Chapter: 7. Integrals

Exercise: 7.4

1. Integrate the function $\frac{3x^2}{1+x^6}$

Solution: The given integral is $\int \frac{3x^2}{1+x^6}dx$

Put $t=x^3$ so that $dt=3x^2dx$

Hence,

$$\begin{aligned}\int \frac{3x^2}{1+x^6}dx &= \int \frac{dt}{1+t^2}\\ &= \tan^{-1}(t)+C\\ &= \tan^{-1}\left(x^3\right)+C\end{aligned}$$

Therefore, $\int \frac{3x^2}{1+x^6}dx = \tan^{-1}\left(x^3\right)+C$

2. Integrate the function $\frac{1}{\sqrt{1+4x^2}}$

Solution: The given integral is $\int \frac{1}{\sqrt{1+4x^2}}dx$

Put $t=2x$ so that $dt=2dx$

Hence,

$$\begin{aligned}\int \frac{1}{\sqrt{1+4x^2}}dx &= \frac{1}{2}\int \frac{2dx}{\sqrt{1+4x^2}}\\ &= \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}\\ &= \frac{1}{2}\tan^{-1}(t)+C\\ &= \frac{1}{2}\tan^{-1}(2x)+C\end{aligned}$$

Therefore, $\int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\tan^{-1}(2x)+C$

3. Integrate the function $\frac{1}{\sqrt{(2-x)^2+1}}$

Solution: The given integral is $\int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{-1}{\sqrt{(2-x)^2+1}}dx$

Put $t = 2-x$ so that $dt = -dx$

Hence,

$$\int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{-1}{\sqrt{(2-x)^2+1}}dx$$

$$= -\int \frac{dt}{\sqrt{t^2+1}} \qquad \bullet \int \frac{1}{\sqrt{1+x^2}}dx = \log\left(x+\sqrt{1+x^2}\right)+C$$

$$= -\log\left((2-x)+\sqrt{(2-x)^2+1}\right)+C$$

$$= \log\left(\frac{1}{2-x+\sqrt{x^2-4x+5}}\right)+C$$

Therefore, $\int \frac{1}{\sqrt{(2-x)^2+1}}dx = \log\left(\frac{1}{2-x+\sqrt{x^2-4x+5}}\right)$

4. Integrate the function $\frac{1}{\sqrt{9-25x^2}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{9-25x^2}}dx$

Put $t = \frac{5x}{3}$ so that $dt = \frac{5}{3}dx$

Hence,

$$\int \frac{1}{\sqrt{9-25x^2}}dx = \frac{3}{5}\int \frac{1}{3\sqrt{1-\left(\frac{5}{3}x\right)^2}}\frac{5}{3}dx$$

$$= \frac{1}{5}\int \frac{1}{\sqrt{1-t^2}}dt$$

$$= \frac{1}{5}\sin^{-1}t + C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Therefore, $\int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$

5. Integrate the function $\frac{3x}{1+2x^4}$

Solution:

The given integral is $\int \frac{3x}{1+2x^4}dx$

Put $t = \sqrt{2}x^2$ so that $dt = 2\sqrt{2}xdx$

Hence,

$$\int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{2\sqrt{2}dx}{1+\left(\sqrt{2}x^2\right)^2}$$

$$= \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}t + C$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Therefore, $\int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$

6. Integrate the function $\frac{x^2}{1-x^6}$

Solution:

The given integral is $\int \frac{x^2}{1-x^6} dx$

Put $t = x^3$ so that $dt = 3x^2 dx$

Hence,

$$\begin{aligned}\int \frac{x^2}{1-x^6} dx &= \frac{1}{3}\int \frac{3x^2}{1-\left(x^3\right)^2} dx \\ &= \frac{1}{3}\int \frac{dt}{1-t^2} \\ &= \frac{1}{3}\log\left(\frac{1+t}{1-t}\right)+C \\ &= \frac{1}{3}\log\left(\frac{1+x^3}{1-x^3}\right)+C\end{aligned}$$

Therefore, $\int \frac{x^2}{1-x^6} dx = \frac{1}{3}\log\left(\frac{1+x^3}{1-x^3}\right)+C$

7. Integrate the function $\frac{x-1}{\sqrt{x^2-1}}$

Solution:

The given integral is $\int \frac{x-1}{\sqrt{x^2-1}} dx$

Put $t = x^2 - 1$ so that $dt = 2x dx$

Hence,

$$\begin{aligned}\int \frac{x-1}{\sqrt{x^2-1}} dx &= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx \\ &= \frac{1}{2}\int \frac{1}{\sqrt{t}} dt - \int \frac{1}{\sqrt{x^2-1}} dx \\ &= \frac{1}{2}\left(2\sqrt{t}\right) - \log\left(x+\sqrt{x^2-1}\right)+C \\ &= \sqrt{1-x^2} - \log\left|x+\sqrt{x^2-1}\right|+C\end{aligned}$$

Therefore, $\int \frac{x-1}{\sqrt{x^2-1}} dx = \sqrt{1-x^2} - \log\left|x+\sqrt{x^2-1}\right|+C$

8. Integrate the function $\frac{x^2}{\sqrt{x^6+a^6}}$

Solution:

The given integral is $\int \frac{x^2}{\sqrt{x^6+a^6}} dx$

Put $t = x^3$ so that $dt = 3x^2 dx$

Hence,

$$\int \frac{x^2}{\sqrt{x^6+a^6}} dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{\left(x^3\right)^2+\left(a^3\right)^2}} dx$$

$$= \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left(t+\sqrt{t^2+a^6}\right)+C$$

$$= \frac{1}{3}\log\left(x^3+\sqrt{x^6+a^6}\right)+C$$

Therefore, $\int \frac{x^2}{\sqrt{x^6+a^6}} dx = \frac{1}{3}\log\left(x^3+\sqrt{x^6+a^6}\right)+C$

9. Integrate the function $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Solution:

The given integral is $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}} dx$

Put $t = \tan x$ so that $dt = \sec^2 x dx$

Hence,

$$\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+2^2}}$$

$$= \log\left(t+\sqrt{t^2+4}\right)+C$$

$$= \log\left(\tan x+\sqrt{\tan^2 x+4}\right)+C$$

Therefore, $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \log\left(\tan x+\sqrt{\tan^2 x+4}\right)$

10. Integrate the function $\frac{1}{\sqrt{x^2+2x+2}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{x^2+2x+2}}\cdot dx = \int \frac{1}{\sqrt{(x+1)^2+1}}\cdot dx$

Put $t = x+1$ so that $dt = dx$

Hence,

$$\int \frac{1}{\sqrt{(x+1)^2+1}}\cdot dx = \int \frac{1}{\sqrt{t^2+1}}\cdot dx$$

$$= \log\left(t+\sqrt{t^2+1}\right)+C$$

$$= \log\left(x+1+\sqrt{x^2+2x+2}\right)+C$$

Therefore, $\int \frac{1}{\sqrt{(x+1)^2+1}}\cdot dx = \log\left(x+1+\sqrt{x^2+2x+1}\right)$

11. Integrate the function $\frac{1}{9x^2+6x+5}$

Solution:

The given integral is $\int \frac{1}{9x^2+6x+5}dx$

$$\int \frac{1}{9x^2+6x+5}dx = \int \frac{1}{(3x+1)^2+2^2}dx$$

Put, $3x+1=t$, so that $3dx = dt$

Hence,

$$\int \frac{1}{9x^2+6x+5}dx = \frac{1}{3}\int \frac{3}{(3x+1)^2+2^2}dx$$

$$= \frac{1}{3}\int \frac{dt}{t^2+2^2}$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{t}{2}\right)+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

Therefore, $\int \frac{1}{9x^2+6x+5}dx = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$

12. Integrate the function $\frac{1}{\sqrt{7-6x+x^2}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{7-6x+x^2}}dx$

Put $t = x^3 - 1$ so that $dt = 3x^2 dx$

Hence,

$$\int \frac{1}{\sqrt{7-6x+x^2}}dx = \int \frac{1}{\sqrt{(x-3)^2-2}}dx$$

$$= \int \frac{1}{\sqrt{(x-3)^2-\left(\sqrt{2}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left(x+\sqrt{x^2-a^2}\right)+c$

$$\int \frac{1}{\sqrt{(x-3)^2-\left(\sqrt{2}\right)^2}}dx = \log\left(x-3+\sqrt{x^2-6x+7}\right)+C$$

Therefore, $\int \frac{1}{\sqrt{7-6x+x^2}}dx = \log\left(x-3+\sqrt{x^2-6x+7}\right)+C$

13. Integrate the function $\frac{1}{\sqrt{(x-1)(x-2)}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \int \frac{1}{\sqrt{x^2-3x+2}}dx$

The quadratic expression x^2-3x+2 can be written as below.

