Chapter: 7. Integrals

Exercise: 7.11

1. $\int_0^{\frac{\pi}{2}} \cos^2 x dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx........(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right)dx \qquad \left(\int_0^0 f(x)dx = \int_0^0 f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x dx.........(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

2. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Consider, $I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx........(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cos}}{\sqrt{\cos}+\sqrt{\sin x}}dx.........(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

3. $\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}xdx}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}xdx}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx.........(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx..........(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

4. $\int_0^{\frac{\pi}{2}}\frac{\cos^5 xdx}{\sin^5 x+\cos^5 x}dx$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}}\frac{\cos^5 xdx}{\sin^5 x+\cos^5 x}dx.........(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx........(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

5. $\int_{-5}^{5}|x+2|\,dx$

Solution:

Let $I=\int_{-5}^{5}|x+2|\,dx$

As, $(x+2)\le 0$ on [-5,-2] and $(x+2)\ge 0$ on [-2,5]

$\therefore \int_{5}^{-2}-(x+2)\,dx+\int_{-2}^{5}(x+2)\,dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$

$$I=-\left[\frac{x^2}{2}+2x\right]_{-5}^{-2}+\left[\frac{x^2}{2}+2x\right]_{-2}^{5}$$

$$=-\left[\frac{(-2)^2}{2}+2(-2)-\frac{(-5)^2}{2}-2(-5)\right]+\left[\frac{(5)^2}{2}+2(5)-\frac{(-2)^2}{2}-2(-2)\right]$$

$$=-\left[2-4-\frac{25}{2}+10\right]+\left[\frac{25}{2}+10-2+4\right]$$

$$=-2+4+\frac{25}{2}-10+\frac{25}{2}+10-2+4$$

$$=29$$

6. $\int_{2}^{8}|x-5|\,dx$

Solution:

Consider, $I=\int_{2}^{8}|x-5|\,dx$

As $(x-5)\le 0$ on $[2,5]$ and $(x-5)\ge 0$ on [5,8]

$I=\int_{2}^{5}-(x-5)\,dx+\int_{2}^{8}(x-5)\,dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$

$$\left[\frac{x^2}{2}-5x\right]_2^5+\left[\frac{x^2}{2}-5x\right]_5^8$$

$$-\left[\frac{25}{2}-25-2+10\right]+\left[32-40-\frac{25}{2}+25\right]=9$$

7. $\int_0^1 x(1-x)^n\,dx$

Solution:

Consider, $I=\int_0^1 x(1-x)^n\,dx$

$$\therefore I=\int_0^1 (1-x)\left(1-(1-x)\right)^n\,dx$$

$$=\int_0^1 (1-x)(x)^n\,dx=\int_0^1\left(x^n-x^{n+1}\right)dx$$

$$=\left[\frac{1}{n+1}-\frac{1}{n+2}\right]=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1 \qquad \left(\int_1^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$

$$=\frac{(n+2)-(n+1)}{(n+1)(n+2)}=\frac{1}{(n+1)(n+2)}$$

8. $\int_0^{\frac{\pi}{4}}\log(1+\tan x)\,dx$

Solution:

Let $I=\int_0^{\frac{\pi}{4}}\log(1+\tan x)\,dx$........(1)

$$\therefore I=\int_0^{\frac{\pi}{4}}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx \qquad \left(\int_1^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left\{1+\frac{1-\tan x}{1+\tan x}\right\}dx \Rightarrow I=\int_0^{\frac{\pi}{4}}\log\frac{2}{(1+\tan x)}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log 2dx-\int_0^{\frac{\pi}{4}}\log(1+\tan x)dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log 2dx \quad \text{(from (1))}$$

$$\Rightarrow 2I=[x\log 2]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I=\frac{\pi}{4}\log 2$$

$$\Rightarrow I=\frac{\pi}{8}\log 2$$

9. $\int_0^2 x\sqrt{2-x}\,dx$

Solution:

Consider, $I=\int_0^2 x\sqrt{2-x}\,dx$

$$I=\int_0^2(2-x)\sqrt{x}\,dx \qquad \left(\int_1^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$=\int_0^2\left\{2x^{\frac{1}{2}}-x^{\frac{3}{2}}\right\}dx=\left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{5}{2}}}{\frac{5}{5}}\right)\right]_0^2$$

$$=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\right]_0^2=\frac{4}{3}(2)^{\frac{3}{2}}-\frac{2}{5}(2)^{\frac{5}{2}}$$

$$=\frac{4\times 2\sqrt{2}}{3}-\frac{2}{5}\times 4\sqrt{2}=\frac{8\sqrt{2}}{3}-\frac{8\sqrt{2}}{5}$$

$$=\frac{40\sqrt{2}-24\sqrt{2}}{15}=\frac{16\sqrt{2}}{15}$$

10. $\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)\,dx$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\log(2\sin x\cos x))\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\text{logsinx}-\text{logcosx}-\log 2)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log\sin x-\log\text{cosx}-\log 2)\,dx\text{.........}(1)$

Since, $\left(\int_0^a f(d)\,dx=\int_0^a f(a-x)\,dx\right)$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{\log\cos x-\log\sin x-\log 2\}\,dx\text{.........}(2)$

Adding (1) and (2), we get

$2I=\int_0^{\frac{\pi}{2}}(-\log 2-\log 2)\,dx$

$\Rightarrow 2I=-2\log 2\int_0^{\frac{\pi}{2}}1.dx$

$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$

$\Rightarrow I=\frac{\pi}{2}(-\log 2)$

$\Rightarrow I=\frac{\pi}{2}\left(\log\frac{1}{2}\right)$

$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$

11. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$

Solution:

Let $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$

As $\sin^2(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x$, therefore, $\sin^2 x$ is an even function

If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$$I = 2\int_0^{\frac{\pi}{2}} \sin^2 x dx = 2\int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$$

$$I = \int_0^{\frac{\pi}{2}} (1-\cos 2x) dx = \left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

12. $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$

Solution:

Let $I = \int_0^{\pi} \frac{x\,dx}{1+\sin x}$..........(1)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x(\pi - x)} dx \qquad \left(\int_1^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx........(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx \Rightarrow 2I = \pi\int_0^{\pi} \{\sec^2 x - \tan x \sec x\} dx$$

$$\Rightarrow 2I = \pi[2] \Rightarrow I = \pi$$

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$

Solution:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\,dx$.........(1)

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, thus, $\sin^2 x$ is an odd function

$f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx = 0$$

14. $\int_0^{2\pi} \cos^5 x\, dx$

Solution:

Let $I = \int_0^{2\pi} \cos^5 x\, dx..........(1)$

$\cos^5(2\pi - x) = \cos^5 x$

We know that, $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$, if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

$\therefore I = 2\int_0^{2\pi} \cos^5 xdx$

$\Rightarrow I = 2(0) = 0$ $\qquad\qquad [\cos^5(\pi - x) = -\cos^5 x]$

15. Find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x} dx$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x} dx..........(1)$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)} dx \qquad \left(\int_1^{a} f(x)dx = \int_0^{a} f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x\cos x} dx.......(2)$$

Adding (1) and (2) we get

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{0}{1+\sin x\cos x} dx \Rightarrow I = 0$$

16. Find the value of $\int_0^{\pi} \log(1+\cos x)dx$

Solution: Use the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

The given integral can be written as

$$I = \int_0^{\pi} \log(1+\cos x)\,dx$$
$$= \int_0^{\pi} \log(1+\cos(\pi - x))\,dx$$
$$I = \int_0^{\pi} \log(1-\cos x)\,dx$$

Adding the above two integrals

$$2I = \int_0^{\pi} \log(1+\cos x)\,dx + \int_0^{\pi} \log(1-\cos x)\,dx$$
$$= \int_0^{\pi} \log(1-\cos^2 x)\,dx$$
$$= \int_0^{\pi} \log(\sin^2 x)\,dx$$
$$= 2\int_0^{\pi} \log(\sin x)\,dx$$

