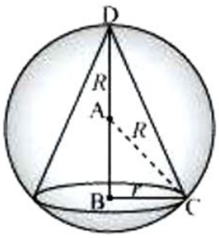
When $l = \frac{112}{\pi+4}, \frac{d^2A}{dl^2} > 0$

The area is minimum when $l = \frac{112}{\pi+4}$

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere

Solution:

Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.



$$V=\frac{1}{3}\pi r^2 h$$

$$h=R+AB=R+\sqrt{R^2-r^2}$$

$$\therefore V=\frac{1}{3}\pi r^2\left(R+\sqrt{R^2-r^2}\right)$$

$$=\frac{1}{3}\pi r^2 R+\frac{1}{3}\pi r^2\sqrt{R^2+r^2}$$

$$\therefore \frac{dV}{dr}=\frac{2}{3}\pi rR+\frac{2}{3}\pi r\sqrt{R^2-r^2}+\frac{1}{3}\pi r^2\frac{(-2r)}{2\sqrt{R^2-r^2}}$$

$$=\frac{2}{3}\pi rR+\frac{2}{3}\pi r\sqrt{R^2-r^2}-\frac{1}{3}\pi\frac{r^3}{\sqrt{R^2-r^2}}$$

$$=\frac{2}{3}\pi rR+\frac{2\pi r\left(R^2-r^2\right)-\pi r^3}{3\sqrt{R^2-r^2}}$$

$$=\frac{2}{3}\pi rR+\frac{2\pi rR^2-3\pi r^3}{3\sqrt{R^2-r^2}}$$

$$\frac{d^2V}{dr^2}=\frac{2\pi R}{3}+\frac{3\sqrt{R^2-r^2}\left(2\pi R^2-9\pi r^2\right)-\left(2\pi rR^2-3\pi r^3\right)\frac{(-2r)}{6\sqrt{R^2-r^2}}}{9\left(R^2-r^2\right)}$$

$$= \frac{2}{3}\pi rR + \frac{9\left(R^2-r^2\right)\left(2\pi R^2-9\pi r^2\right)+2\pi r^2R^2+3\pi r^4}{27\left(R^2-r^2\right)^{\frac{3}{2}}}$$

$$\frac{dV}{dr}=0 \Rightarrow \pi\frac{2}{3}rR=\frac{3\pi r^3-2\pi R^2}{3\sqrt{R^2-r^2}}$$

$$\Rightarrow 2R=\frac{3\pi r^3-2\pi R^2}{\sqrt{R^2-r^2}}=2R\sqrt{R^2-r^2}=3r^2-2R^2$$

$$\Rightarrow 4R^2\left(R^2-r^2\right)=\left(3r^2-2R^2\right)^2$$

$$\Rightarrow 4R^4-4R^2r^2=9r^4+4R^4-12r^2R^2$$

$$\Rightarrow 9r^4=8R^2r^2$$

$$\Rightarrow r^2=\frac{8}{9}R^2$$

$$r^2=\frac{8}{9}R^2, \frac{d^2V}{dr^2}<0$$

Volume of the cone is the maximum when $r^2=\frac{8}{9}R^2$

$$r^2=\frac{8}{9}R^2, h=R+\sqrt{R^2-\frac{8}{9}R^2}=R+\sqrt{\frac{1}{9}R^2}=R+\frac{R}{3}=\frac{4}{3}R$$

$$=\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$$

$$=\frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$=\frac{8}{27}\times\left(volume\ of\ the\ sphere\right)$$