$$\begin{aligned} x^2-3x+2 &= \left(x-\frac{3}{2}\right)^2+2-\frac{9}{4} \\ &= \left(x-\frac{3}{2}\right)^2-\frac{1}{4} \\ &= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2 \end{aligned}$$

Hence,

$$\begin{aligned} \int \frac{1}{\sqrt{(x-1)(x-2)}}dx &= \int \frac{1}{\sqrt{x^2-3x+2}}dx \\ &= \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx \\ &= \log\left(x-\frac{3}{2}+\sqrt{x^2-3x+2}\right)+C \end{aligned}$$

Therefore, $\int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \log\left(x-\frac{3}{2}+\sqrt{x^2-3x+2}\right)+C$

14. Find the integral of the function $\frac{1}{\sqrt{8+3x-x^2}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{8+3x-x^2}}dx$

The quadratic expression $8+3x-x^2$ can be written as

$$8+3x-x^2=\left(8+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2$$
$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

Hence, the integral becomes

$$\int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$

It gives

$$\int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}dx$$
$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$
$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Therefore, $\int \frac{1}{\sqrt{8+3x-x^2}}dx=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$

15. Integrate the function $\frac{1}{\sqrt{(x-a)(x-b)}}$

Solution:

The given integral is $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx$

The quadratic expression can be written as

$$(x-a)(x-b)=x^2-(a+b)+ab$$
$$=\left(x-\left(\frac{a+b}{2}\right)\right)^2+\left(ab-\left(\frac{a+b}{2}\right)^2\right)$$

Hence, the integral can be rewrite as

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}dx=\int \frac{1}{\sqrt{\left(x-\left(\frac{a+b}{2}\right)\right)^2+\left(ab-\left(\frac{a+b}{2}\right)^2\right)}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2+a^2}}dx=\log\left(x+\sqrt{x^2+a^2}\right)+C$

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}dx=\int \frac{1}{\sqrt{\left(x-\left(\frac{a+b}{2}\right)\right)^2+\left(ab-\left(\frac{a+b}{2}\right)^2\right)}}dx$$
$$=\log\left(x-\frac{a+b}{2}+\sqrt{(x-a)(x-b)}\right)+C$$

Therefore, $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx=\log\left(x-\frac{a+b}{2}+\sqrt{(x-a)(x-b)}\right)+C$

16. Find the integral of the function $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Solution:

The given integral is $\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx$

Put $t=2x^2+x-3$, so that $dt=(4x+1)dx$

Hence, the given integral becomes

$$\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = \int \frac{dt}{\sqrt{t}}$$
$$= 2\sqrt{t}+C$$
$$= 2\sqrt{2x^2+x-3}+C$$

Therefore, $\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx = 2\sqrt{2x^2+x-3}+C$

17. Find the integral of the function $\frac{x+2}{\sqrt{x^2-1}}$

Solution:

The given integral is $\int \frac{x+2}{\sqrt{x^2-1}}dx$

Put $t = x^2$, so that $dt = 2xdx$

Hence the integral can be rewrite as

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx$$
$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} + 2\int \frac{1}{\sqrt{x^2-1}}dx$$
$$= \sqrt{t} + 2\sin^{-1} x + C$$
$$= \sqrt{x^2-1} + 2\sin^{-1} x + C$$

Therefore, $\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left(x+\sqrt{x^2-1}\right)+C$

18. Find the integral of the function $\frac{5x-2}{1+2x+3x^2}$

Solution:

The given integral is $\int \frac{5x-2}{1+2x+3x^2}dx$

Suppose that $5x-2 = P\left(\frac{d}{dx}\left(1+2x+3x^2\right)\right)+Q$

It gives $5x-2=P(6x+2)+Q$

Comparing coefficient of x from both sides

$$6P=5\Rightarrow P=\frac{5}{6}$$

Comparing the constant terms from both sides

$$-2=2P+Q$$
$$2\left(\frac{5}{6}\right)+Q=-2$$
$$Q=-2-\frac{5}{3}$$
$$=-\frac{11}{3}$$

Hence, the given integral becomes

$$\int\frac{5x-2}{1+2x+3x^2}dx=\int\frac{\frac{5}{6}\left(\frac{d}{dx}\left(3x^2+2x+1\right)\right)-\frac{11}{3}}{1+2x+3x^2}dx$$
$$=\frac{5}{6}\int\frac{6x+2}{1+2x+3x^2}dx-\frac{11}{3}\int\frac{1}{3\left(x^2+\frac{2}{3}x+\frac{1}{3}\right)}dx$$