Hence, $I = \int_0^{\pi} \log(\sin x)\,dx$

Use the property $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{If } f(2a-x) = f(x) \\ 0 & \text{If } f(2a-x) = -f(x) \end{cases}$

The above integral becomes

$$I = 2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} \log(\cos x)\,dx$$

$$2I = 2\int_0^{\frac{\pi}{2}} \log(\sin x \cos x)\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \int_0^{\frac{\pi}{2}} \log 2dx$$

$$= 2I - 2\left(\frac{\pi}{2}\log 2\right)$$

$$I = \pi \log 2$$

Therefore, $\int_0^{\pi} \log(1+\cos x)\,dx = \pi \log 2$

17. Find the value of $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

Solution: Consider the integral $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$

Use the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

The given integral can be rewrite as

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}\,dx$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx$$

The sum of the above two integrals

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

$$= \int_0^a 1dx$$

$$= [x]_0^a$$

$$= a$$

Divide both sides by 2

$$I = \frac{a}{2}$$

Therefore, $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx = \frac{a}{2}$

18. Find the value of $\int_0^4 |x-1|dx$

Solution: Consider the integrand

$$|x-1| = \begin{cases} 1-x & 0<x<1 \\ x-1 & 1<x<4 \end{cases}$$

Hence, the integral becomes

$$\int_0^4 |x-1|dx = \int_0^1 (1-x)dx + \int_1^4 (x-1)dx$$

$$= \left(x-\frac{x^2}{2}\right)_0^1 + \left(\frac{x^2}{2}-x\right)_1^4$$

$$= 1-\frac{1}{2}+\frac{16}{2}-4-\frac{1}{2}+1$$

$$= 5$$

Therefore, $\int_0^4 |x-1|dx = 5$

19. Show that $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$, if $f(x)$ and $g(x)$ are defined $f(x) = f(a-x)$ and $g(x)+g(a-x) = 4$

Solution:

Consider the given integral $\int_0^a f(x)g(x)dx$

Use the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, the above integral rewrite as

$$
\begin{aligned}
I &= \int_0^a f(x)g(x)dx \\
&= \int_0^a f(a-x)g(a-x)dx \\
&= \int_0^a f(x)g(a-x)dx
\end{aligned}
$$

Adding both integrals

$$
\begin{aligned}
2I &= \int_0^a f(x)g(x)dx + \int_0^a f(x)g(a-x)dx \\
&= \int_0^a f(x)\left(g(x)+g(a-x)\right)dx \\
&= 4\int_0^a f(x)dx
\end{aligned}
$$

Divide both sides with 2

$$I = 2\int_0^a f(x)dx$$

Therefore, $I = 2\int_0^a f(x)dx$

20. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$ is

A) 0 B) 2 C) π D) 1

Solution: Use the property that

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{If } f(x) \text{ is odd function} \\ 2\int_0^a f(x)dx & \text{If } f(x) \text{is even function} \end{cases}$$

In the given integrand $x^3, x\cos x, \tan^3 x$ are odd functions.

Hence, the given integral can be rewrite it as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 + x\cos x + \tan^5 x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1dx$$

$$= 0 + 0 + 0 + 2\int_0^{\frac{\pi}{2}} 1dx$$

$$= 2[x]_0^{\frac{\pi}{2}}$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi$$

Consider, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx = \pi$

This is matching with the option (C)

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

A) 2 B) $\frac{3}{4}$ C) 0 D) -2

Solution: Suppose that the integral $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$

Use the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

The integral becomes

$$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

Adding the above two integrals

$$2I=\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx+\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\cos x}{4+3\sin x}\times\frac{4+3\sin x}{4+3\cos x}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}\log(1)\,dx$$

$$=0$$

This is matching with the option (C)