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Chapter: 7. Integrals

Exercise: Miscellaneous.

1. Find the integral of the function $\int \frac{1}{x-x^3}dx$

Solution: The given integrand is $\frac{1}{x-x^3}$

The given integrand can be written as

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}$$

$$=\frac{1}{x(1-x)(1+x)}$$

Use the concept of partial fractions, suppose that $\frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x}$

To get the values of constants, cancel out the common denominator.

Hence, $1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$

Plug in $x=1$

$$1=B(2)\Rightarrow B=\frac{1}{2}$$

Plug in $x=-1$

$$1=-C(2)\Rightarrow C=-\frac{1}{2}$$

Plug in $x=0$

$$A=1$$

Hence,

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

Integrate both sides

$$\int \frac{1}{x(1-x)(1+x)}dx = \int \frac{1}{x}dx + \int \frac{1}{2(1-x)}dx - \int \frac{1}{2(1+x)}dx$$

$$\int \frac{1}{x-x^3}dx = \log|x| - \frac{1}{2}\log|1-x| - \frac{1}{2}\log|1+x| + C$$

$$= \frac{1}{2}\log\left|x^2\right| - \frac{1}{2}\log\left|1-x^2\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$$

Therefore, $\int \frac{1}{x-x^3}dx = \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$

2. Find the integral of the function $\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}$

Solution:

$$\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}} = \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+b)-(x-b)} = \frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow \int \frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}dx = \frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$= \frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right] = \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + C$$

3. $\frac{1}{x\sqrt{ax-x^2}}$ $\left[\text{Hint } x = \frac{a}{t}\right]$

Solution:

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t} \Rightarrow dx=-\frac{a}{t^2}dh$

$$\Rightarrow \int \frac{1}{x\sqrt{ax-x^2}}dx=\int \frac{1}{\frac{a}{t}\sqrt{a.\frac{a}{t}-\left(\frac{a}{t}\right)^2}}-\left(-\frac{a}{t^2}dt\right)$$

$$=-\int \frac{1}{at}\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt=-\frac{1}{a}\int \frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$=-\frac{1}{a}\int \frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$=-\frac{2}{a}\left[\frac{\sqrt{a-x}}{\sqrt{x}}\right]+C$$

$$=-\frac{2}{a}\left[\sqrt{\frac{a-x}{x}}\right]+C$$

4. $\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$

Solution:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we get

$$\frac{x^{-3}}{x^2.x^{-3}\left(x^4+1\right)^{\frac{3}{4}}}=\frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2-x^{-3}}$$

$$\frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5.\left(x^4\right)^{-\frac{3}{4}}}=\frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{\frac{3}{4}}$$

$$=\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{\frac{3}{4}}$$

Let $\frac{1}{x^4}=t \Rightarrow -\frac{4}{x^5}dx=dt \Rightarrow \frac{1}{x^5}dx=-\frac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx=\int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}=-\frac{1}{4}\int (1+t)^{-\frac{3}{4}}dt$$

$$=-\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C=-\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C}{\frac{1}{4}}$$

$$-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

5. $\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\left[\text{Hint}: \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}\ Put\ x=t^6\right]$

Solution:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Let $x=t^6 \Rightarrow dx=6t^5dt$

$$\therefore \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=\int \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx=\int \frac{6t^5}{t^2(1+t)}dt$$

$$=6\int \frac{t^3}{(1+t)}dt$$

On dividing, we get

$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = 6\int \left\{\left(t^2-t+1\right)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1-x^{\frac{1}{6}}\right)+C$$

6. $\dfrac{5x}{(x+1)\left(x^2+9\right)}$

Solution:

Consider, $\dfrac{5x}{(x+1)\left(x^2+9\right)}=\dfrac{A}{(x+1)}+\dfrac{Bx+C}{\left(x^2+9\right)}$.........(1)

$$\Rightarrow 5x = A\left(x^2+9\right)+\left(Bx+C\right)\left(x+1\right)$$

$$\Rightarrow 5x = Ax^2+9A+Bx^2+Bx+Cx+C$$

Equating the coefficients of x^2, x and constant term, we get

$A+B=0$

$B+C=5$

$9A+C=0$

On solving these equations, we get

$$A=-\frac{1}{2}, B=\frac{1}{2}\ and\ C=\frac{9}{2}$$

From equation (1), we get

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int \frac{5x}{(x+1)(x+9)}dx = \int\left\{\frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)}\right\}dx$$

$$= \frac{1}{2}\log|x+1| + \frac{1}{2}\int\frac{x}{x^2+9}dx + \frac{9}{2}\int\frac{1}{x^2+9}dx = -\frac{1}{2}\log|x+1| + \frac{1}{4}\int\frac{2x}{x^2+9}dx + \frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$= \frac{1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{9}{2}.\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$= \frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$$