Use the formula $\int\frac{f'(x)}{f(x)}dx=\log\left|f(x)\right|+c$ to find the first integral

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\int\frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{1}{3}-\frac{1}{9}}dx$$
$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\int\frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}dx$$

Use the formula $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{9}\left(\frac{3}{\sqrt{2}}\right)\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + C$$

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

Therefore, $\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\log\left|3x^2+2x+1\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

19. Find the integral of the function $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

Solution:

The given integral is $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$

Suppose that $6x+7 = P\left(\frac{d}{dx}\left(x^2-9x+20\right)\right)+Q$

It gives $6x+7 = P(2x-9)+Q$

Comparing coefficient of x from both sides

$$2P = 6 \Rightarrow P = 3$$

Comparing the constant terms from both sides

$$7 = -9P + Q$$
$$-9(3)+Q = 7$$
$$Q = 7+27$$
$$= 34$$

Hence, the given integral becomes

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = \int \frac{3\left(\frac{d}{dx}\left(x^2-9x+20\right)\right)+34}{\sqrt{x^2-9x+20}}dx$$

$$=\int \frac{3(2x-9)}{\sqrt{x^2-9x+20}}dx+\int \frac{34}{\sqrt{\left(x-\frac{9}{2}\right)^2+20-\frac{81}{4}}}dx$$

Use the formula $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}+c$ to find the first integral

$$=6\sqrt{x^2-9x+20}+\int \frac{34}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left(x+\sqrt{x^2-a^2}\right)+C$ to find the second integral

$$=6\sqrt{x^2-9x+20}+34\log\left(x-\frac{9}{2}+\sqrt{x^2-9x+20}\right)+C$$

Therefore,

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = 6\sqrt{x^2-9x+20}+34\log\left(x-\frac{9}{2}+\sqrt{x^2-9x+20}\right)+C$$

20. Find the integral of the function $\frac{x+2}{\sqrt{4x-x^2}}$

Solution:

The given integral is $\int \frac{x+2}{\sqrt{4x-x^2}}dx$

Suppose that $x+2 = P\left(\frac{d}{dx}\left(4x-x^2\right)\right)+Q$

It gives $x+2 = P(4-2x)+Q$

Comparing coefficient of x from both sides

$$-2P = 1 \Rightarrow P = -\frac{1}{2}$$

Comparing the constant terms from both sides

$$2 = 4P + Q$$

$$4\left(-\frac{1}{2}\right) + Q = 2$$

$$Q = 2 + 2$$
$$= 4$$

Hence, the given integral becomes

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}\left(\frac{d}{dx}\left(4x-x^2\right)\right)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + \int \frac{4}{\sqrt{4x-x^2}}dx$$

Use the formula $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$ to find the first integral

$$= -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + \int \frac{4}{\sqrt{2^2-(x-2)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ to find the second integral

$$= -\sqrt{4x-x^2} + \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Therefore, $\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\sqrt{4x-x^2} + \sin^{-1}\left(\frac{x-2}{2}\right) + C$

21. Find the integral of the function $\frac{x+2}{\sqrt{x^2+2x+3}}$

Solution:

The given integral is $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$

Suppose that $x + 2 = P\left(\frac{d}{dx}\left(x^2+2x+3\right)\right) + Q$

It gives $x+2=P(2x+2)+Q$

Comparing coefficient of x from both sides

$$2P=1 \Rightarrow P=\frac{1}{2}$$

Comparing the constant terms from both sides

$$\begin{aligned} 2&=2P+Q \\ 2&=2\left(\frac{1}{2}\right)+Q \\ Q&=2-1 \\ &=1 \end{aligned}$$

Hence, the given integral becomes

$$\frac{x+2}{\sqrt{x^2+2x+3}}=\int \frac{\frac{1}{2}\left(\frac{d}{dx}\left(x^2+2x+3\right)\right)+1}{\sqrt{x^2+2x+3}}dx$$

$$=\int \frac{\frac{1}{2}(2x+2)}{\sqrt{x^2+2x+3}}dx+\int \frac{1}{\sqrt{(x+1)^2+3-1}}dx$$

Use the formula $\int \frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c$ to find the first integral

$$=\frac{1}{2}\left(2\sqrt{x^2+2x+3}\right)+\int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2+a^2}}dx=\log\left(x+\sqrt{x^2+a^2}\right)+C$ to find the second integral

$$=\sqrt{x^2+2x+3}+\log\left(x+1+\sqrt{x^2+2x+3}\right)+C$$

Therefore, $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx=\sqrt{x^2+2x+3}+\log\left(x+1+\sqrt{x^2+2x+3}\right)+C$

