Chapter: 7. Integrals

Exercise: 7.5

1. Find the integral of the function $\frac{x}{(x+1)(x+2)}$

Solution:

The integrand is $\frac{x}{(x+1)(x+2)}$,

- It has two linear factors in the denominator
- It is proper fraction

Suppose that $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

Simplifying, we get

$$x=A(x+2)+B(x+1) \qquad \ldots\ldots (1)$$

Plug in $x=-2$ in the equation (1),

$$-2=B(-1)$$
$$B=2$$

Plug in $x=-1$ in the equation (1),

$$-1=A(1)$$
$$A=-1$$

Hence, $\frac{x}{(x+1)(x+2)}=\frac{-1}{(x+1)}+\frac{2}{(x+2)}$

Consider the integral

$$\int \frac{x}{(x+1)(x+2)}dx = \int \frac{-1}{(x+1)}dx + \int \frac{2}{(x+2)}dx$$
$$= -\log(x+1) + 2\log(x+2) + C$$
$$= \log \frac{(x+2)^2}{|x+1|} + C$$

Therefore, $\int \frac{x}{(x+1)(x+2)}dx = \log \frac{(x+2)^2}{|x+1|} + C$

2. Find the integral of the function $\frac{1}{x^2-9}$

Solution:

The integrand is $\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)}$,

- It has two linear factors in the denominator
- It is proper fraction

Suppose that $\frac{1}{(x-3)(x+3)} = \frac{A}{(x-3)} + \frac{B}{(x+3)}$

It implies

$$1 = A(x+3) + B(x-3) \qquad \text{….. (1)}$$

Plug in $x = -3$ in equation (1)

$$1 = A(0) + B(-6)$$
$$B = -\frac{1}{6}$$

Plug in $x = 3$ in equation (1)

$$1 = A(6) + B(0)$$
$$A = \frac{1}{6}$$

Consider the integral

$$\int \frac{1}{x^2-9}dx = \int \frac{1}{(x-3)(x+3)}dx$$
$$= \frac{1}{6}\int \frac{1}{x+3} - \frac{1}{x-3}dx$$
$$= \frac{1}{6}\left[\log|x+3| - \log|x-3|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{x+3}{x-3}\right| + C$$

Therefore, $\int \frac{1}{x^2-9}dx = \frac{1}{6}\log\left|\frac{x+3}{x-3}\right| + C$

3. Find the integral of the function $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Solution:

The integrand is $\frac{3x-1}{(x-1)(x-2)(x-3)}$

- It has three linear factors in the denominator
- It is proper fraction

Suppose that $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

It implies

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \ldots\ldots (1)$$

Plug in $x=1$ in equation (1)

$$2 = A(-1)(-2)$$
$$A = \frac{2}{2}$$
$$= 1$$

Plug in $x=2$ in equation (1)

$$5 = A(0) + B(1)(-1)$$
$$B = -5$$

Plug in $x = 3$ in equation (1)

$$8 = A(0) + B(0) + C(2)(1)$$
$$C = \frac{8}{2}$$
$$= 4$$

Consider the integral

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \int \frac{2}{x-1} - \frac{5}{x-2} + \frac{4}{x-3} dx$$
$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

Therefore, $\int \frac{3x-1}{(x-1)(x-2)(x-3)} dx = \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

4. Find the integral of the function $\frac{x}{(x-1)(x-2)(x-3)}$

Solution:

The integrand is $\frac{x}{(x-1)(x-2)(x-3)}$

- It has three linear factors in the denominator
- It is proper fraction

Suppose that $\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

It implies

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \qquad \text{..... (1)}$$

Plug in $x = 1$ in equation (1)

$$1 = A(-1)(-2)$$
$$A = \frac{1}{2}$$

Plug in $x = 2$ in equation (1)

$$2 = A(0) + B(1)(-1)$$
$$B = -2$$

Plug in $x = 3$ in equation (1)

$$3 = A(0) + B(0) + C(2)(1)$$
$$C = \frac{3}{2}$$

Consider the integral

$$\int \frac{x}{(x-1)(x-2)(x-3)} dx = \int \frac{\frac{1}{2}}{x-1} - \frac{2}{x-2} + \frac{\frac{3}{2}}{x-3} dx$$
$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$$

Therefore, $\int \frac{x}{(x-1)(x-2)(x-3)} dx = \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$

5. Find the integral of the function $\frac{2x}{x^2+3x+2}$

Solution:

The integrand is $\frac{2x}{x^2+3x+2} = \frac{2x}{(x+1)(x+2)}$

- It has two linear factors in the denominator
- It is proper fraction

Suppose that $\frac{2x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

It implies

$$2x = A(x+2) + B(x+1) \quad \text{….. (1)}$$

Plug in $x = -1$ in equation (1)

$$-2 = A(1)$$
$$A = -2$$

Plug in $x = -2$ in equation (1)

$$-4 = A(0) + B(-1)$$
$$B = 4$$

Consider the integral

$$\int \frac{2x}{x^2+3x+2}dx = \int \frac{-2}{x+1} + \frac{4}{x+2}dx$$
$$= -2\log|x+1| + 4\log|x+2| + C$$