24. Show that he right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base

Solution:

$$V=\frac{1}{3\pi}\pi r^2 h \Rightarrow h=\frac{3V}{r^2}$$

$$S=\pi rl$$

$$=\pi r\sqrt{r^2+h^2}$$

$$=\pi r\sqrt{r^2+\frac{9V^2}{\pi^2 r^4}}\pi=\frac{r\sqrt{9^2 r^6+V^2}}{\pi r^2}$$

$$=\frac{1}{r}\sqrt{\pi^2 r^6+9V^2}$$

$$\therefore \frac{dS}{dr}=\frac{r.\frac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6+9V^2}}-\sqrt{\pi^2 r^6+9V^2}}{r^2}$$

$$=\frac{3\pi^2 r^6-\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

$$=\frac{2\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

$$\frac{dS}{dr}=0 \Rightarrow 2\pi^2 r^6=9V^2 \Rightarrow r^6=\frac{9V^2}{2\pi^2}$$

$$r^6=\frac{9V^2}{2\pi^2},\frac{d^2S}{dr^2}>0$$

Surface area of the cone is least when $r^6=\frac{9V^2}{2\pi^2}$

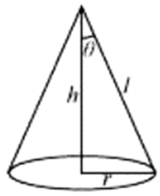
$$r^6=\frac{9V^2}{2\pi^2},h=\frac{3V}{\pi r^2}=\frac{3V}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}}=\frac{3}{\pi r^2}\frac{\sqrt{2}\pi r^3}{3}=\sqrt{2}r$$

25. Show that the semi – vertical angle of the cone of the maximum volume and of the given slant height is $\tan^{-1}\sqrt{2}$

Solution:

Let θ be semi – vertical angle of cone

$\theta \in \left[0, \frac{\pi}{2}\right]$



$r = l\sin\theta \; and \; h = l\cos\theta$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(l^2\sin^2\theta\right)\left(l\cos\theta\right)$

$= \frac{1}{3}\pi l^3\sin^2\theta\cos\theta$

$\therefore \frac{dV}{d\theta} = \frac{l^3\pi}{3}\left[\sin^2\theta\left(-\sin\theta\right) + \cos\theta\left(2\sin\theta\cos\theta\right)\right]$

$= \frac{l^3\pi}{3}\left[-\sin^3 + 2\sin\theta\cos^2\theta\right]$

$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta\right]$

$= \frac{l^3\pi}{3}\left[2\cos^2\theta - 7\sin^2\theta\cos\theta\right]$

$\frac{dV}{d\theta} = 0$

$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta$

$\Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}$

$\Rightarrow \theta = \tan^{-1}\sqrt{2}$

When $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2 \; or \; \sin^2\theta = 2\cos^2\theta$

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[2\cos^3\theta - 14\cos^3\theta\right] = -4\pi l^3\cos^3\theta < 0 \;\; for \; \theta \in \left[0, \frac{\pi}{2}\right]$$

Volume is the maximum when $\theta = \tan^{-1}\sqrt{2}$

26. The point on the curve $x^2 = 2y$ which is nearest to the point $(0,5)$ is

$(A)\left(2\sqrt{2},4\right),(B)\left(2\sqrt{2},0\right),(C)=(0,0),(D)=(2,2)$

Solution:

Position of point is $\left(x, \frac{x^2}{2}\right)$

Distance $d(x)$ between points $\left(x, \frac{x^2}{2}\right)$ *and* $(0,5)$ is

$$d(x) = \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 5\right)^2} = \sqrt{x^2 + \frac{x^4}{4} + 25 - 5x^2} = \sqrt{\frac{x^4}{4} - 4x^2 + 25}$$

$$\therefore d'(x) = \frac{\left(x^3 - 8x\right)}{2\sqrt{\frac{x^4}{4} - 4x^2 + 25}} = \frac{\left(x^3 - 8x\right)}{\sqrt{x^4 - 16x^2 + 100}}$$

$$d'(x) = 0 \Rightarrow x^3 - 8x = 0$$

$$\Rightarrow x\left(x^2 - 8\right) = 0$$

$$\Rightarrow x = 0, \pm 2\sqrt{2}$$

$$d''(x) = \frac{\sqrt{x^2 - 16x^2 + 100}\,(3x^2 - 8) - \left(x^3 - 8x\right)\dfrac{4x^3 - 32x}{2\sqrt{x^4 - 16x^2 + 100}}}{\left(x^2 - 16x^2 + 100\right)}$$