7. $\frac{\sin x}{\sin(x-a)}$

Solution:

$$\frac{\sin x}{\sin(x-a)}$$

Put, $x-a=t \Rightarrow dx=dt$

$$\int \frac{\sin x}{\sin(x-a)}dx = \int\frac{\sin(t+a)}{\sin t}dt$$

$$= \int\frac{\sin t\cos a + \cos t\sin a}{\sin t}dt = \int(\cos a + \cot t\sin a)dt$$

$$= t\cos a + \sin a\log|\sin t| + C_1$$

$$= (x-a)\cos a + \sin a\log|\sin(x-a)| + C_1$$

$$= x\cos a + \sin a\log|\sin(x-a)| - a\cos a + C_1$$

$$= \sin a\log|\sin(x-a)| + x\cos a + C$$

8. $\dfrac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$

Solution:

Let $\dfrac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}=\dfrac{e^{4\log x}\left(e^{\log x}-1\right)}{e^{2\log x}\left(e^{\log x}-1\right)}$

$=e^{2\log x}$

$=e^{\log x^2}$

$=x^2$

$\therefore \int \dfrac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}dx=\int x^2 dx=\dfrac{x^3}{3}+C$

9. $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

Solution:

$\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

Put, $\sin x=t \Rightarrow \cos x\, dx=dt$

$\Rightarrow \int \dfrac{\cos x}{\sqrt{4-\sin^2 x}}dx=\int \dfrac{dt}{\sqrt{(2)^2-(t)^2}}$

$=\sin^{-1}\left(\dfrac{t}{2}\right)+C$

$=\sin^{-1}\left(\dfrac{\sin x}{2}\right)+C$

10. $\dfrac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

Solution:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}=\frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{\sin^2 x\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$$

$$=\frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{\left(\sin^2 x-\sin^2 x\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\cos^2 x\right)}$$

$$=\frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{\sin^2 x\left(1-\cos^2 x\right)+\cos^2 x\left(1-\sin^2 x\right)}$$

$$=\frac{-\left(\sin^4 x-\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$

$$=-\cos 2x$$

$$\therefore \int\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=\int -\cos 2x\,dx=-\frac{\sin 2x}{2}+C$$

11. $$\frac{1}{\cos(x+a)\cos(x+b)}$$

Solution:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we get

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a).\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int \frac{1}{\cos(x+a)\cos(x+b)}dx = \frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$

$$= \frac{1}{\sin(a-b)}\left[-\log\left|\cos(x-a)\right|+\log\left|\cos(x+b)\right|\right]+C$$

$$= \frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

12. $$\frac{x^3}{\sqrt{1-x^8}}$$

Solution:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Put $x^4 = t \Rightarrow 4x^3dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{4}\sin^{-1} t + C$$

$$= \frac{1}{4}\sin^{-1}\left(x^4\right)+C$$

13. $$\frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}$$

Solution:

Put, $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow \int \frac{e^x}{\left(1+e^x\right)\left(2+e^x\right)}dx = \int \frac{dt}{(t+1)(t+2)}$$

$$= \int\left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right]dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

14. $\dfrac{1}{\left(x^2+1\right)\left(x^2+4\right)}$

Solution:

$$\therefore \frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{Ax+B}{\left(x^2+1\right)} + \frac{Cx+D}{\left(x^2+4\right)}$$

$$\Rightarrow 1 = (Ax+B)\left(x^2+4\right) + (Cx+D)\left(x^2+1\right)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equation the coefficients of x^3, x^2, x and constant term, we get

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$

On solving these equations, we get

$A = 0, B = \frac{1}{3}, C = 0$ and $D = -\frac{1}{3}$

From equation (1), we get

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{1}{3\left(x^2+1\right)} - \frac{1}{3\left(x^2+4\right)}$$

$$\int \frac{1}{\left(x^2+1\right)\left(x^2+4\right)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