22. Find the integral of the function $\frac{x+3}{x^2-2x-5}$

Solution:

The given integral is $\int \frac{x+3}{x^2-2x-5}dx$

Suppose that $x+3=P\left(\frac{d}{dx}\left(x^2-2x-5\right)\right)+Q$

It gives $x+3=P(2x-2)+Q$

Comparing coefficient of x from both sides

$$2P=1 \Rightarrow P=\frac{1}{2}$$

Comparing the constant terms from both sides

$$3=-2P+Q$$

$$-2\left(\frac{1}{2}\right)+Q=3$$

$$Q=3+1$$

$$=4$$

Hence, the given integral becomes

$$\int \frac{x+3}{x^2-2x-5}dx=\int \frac{\frac{1}{2}\left(\frac{d}{dx}\left(x^2-2x-5\right)\right)+4}{x^2-2x-5}dx$$

$$=\frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx+4\int \frac{1}{(x-1)^2-5-1}dx$$

Use the formula $\int \frac{f'(x)}{f(x)}dx=\log|f(x)|+c$ to find the first integral

$$\frac{1}{2}\log\left|x^2-2x-5\right|+4\int \frac{1}{(x-1)^2-\left(\sqrt{6}\right)^2}dx$$

Use the formula $\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+C$

$$= \frac{1}{2}\log\left|x^2 - 2x - 5\right| + \frac{4}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) + C$$

$$= \frac{1}{2}\log\left|x^2 - 2x - 5\right| + \frac{2}{\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) + C$$

Therefore, $\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) + C$

23. Find the integral of the function $\frac{5x+3}{\sqrt{x^2+4x+10}}$

Solution:

The given integral is $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

Suppose that $5x+3 = P\left(\frac{d}{dx}\left(x^2+4x+10\right)\right) + Q$

It gives $5x+3 = P(2x+4)+Q$

Comparing coefficient of x from both sides

$$2P = 5 \Rightarrow P = \frac{5}{2}$$

Comparing the constant terms from both sides

$$\begin{aligned} 3 &= 4P + Q \\ &= 4\left(\frac{5}{2}\right) + Q \\ &= 10 + Q \\ Q &= -7 \end{aligned}$$

Hence, the given integral becomes

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int \frac{\frac{5}{2}\left(\frac{d}{dx}\left(x^2+4x+10\right)\right)-7}{\sqrt{x^2+4x+10}}dx$$

$$=\frac{5}{2}\int \frac{(2x+4)}{\sqrt{x^2+4x+10}}dx-\int \frac{7}{\sqrt{(x+2)^2+6}}dx$$

Use the formula $\int \frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c$ to find the first integral

$$=\frac{5}{2}\left(2\sqrt{x^2+4x+10}\right)-7\int \frac{1}{\sqrt{(x+2)^2+\left(\sqrt{6}\right)^2}}dx$$

Use the formula $\int \frac{1}{\sqrt{x^2+a^2}}dx=\log\left(x+\sqrt{x^2+a^2}\right)+C$ to find the second integral

$$=5\left(\sqrt{x^2+4x+10}\right)-7\log\left|x+2+\sqrt{x^2+4x+10}\right|+C$$

Therefore, $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx=5\left(\sqrt{x^2+4x+10}\right)-7\log\left|x+2+\sqrt{x^2+4x+10}\right|+C$

24. $\int \frac{dx}{x^2+2x+2}$ equals

A) $x\tan^{-1}(x+1)+C$ B) $\tan^{-1}(x+1)+C$

C) $(x+1)\tan^{-1}x+C$ D) $\tan^{-1}x+C$

Solution:

The given integral is $\int \frac{dx}{x^2+2x+2}=\int \frac{dx}{(x+1)^2+1}$

Use the formula $\int \frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

Hence, $\int \frac{dx}{(x+1)^2+1}=\tan^{-1}(x+1)+C$

Therefore, the option (B), is correct.

25. $\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$

D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution:

The given integral is $\int \frac{dx}{\sqrt{9x-4x^2}} = \int \frac{dx}{2\sqrt{\frac{9}{4}x-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}$

Use the formula $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+C$

Hence,

$$\int \frac{dx}{\sqrt{9x-4x^2}} = \int \frac{dx}{2\sqrt{\frac{9}{4}x-x^2}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Therefore, option (B) is correct.