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Chapter: 7. Integrals

Exercise: 7.10

1. Find the value of $\int_0^1 \frac{x}{x^2+1}dx$

Solution: Consider the integral $\int_0^1 \frac{x}{x^2+1}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Hence the integral becomes

$$\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}dx$$
$$= \frac{1}{2}\left[\log\left|x^2+1\right|\right]_0^1$$
$$= \frac{1}{2}\log 2$$

Therefore, $\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\log 2$

2. Find the value of $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Solution: Consider the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Suppose that $\sin\phi = t$, so that $\cos\phi d\phi = dt$

Hence the integral becomes

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\left(\cos\phi\, d\phi\right)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\left(1-\sin^2\phi\right)\left(\cos\phi\, d\phi\right)$$

$$= \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$

$$= \int_0^1 \sqrt{t}\left(1+t^4-2t^2\right)dt$$

$$= \int_0^1 t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}} dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - 2\frac{t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154+42-132}{231}$$

$$= \frac{64}{231}$$

Therefore, $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \frac{64}{231}$

3. Find the value of $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Solution: Consider the integral $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Suppose that $x = \tan\theta$, so that $dx = \sec^2\theta d\theta$

Hence the integral becomes

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta d\theta$$

$$= [2\theta\tan\theta]_0^{\frac{\pi}{4}} + [2\log(\cos\theta)]_0^{\frac{\pi}{4}}$$

$$= 2\left(\frac{\pi}{4}\right) - \log 2$$

$$= \frac{\pi}{2} - \log 2$$

Therefore, $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \frac{\pi}{2} - \log 2$

4. Find the value of $\int_0^2 x\sqrt{x+2}dx$

Solution: Consider the integral $\int_0^2 x\sqrt{x+2}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x) + C$ then $\int_a^b f(x)dx = F(b) - F(a)$"

Suppose that $x+2=t$, so that $dx = dt$

When $x = 0 \Rightarrow t = 2$ and when $x = 2 \Rightarrow t = 4$

Hence the integral becomes

$$\int_0^2 x\sqrt{x+2}dx=\int_2^4 (t-2)\sqrt{t}dt$$

$$=\int_2^4\left(t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)dt$$

$$=\left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}}-2\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$=\frac{2}{5}\left(32-4\sqrt{2}\right)-\frac{4}{3}\left(8-2\sqrt{2}\right)$$

$$=\frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Therefore, $\int_0^2 x\sqrt{x+2}dx=\frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$

5. Find the value of $\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx$

Solution: Consider the integral $\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx=F(x)+C$ then $\int_a^b f(x)dx=F(b)-F(a)$"

Suppose that $t=\cos x$, so that $dt=-\sin xdx$

When $x=0\Rightarrow t=1$ and when $x=\frac{\pi}{2}\Rightarrow t=0$

Hence the integral becomes

$$\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx=-\int_1^0\frac{dt}{1+t^2}$$

$$=\left[-\tan^{-1}t\right]_1^0$$

$$=\frac{\pi}{4}$$

Therefore, $\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx=\frac{\pi}{4}$

6. Find the value of $\int_0^2 \frac{dx}{x+4-x^2}$

Solution: Consider the integral $\int_0^2 \frac{dx}{x+4-x^2}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx = F(x)+C$ then $\int_a^b f(x)dx = F(b)-F(a)$"

Rewrite the quadratic expression $x+4-x^2$ as

$$\begin{aligned} x+4-x^2 &= -(x^2-x-4) \\ &= \frac{17}{4}-\left(x-\frac{1}{2}\right)^2 \\ &= \left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2 \end{aligned}$$

Hence the integral becomes

$$\begin{aligned} \int_0^2 \frac{dx}{x+4-x^2} &= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} \\ &= \frac{1}{\sqrt{17}}\log\left(\frac{\frac{\sqrt{17}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{17}}{2}-\left(x-\frac{1}{2}\right)}\right)_0^2 \\ &= \frac{1}{\sqrt{17}}\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\right)-\log\left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right) \\ &= \frac{1}{\sqrt{17}}\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}\right) \\ &= \frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right) \end{aligned}$$