15. $\cos^3 x e^{\log \sin x}$

Solution:

$$\cos^3 x e^{\log \sin x} = \cos^3 x \times \sin x$$

Let $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\int \cos^3 x\, e^{\log \sin x} dx = \int \cos^3 x \sin x\, dx$$

$$= -\int t.dt$$

$$= -\frac{t^4}{4} + C$$

$$-\frac{\cos^4 x}{4} + C$$

16. $e^{3\log x} = \left(x^4+1\right)^{-1}$

Solution:

$$e^{3\log x} = \left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

Let $x^4+1 = t \Rightarrow 4x^3 dx = dt$

$$\Rightarrow \int e^{3\log x} = \left(x^4+1\right)^{-1} dx = \int \frac{x^3}{\left(x^4+1\right)} dx$$

$$= \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log\left|x^4+1\right| + C$$

$$= \frac{1}{4}\log\left|x^4+1\right| + C$$

17. $f'(ax+b)\left[f(ax+b)\right]^n$

Solution:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Put, $f(ax+b)=t \Rightarrow a f'(ax+b)dx=dt$

$$\Rightarrow f'(ax+b)\left[f(ax+b)\right]^n dx=\frac{1}{a}\int t^n dt$$

$$=\frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]=\frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1}+C$$

18. $\dfrac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$

Solution:

$$\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}=\frac{1}{\sqrt{\sin^3 x(\sin x\cos\alpha+\cos x\sin\alpha)}}$$

$$=\frac{1}{\sqrt{\sin^4 x\cos\alpha+\sin^3 x\cos x\sin\alpha}}$$

$$=\frac{1}{\sin^2 x\sqrt{\cos\alpha+\text{cotxsin}\,\alpha}}=\frac{\cos ec^2 x}{\sqrt{\cos\alpha+\cot x\sin\alpha}}$$

Put, $\cos\alpha+\cot x\sin\alpha=\text{t}\Rightarrow-\text{cosec}^2 x\sin\alpha\,\text{dx}=\text{dt}$

$$\therefore\int\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}dx=\int\frac{\cos ec^2 x}{\sqrt{\cos\alpha+\text{cotxsin}\,\alpha}}dx$$

$$=\frac{-1}{\sin\alpha}\int\frac{dt}{\sqrt{t}}$$

$$=\frac{-1}{\sin\alpha}\left[2\sqrt{t}\right]+C$$

$$=\frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha+\cot x\sin\alpha}\right]+C$$

$$=\frac{-2}{\sin\alpha}\sqrt{\cos\alpha+\frac{\cos x\sin\alpha}{\sin x}}+C$$

$$=\frac{-2}{\sin\alpha}\sqrt{\frac{\sin x\cos\alpha+\cos x\sin\alpha}{\sin x}}+C=\frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}}+C$$

19. $\dfrac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x\in[0,1]$

Solution:

Let $I=\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}$

As we know that, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$

$$\Rightarrow I=\int\frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$=\frac{2}{\pi}\int\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

$$=\frac{2}{\pi}.\frac{\pi}{2}\int 1.dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx$$

$$=x-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx..........(1)$$

Let $I_1=2\int\cos^{-1}t.dt$

$$=2\left[\cos^{-1}t.\frac{t^2}{2}-\int\frac{-1}{\sqrt{1-t^2}}.\frac{t^2}{2}dt\right]$$

$$=t^2\cos^{-1}t+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\sqrt{1-t^2}dt+\int\frac{1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\frac{1}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t$$

From equation (1), we get

$$I=x-\frac{4}{\pi}\left[t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]$$

$$=x-\frac{4}{\pi}\left[x\cos^{-1}\sqrt{x}-\frac{\sqrt{x}}{2}\sqrt{1-x}+\frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-\frac{4}{\pi}\left[x\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)-\frac{\sqrt{x-x^2}}{2}+\frac{\pi}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-2x+\frac{4x}{\pi}\sin^{-1}\sqrt{x}+\frac{2}{\pi}\sqrt{x-x^2}-\frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$= -x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

20. $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Solution:

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Put $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta d\theta$

$$I = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta = -\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta d\theta$$

$$= -\int \tan\frac{\theta}{2}.2\sin\theta\cos\theta d\theta = -2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta d\theta$$

$$= -4\int\sin^2\frac{\theta}{2}\cos\theta d\theta$$

$$= -4\int\sin^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$= -4\int 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}d\theta$$

$$= -8\int\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}d\theta + 4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\sin^2\frac{\theta}{2}d\theta + 4\int\sin^2\frac{\theta}{2}d\theta$$

$$= -2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta + 4\int\frac{1-\cos\theta}{2}d\theta$$

$$= -2\int\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] + 4\left[\frac{\theta}{2} - \frac{\sin\theta}{2}\right] + C$$

$$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin\theta + C$$

$$=\theta+\frac{\sin 2\theta}{2}+2\sin\theta+C$$

$$=\theta+\frac{2\sin\theta\cos\theta}{2}+2\sin\theta+C$$

$$=\theta+\sqrt{1-\cos^2\theta}.\cos\theta-2\sqrt{1-\cos^2\theta}+C$$

$$=\cos^{-1}\sqrt{x}+\sqrt{1-x}.\sqrt{x}-2\sqrt{1-x}+C$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+C$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$$

21. $\dfrac{2+\sin 2x}{1+\cos 2x}e^x$

Solution:

$$I=\int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$$

$$=\int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$$

$$=\int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$$

$$=\int\left(\sec^2 x+\tan x\right)e^x$$

Let $f(x)=\tan x\Rightarrow f'(x)=\sec^2 x$

$$\therefore I=\int f(x)+f'(x)e^x dx$$

$$=e^x f(x)+C$$

$$=e^x\tan x+C$$

22. $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$

Solution:

Let $\dfrac{x^2+x+1}{(x+1)^2(x+2)}=\dfrac{A}{(x+1)}+\dfrac{B}{(x+1)^2}+\dfrac{C}{(x+2)}$..........(1)

$$\Rightarrow x^2+x+1=A\left(x^2+3x+2\right)+B(x+2)+C\left(x^2+2x+1\right)$$

$$\Rightarrow x^2+x+1=(A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

Equating the coefficients of x^2, x and constant term, we get

$$A+C=1$$
$$3A+B+2C=1$$
$$2A+2B+C=1$$

On solving these equations, we get

From equation (1), we get

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}=-2\int\frac{1}{(x+1)}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

23. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Solution:

$$\mathrm{I}=\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$$

Let $x=\cos\theta\Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(\sin\theta d\theta)$$

$$=-\int\tan^{-1}\sqrt{\frac{3\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta=-\int\tan^{-1}\tan\frac{\theta}{2}.\sin\theta d\theta$$

$$=-\frac{1}{2}\int \theta.\sin\theta d\theta=-\frac{1}{2}\left[\theta.(-\cos\theta)-\int 1.(-\cos\theta)d\theta\right]$$

$$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]$$

$$=+\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x.x-\frac{1}{2}\sqrt{1-x^2}+C=\frac{x}{2}\cos^{-1}-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

24. $$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$$

Solution:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

Let $1+\frac{1}{x^2}=t\Rightarrow\frac{-2}{x^3}dx=dt$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$= -\frac{1}{2}\int \sqrt{t}\log t dt = -\frac{1}{2}\int t^{\frac{1}{2}}.\log t dt$$

Using integration by parts, we get

$$I = -\frac{1}{2}\left[\log t.\int t^{\frac{1}{2}}dt - \left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$= -\frac{1}{2}\left[\log t\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{4}{9}t^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}}$$

$$= -\frac{1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C$$

25. $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Solution:

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx = \int_{\frac{\pi}{2}}^{\pi}\left(\frac{\cos ec^2\frac{x}{2}}{2} - \cot\frac{x}{2}\right)dx$$

Let $f(x) = -\cot\frac{x}{2}$

$$\Rightarrow f'(x) = -\left(-\frac{1}{2}\cos ec^2\frac{x}{2}\right) = \frac{1}{2}\cos ec^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(f(x) + f'(x)\right)dx$$

$$= \left[e^x . f(x)dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x . \cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x \times \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot\frac{\pi}{4}\right]$$

$$= -\left[e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1\right]$$

$$= e^{\frac{\pi}{2}}$$

26. $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}dx$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\left(\frac{\sin x \cos x}{\cos^4 x}\right)}{\left(\frac{\cos^4 x + \sin^4 x}{\cos^4 x}\right)}dx$$

$$\Rightarrow I = \int_{\pi}^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x}dx$$