$$= \frac{\left(x^4 - 16x^2 + 100\right)\left(3x^2 - 8\right) - 2\left(x^3 - 8x\right)\left(x^3 - 8x\right)}{\left(x^2 - 16x^2 + 100\right)^{\frac{3}{2}}}$$

$$=\frac{\left(x^4-16x^2+100\right)\left(3x^2-8\right)-2\left(x^3-8x\right)^2}{\left(x^2-16x^2+100\right)^{\frac{3}{2}}}$$

$$x=0, \textit{ then } d''(x)=\frac{36(-8)}{6^3}<0$$

$$x=\pm 2\sqrt{2}, d''(x)>0$$

d(x) is the minimum at $x=\pm 2\sqrt{2}$

$$x=\pm 2\sqrt{2}, y=\frac{\left(2\sqrt{2}\right)^2}{2}=4$$

The correct answer is A

27. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

$(A)0, (B)1, (C)3, (D)\frac{1}{3}$

Solution:

$$f(x)=\frac{1-x+x^2}{1+x+x^2}$$

$$\therefore f'(x)=\frac{\left(1-x+x^2\right)(-1+2x)-\left(1-x+x^2\right)(1+2x)}{\left(1+x+x^2\right)^2}$$

$$=\frac{-1+2x-x+2x^2-x^2+2x^2-1-2x+x+2x^2-x^2-2x^3}{\left(1+x+x^2\right)^2}$$

$$=\frac{2x^2-2}{\left(1+x+x^2\right)^2}=\frac{2\left(x^2-1\right)}{\left(1+x+x^2\right)^2}$$

$$\therefore f'(x)=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$$

$$f''(x)=\frac{2\left[\left(1+x+x^2\right)(2x)-\left(x^2-1\right)(2)\left(1+x+x^2\right)(1+2x)\right]}{\left(1+x+x^2\right)^4}$$

$$= \frac{4(1+x+x^2)\left[(1+x+x^2)x-(x^2-1)(1+2x)\right]}{(1+x+x^2)^4}$$

$$= 4\frac{\left[x+x^2+x^3-x^2-2x^3+1+2x\right]}{(1+x+x^2)^3}$$

$$= \frac{4(1+x+x^2)^3}{(1+x+x^2)^3}$$

$$f''(1) = \frac{4(1+3-1)}{(1+1+1)^3} = \frac{4(3)}{(3)^3} = \frac{4}{9} > 0$$

$$f''(-1) = \frac{4(1+3-1)}{(1+1+1)^3} = 4(-1) = 4 < 0$$

f is the minimum at x = 1 and the minimum value is given by

$$f(1) = \frac{1-1+1}{1+1+1} = \frac{1}{3}$$

The correct answer is D

28. The maximum value of $\left[x(x+1)+1\right]^{\frac{1}{3}}, 0 \le x \le 1$ is

$(A)\left(\frac{1}{3}\right)^{\frac{1}{3}}, (B)\frac{1}{2}, (C)1, (D)0$

Solution:

$$f(x) = \left[x(x+1)+1\right]^{\frac{1}{3}}$$

$$\therefore f'(x) = \frac{2x-1}{3\left[x(x+1)+1\right]^{\frac{2}{3}}}$$

$$f'(x) = 0 \Rightarrow x = \frac{1}{2}$$

$$f(0) = \left[0(0-1)+1\right]^{\frac{1}{3}} = 1$$

$$f(2)=[1(1-1)+1]^{\frac{1}{3}}=1$$

$$f\left(\frac{1}{2}\right)=\left[\frac{1}{2}\left(\frac{-1}{2}\right)+1\right]^{\frac{1}{3}}=\left(\frac{3}{4}\right)^{\frac{1}{3}}$$

Maximum value of f in $[0,1]$ is 1

The correct answer is C