Therefore, $\int_0^2 \frac{dx}{x+4-x^2} = \frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$

7. Find the value of $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

Solution: Consider the integral $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx=F(x)+C$ then $\int_a^b f(x)dx=F(b)-F(a)$"

Rewrite the quadratic expression x^2+2x+5 as

$$\begin{aligned}x^2+2x+5&=(x+1)^2+4\\&=(x+1)^2+2^2\end{aligned}$$

Hence the integral becomes

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{x^2+2x+5}&=\int_{-1}^{1}\frac{dx}{(x+1)^2+2^2}\\&=\frac{1}{2}\left[\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1}\\&=\frac{1}{2}\tan^{-1}(1)\\&=\frac{\pi}{8}\end{aligned}$$

Therefore, $\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\frac{\pi}{8}$

8. Find the value of $\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$

Solution: Consider the integral $\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$

Use the fundamental theorem of integral calculus which states that "if $\int f(x)dx=F(x)+C$ then $\int_a^b f(x)dx=F(b)-F(a)$"

Suppose that $2x=t$, so that $2dx=dt$

When $x=1 \Rightarrow t=2$ and $x=2 \Rightarrow t=4$

Hence the integral becomes

$$\int_1^2 \left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx = \frac{1}{2}\int_2^4 \left(\frac{2}{t}-\frac{2}{t^2}\right)e^t dx$$

$$= \int_2^4 \left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dx$$

$$= \left[\frac{e^4}{4}-\frac{e^2}{2}\right]$$

$$= \frac{e^2\left(e^2-2\right)}{4}$$

Therefore, $\int_1^2 \left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx = \frac{e^2\left(e^2-2\right)}{4}$

9. The value of the integral $\int_{\frac{1}{3}}^{1} \frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ is

A) 6 B) 0 C) 3 D) 4

Solution:

Consider $I = \int_{\frac{1}{3}}^{1} \frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$

Let $x=\sin\theta \Rightarrow dx=\cos\theta d\theta$

When $x=\frac{1}{3}, \theta=\sin^{-1}\left(\frac{1}{3}\right)$ and when $x=1, \theta=\frac{\pi}{2}$

$$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\left(\sin\theta-\sin^3\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{1}{3}}\left(1-\sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{1}{3}}}{\sin^2\theta\sin^2\theta}\cos\theta d\theta = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}}\operatorname{cosec}^2\theta d\theta$$

$$\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} (\cot\theta)^{\frac{5}{3}}\operatorname{cosec}^2\theta d\theta$$

Put $\cot\theta = \mathrm{t} \Rightarrow -\operatorname{cosec}^2\theta d\theta = dt$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right), t = 2\sqrt{2}$ and when $\theta = \frac{\pi}{2}, t = 0$

$$\therefore I = \int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$$

$$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$= -\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0} = \frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$= \frac{3}{8}[16]$$

$$= 3\times 2$$

=6

Thus, the correct answer is A

10. If $f(x) = \int_0^x t\sin t\, dt$, then $f'(x)$ is

A) $\cos x + x\sin x$ B) $x\sin x$ C) $x\cos x$ D) $\sin x + x\cos x$

Solution:

$$f(x) = \int_0^x t\sin t dt$$

Using integration by parts, we get

$$f(x) = t\int_0^x \sin t dt - \int_0^x \left\{\left(\frac{d}{dt}t\right)\int \sin t dt\right\} dt$$

$$=\left[t(-\cos t)\right]_0^x-\int_0^x(-\cot t)\,dt$$

$$=\left[-t\cos t+\sin t\right]_0^x$$

$$=-x\cos x+\sin x$$

$$\Rightarrow f'(x)=-\left[\{x(-\sin x)\}+\cos x\right]+\cos x$$

$$=x\sin x-\cos x+\cos x$$

$$=x\sin x$$

Thus, the correct answer is B