Put, $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x dx = dt$

When $x=0, t=0$ and when $x=\frac{\pi}{4}, t=1$

$$\therefore I=\frac{1}{2}\int_0^4 \frac{dt}{1+t^2}=\frac{1}{2}\left[\tan^{-1} t\right]_0^1$$

$$=\frac{1}{2}\left[\tan^{-1} 1-\tan^{-1} 0\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}\right]$$

$$=\frac{\pi}{8}$$

27. $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\,dx}{\cos^2 x+4\sin^2 x}$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}} \frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4}{4-3\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}} 1.dx+\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x-3}dx$$

$$\Rightarrow I=\frac{-1}{3}[x]_0^{\frac{\pi}{2}}+\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\left(1+\tan^2 x\right)-3}dx$$

$$\Rightarrow I=-\frac{\pi}{6}+\frac{2}{3}\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx \;\ldots\ldots\ldots(1)$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx$

Put, $2\tan x = t \Rightarrow 2\sec^2 x dx = dt$

When x=0, t = 0 and when $x=\frac{\pi}{2}, t=\infty$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx = \int_0^{\infty}\frac{dt}{1+t^2}$$

$$=\left[\tan^{-1} t\right]_0^{\infty}$$

$$=\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]$$

$$=\frac{\pi}{2}$$

Therefore , from (1)we, get

$$I=-\frac{\pi}{6}+\frac{2}{3}\left[\frac{\pi}{2}\right]=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

28. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$

Solution:

Consider, $I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{-(-\sin 2x)}}dx \Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{-(-1+1-2\sin\cos x)}}dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{1-(\sin^2 x\cos^2 x-2\sin x\cos x)}}dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}$$

Let $(\sin x-\cos x)=t-(\sin x+\cos x)dx=dt$

When $x=\frac{\pi}{6},t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when $x=\frac{\pi}{3},t=\left(\frac{\sqrt{3}-1}{2}\right)$

$$I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}\frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I=\int_{-\left(\frac{1-\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}}\frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}}=\frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function

$$\int_{-a}^{a}f(x)dx=2\int_0^a f(x)dx$$

We know that if f(x) is an even function, then

$$\Rightarrow I=2\int_0^{\frac{\sqrt{3}-1}{2}}\frac{dt}{\sqrt{1-t^2}}$$

$$=\left[2\sin^{-1}t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

29. $\int_0^1\frac{dx}{\sqrt{1+x}-\sqrt{x}}$

Solution:

Consider, $I=\int_0^1\frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$I=\int_0^1\frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)}\times\frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)}dx$$

$$=\int_0^1 \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{1+x-x}dx$$

$$=\int_0^1 \sqrt{1+x}dx+\int_0^1 \sqrt{x}dx$$

$$=\left[\frac{2}{3}(1+x)^{\frac{2}{3}}\right]_0^1\left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{3}\left[(2)^{\frac{2}{3}}-1\right]+\frac{2}{3}[1]$$

$$=\frac{2}{3}(2)^{\frac{2}{3}}=\frac{2.2\sqrt{2}}{3}$$

$$=\frac{4\sqrt{2}}{3}$$

30. $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

Solution:

Consider, $I=\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

Put, $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)dx=dt$

Where $x=0, t=-1$ and when $x=\frac{\pi}{4}, t=0$

$$\Rightarrow (\sin x-\cos x)^2=t^2$$

$$\Rightarrow \sin^2+\cos^2-2\sin x\cos x=t^2$$

$$\Rightarrow 1-\sin 2x=t^2$$

$$\Rightarrow \sin 2x=1-t^2$$

$$\therefore I=\int_{-1}^{0} \frac{dt}{9+16\left(1-t^2\right)}$$

$$= \int_{-1}^{0} \frac{dt}{9+16-16t^2}$$

$$= \int_{-1}^{0} \frac{dt}{25-16t^2} = \int_{-1}^{0} \frac{dt}{(5)^2-(4t)^2}$$

$$= \frac{1}{4}\left[\frac{1}{2(5)} \log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log(1) - \log\left|\frac{1}{9}\right|\right]$$

$$= \frac{1}{40}\log 9$$

31. $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\,dx$

Put, $\sin x = t \Rightarrow \cos x dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$\Rightarrow I = \int_0^1 t \tan^{-1}(t)\,dt.........(1)$

Consider, $\int t \tan^{-1} t dt = \tan^{-1} t. \int t dt - \int \left\{\frac{d}{dt}(\tan^{-1} t) t dt\right\} dt$

$$= \tan^{-1} t. \frac{t^2}{2} - \int \frac{1}{1+t^2}.\frac{t^2}{2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}\int \frac{t^2+1-1}{1+t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}\int 1.dt + \frac{1}{2}\int \frac{1}{1+t^2} dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}.t+\frac{1}{2}\int\frac{1}{1+t^2}dt$$

$$=\frac{t^2\tan^{-1}t}{2}-\frac{1}{2}.t+\frac{1}{2}\tan^{-1}t$$

$$\Rightarrow\int_0^1 t.\tan^{-1}t dt=\left[\frac{t^2\tan^{-1}t}{2}-\frac{t}{2}+\frac{1}{2}\tan^{-1}t\right]_0^1$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-1+\frac{\pi}{4}\right]=\frac{1}{2}\left[\frac{\pi}{2}-1\right]=\frac{\pi}{4}-\frac{1}{2}$$

From equation (1), we get

$$I=2\left[\frac{\pi}{2}-\frac{1}{2}\right]=\frac{\pi}{2}-1$$

32. $\int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx$

Solution:

Let $\int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx........(1)$

$$I=\int_0^\pi\left\{\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}\right\}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^\pi\left\{\frac{-(\pi-x)\tan x}{-(\sec x+\tan x)}\right\}dx$$

$$\Rightarrow I=\int_0^\pi\frac{(\pi-x)\tan x}{(\sec x+\tan x)}dx..........(2)$$

Adding (1) and (2), we get

$$2I=\int_0^\pi\frac{\pi\tan x}{\sec x+\tan x}dx\Rightarrow 2I=\pi\int_0^\pi\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx$$

$$\Rightarrow 2I=\pi\int_0^\pi \frac{\sin x+1-1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^\pi 1.dx=\pi\int_0^\pi \frac{1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi[x]_0^\pi-\pi\int_0^\pi \frac{1-\sin x}{\cos^2 x}dx$$

$$\Rightarrow 2I=\pi^2-\pi\int_0^\pi\left(\sec^2 x-\tan x\sec x\right)dx$$

$$\Rightarrow 2I=\pi^2-\pi[\tan x-\sec x]_0^\pi$$

$$\Rightarrow 2I=\pi^2-\pi[\tan x-\sec x-\tan 0+\sec 0]$$

$$\Rightarrow 2I=\pi^2-\pi[0-(-1)-0+1]$$

$$\Rightarrow 2I=\pi^2-2\pi$$

$$\Rightarrow 2I=\pi(\pi-2)$$

$$\Rightarrow I=\frac{\pi}{2}(\pi-2)$$

33. $\int_1^4[|x-1|+|x-2|+|x-3|]dx$

Solution:

Consider, $I=\int_1^4[|x-1|+|x-2|+|x-3|]dx$

$$\Rightarrow I=\int_1^4|x-1|dx+\int_1^4|x-2|dx+\int_1^4|x-3|dx$$

$I=I_1+I_2+I_3.........(1)$

Where, $I_1=\int_1^4|x-1|dx$

$(x-1)\geq 0 \; for\, 1\leq x\leq 4$

$$\therefore I_1=\int_1^4 (x-1)\,dx$$

$$\Rightarrow I_1=\left[\frac{x^2}{x}-x\right]_1^4$$

$$\Rightarrow I_1=\left[8-4-\frac{1}{2}+1\right]=\frac{9}{2}.........(2)$$

$$I_2=\int_1^4 |x-2|\,dx$$

$x-2\geq 0$ for $2\leq x\leq 4$ and $x-2\leq 0$ for $1\leq x\leq 2$

$$\therefore I_2=\int_1^2 (2-x)\,dx+\int_2^4 (x-2)\,dx$$

$$\Rightarrow I_2=\left[2x-\frac{x^2}{2}\right]_1^2+\left[\frac{x^2}{2}-2x\right]_2^4\Rightarrow I_2=\left[4-2-2+\frac{1}{2}\right]+[8-8-2+4]$$

$$\Rightarrow I_2=\frac{1}{2}+2=\frac{5}{2}..........(3)$$

$$I_3=\int_1^4 |x-3|\,dx$$

$x-3\geq 0$ for $3\leq x\leq 4$ and $x-3\leq 0$ for $1\leq x\leq 3$

$$\therefore I_3=\int_1^3 (3-x)\,dx+\int_3^4 (x-3)\,dx$$

$$\Rightarrow I_3=\left[3-\frac{x^2}{2}\right]_1^2+\left[\frac{x^2}{2}-3x\right]_3^4$$

$$\Rightarrow I_3=\left[9-\frac{9}{2}-3+\frac{1}{2}\right]+\left[8-12-\frac{9}{2}+9\right]$$

$$\Rightarrow I_3=[6-4]+\left[\frac{1}{5}\right]=\frac{5}{2}.........(4)$$

From equations (1), (2), (3), and (4), we get

$$I=\frac{9}{2}+\frac{5}{2}+\frac{5}{2}=\frac{19}{2}$$

34. $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Solution:

Consider, $I = \int_1^3 \frac{dx}{x^2(x+1)}$

Let, $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$

$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$

Equating the coefficients of x^2, x an constant term, we get

$A + C = 0$
$A + B = 0$
$B = 1$

On solving these equations, we get

$A = -1, C = 1$, and $B = 1$

$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$

$\Rightarrow I = \int_1^3 \left\{ \frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx = \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$

$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3 = \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$

$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$

$= \log 2 - \log 3 + \frac{2}{3}$

$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$

Hence proved

35. $\int_0^4 xe^x dx = 1$

Solution:

Let $I = \int_0^4 xe^x dx$

Using integration by pars, we get

$I = \int_0^4 xe^x dx - \int_0^1 \left\{\left(\frac{d}{dx}(x)\right)\int e^x dx\right\} dx$

$= \left[xe^x\right]_0^1 - \int_0^1 e^x dx$

$= \left[xe^x\right]_0^1 - \left[e^x\right]_0^1$

$= e - e + 1$

$= 1$

Hence proved

36. $\int_{-1}^1 x^{17} \cos^4 x\, dx = 0$

Solution:

Consider, $I = \int_{-1}^1 x^{17} \cos^4 x\, dx$

Let $f(x) = x^{17} \cos^4 x$

$\Rightarrow f(x) = (-x)^{17} \cos^4 (-x) = -x^{17} \cos^4 x = -f(x)$

$f(x)$ is an odd function

We know that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$$\therefore \int_{-1}^{1} x^{17}\cos^4 x\,dx = 0$$

Hence proved

37. $\int_0^{\frac{\pi}{2}} \sin^3 xdx = \frac{2}{3}$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \sin^3 xdx$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x.\sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x\,dx - \int_0^{\frac{\pi}{2}} \cos^2 x.\sin x\,dx$$

$$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$$

$$1+\frac{1}{3}[-1] = 1-\frac{1}{3} = \frac{2}{3}$$

Hence proved

38. $\int_0^{\frac{\pi}{4}} 2\tan^3 xdx = 1-\log 2$

Solution:

Consider, $I = \int_0^{\frac{\pi}{4}} 2\tan^3 xdx$

$$I = \int_0^{\frac{\pi}{4}} 2\tan^2 x\tan x\,dx = 2\int_0^{\frac{\pi}{4}} (\sec^2 x-1)\tan x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, dx - 2\int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2[\log\cos x]_0^{\frac{\pi}{4}} = 1 + 2\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$$

$$= 1 + 2\left[\log\frac{1}{\sqrt{2}} - \log 1\right] = 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence proved

39. $\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1$

Solution:

Let $\int_0^1 \sin^{-1} x\, dx$

$$\Rightarrow I = \int_0^1 \sin^{-1} x.1.dx$$

Using integration by parts, we get

$$I = \left[\sin^{-1} x.x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} x\, dx$$

$$= \left[\sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} dx$$

Put $1 - x^2 = t \Rightarrow -2xdx = dt$

When $x = 0, t = 1$ and when x= 1 , t = 0

$$I = \left[x\sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t}}$$

$$I = \left[x\sin^{-1} x\right]_0^1 + \frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$= \sin^{-1}(1) + \left[-\sqrt{1}\right]$$

$$=\frac{\pi}{2}-1$$

Hence proved

40. Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum

Solution:

Let $I=\int_0^1 e^{2-3x}dx$

We know that,

$$\int_a^b f(x)dx=(b-c)\lim_{x\to n}\frac{1}{n}\left[f(a)+f(a+h)+......+f(a+(n-1)h)\right]$$

Where, $h=\frac{b-a}{n}$

Here, $a=0, b=1,$ and $f(x)=e^{2-3x}$

$$\Rightarrow h=\frac{1-0}{n}=\frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx=(1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+......+f(0+(n-1)h)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2+e^{2-3x}+......+e^{2-3(n-1)h}\right]=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+......+e^{-3(n-1)h}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)}{1-\left(e^{-3h}\right)}\right\}\right]=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{e^2-\left(1-e^3\right)}{1-e^{-\frac{3}{n}}}\right\}\right]=e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\left\{\frac{1}{e^{-\frac{3}{n}}-1}\right\}\right]$$

$$e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]=\frac{e^2\left(e^{-3}-1\right)}{3}\lim_{n\to 8}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$\frac{-e^2\left(e^{-3}-1\right)}{3}(1) \qquad \left[\lim_{x\to\infty}\frac{x}{e^x-1}\right]$$

$$=\frac{e^{-1}+e^2}{3}$$

$$=\frac{1}{3}\left(e^2-\frac{1}{e}\right)$$

41. $\int \frac{dx}{e^x+e^{-x}}$ is equal to

A) $\tan^{-1}\left(e^x\right)+C$ B) $\tan^{-1}\left(e^{-x}\right)+C$ C) $\log\left(e^x-e^{-x}\right)+C$ D) $\log\left(e^x+e^{-x}\right)+C$

Solution:

Consider, $I=\int \frac{dx}{e^x+e^{-x}}dx=\int \frac{e^x}{e^{2x}+1}dx$

Put, $e^x=t \Rightarrow e^x dx=dt$

$$\therefore I=\int \frac{dx}{1+t^2}$$

$$=\tan^{-1}t+C$$

$$=\tan^{-1}\left(e^x\right)+C$$

Thus, the correct answer is A

42. $\frac{\cos 2x}{(\sin x+\cos x)^2}dx$ is

A) $\frac{-1}{\sin x+\cos x}+C$ B) $\log|\sin x+\cos x|+C$

C) $\log|\sin x-\cos x|+C$ D) $\frac{1}{(\sin x+\cos x)}+C$ is equal to

Solution:

Consider, $I=\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx \;\; I=\int \frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)^2}dx$

$$\int \frac{(\cos x+\sin x)(\cos x-\sin x)}{(\sin x+\cos x)^2}dx=\int \frac{\cos x-\sin x}{\cos x+\sin x}dx$$

Let $\cos x+\sin x=t \Rightarrow (\cos x-\sin x)dx=dt$

$\therefore I=\int \frac{dt}{t}$

$=\log|t|+C$

$=\log|\cos x+\sin x|+C$

Thus, the correct answer is B

43. If $f(a+b-x)=f(x)$, then $\int_a^b x f(x)dx$ is equal to

A) $\frac{a+b}{2}\int_a^b f(b-x)dx$

B) $\frac{a+b}{2}\int_a^b f(b+x)dx$

C) $\frac{b-a}{2}\int_a^b f(x)dx$

D) $\frac{a+b}{2}\int_a^b f(x)dx$

Solution:

Consider, $I=\int_a^b x f(x)dx$.........(1)

$I=\int_a^b (a+b-x) f(a+b-x)dx \qquad \left(\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$

$\Rightarrow I=\int_a^b (a+b-x) f(x)dx$

$\Rightarrow I=(a+b)\int_a^b f(x)dx-I$ (Using (1))

$\Rightarrow I+I=(a+b)\int_a^b f(x)dx$

$$\Rightarrow 2I = (a+b)\int_a^b f(x)\,dx$$

$$\Rightarrow I = \left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx$$

Thus, the correct answer is D

44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A) 1 B) 0 C) -1 D) $\frac{\pi}{4}$

Solution:

Consider, $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$$\Rightarrow I = \int_0^1 \tan^{-1}\left(\frac{x(1-x)}{1+x(1-x)}\right)dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1} x - \tan^{-1}(1-x)\right]dx.........(1)$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(1-1+x)\right]dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx.........(2)$$

Adding (1) and (2) we get

$$\Rightarrow 2I = \int_0^1 \left(\tan^{-1} x - \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x\right)dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Thus, the correct answer is B