Therefore, $\int \frac{2x}{x^2+3x+2}dx = -2\log|x+1| + 4\log|x+2| + C$

6. Find the integral of the function $\frac{1-x^2}{x(1-2x)}$

Solution:

The integrand is $\frac{1-x^2}{x(1-2x)} = \frac{x^2-1}{(x)(2x-1)}$

- It has two linear factors in the denominator
- It is improper fraction

Suppose that $\frac{x^2-1}{(x)(2x-1)} = \frac{1}{2} + \frac{A}{x} + \frac{B}{2x-1}$

It implies

$$x^2 - 1 = x(2x-1) + A(2x-1) + B(x) \quad \text{..... (1)}$$

Plug in $x = 0$ in equation (1)

$$-1 = A(-1)$$
$$A = 1$$

Plug in $x = \frac{1}{2}$ in equation (1)

$$\frac{1}{4}-1=A(0)+B\left(\frac{1}{2}\right)$$

$$-\frac{3}{4}=B\left(\frac{1}{2}\right)$$

$$B=-\frac{3}{2}$$

Consider the integral

$$\int \frac{x^2-1}{(x)(2x-1)}dx=\int \frac{1}{2}+\frac{1}{x}+\frac{-\frac{3}{2}}{2x-1}dx$$

$$=\frac{x}{2}+\log|x|-\frac{3}{2}\log|2x-1|\left(\frac{1}{2}\right)+C$$

$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|2x-1|+C$$

Therefore, $\int \frac{x^2-1}{(x)(2x-1)}dx==\frac{x}{2}+\log|x|-\frac{3}{4}\log|2x-1|+C$

7. Find the integral of the function $\frac{x}{(x^2+1)(x-1)}$

Solution:

The integrand is $\frac{x}{(x^2+1)(x-1)}$

- It has one linear factor and one irreducible quadratic factor in the denominator
- It is proper fraction

Suppose that $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$

It implies

$$x=(Ax+B)(x-1)+C(x^2+1) \qquad \ldots\ldots (1)$$

Plug in $x=1$ in equation (1)

$$1 = C(2)$$
$$C = \frac{1}{2}$$

Compare the coefficient of x^2

$$0 = A + C$$
$$A = -C$$
$$= -\frac{1}{2}$$

Compare the constant terms

$$0 = -B + C$$
$$B = C$$
$$= \frac{1}{2}$$

Consider the integral

$$\int \frac{x}{(x^2+1)(x-1)}dx = \int \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x-1}dx$$
$$= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x-1}{x^2+1}dx$$
$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$$
$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

Therefore, $\int \frac{x}{(x^2+1)(x-1)}dx = \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$

8. Find the integral of the function $\frac{x}{(x-1)^2(x+2)}$

Solution:

The integrand is $\frac{x}{(x-1)^2(x+2)}$

- It has one linear factors and one repeated linear factor in the denominator
- It is proper fraction

Suppose that $\frac{x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2}$

It implies

$$x=A(x-1)(x+2)+B(x+2)+C(x-1)^2 \qquad \text{….. (1)}$$

Plug in $x=1$ in equation (1)

$$1=B(3)$$
$$B=\frac{1}{3}$$

Plug in $x=-2$ in equation (1)

$$-2=C(9)$$
$$C=-\frac{2}{9}$$

Compare the coefficient of x^2 in the equation (1)

$$0=A+C$$
$$A=-C$$
$$=\frac{2}{9}$$

Consider the integral

$$\int\frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\int\frac{1}{x-1}dx+\frac{1}{3}\int\frac{1}{(x-1)^2}dx-\frac{2}{9}\int\frac{1}{x+2}dx$$
$$=\frac{2}{9}\log|x-1|-\frac{1}{3}\frac{1}{(x-1)}-\frac{2}{9}\log|x+2|+C$$
$$=\frac{2}{9}\log\left(\frac{x-1}{x+2}\right)-\frac{1}{3(x-1)}+C$$

Therefore, $\int\frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\log\left(\frac{x-1}{x+2}\right)-\frac{1}{3(x-1)}+C$

9. Find the integral of the function $\frac{3x+5}{x^3-x^2-x+1}$

Solution:

The integrand is $\frac{3x+5}{x^2-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$

- It has one linear factors and one repeated linear factor in the denominator
- It is proper fraction

Suppose that $\frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$

It implies

$$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2 \qquad \text{….. (1)}$$

Plug in $x=1$ in equation (1)

$$8=B(2)$$
$$B=4$$

Plug in $x=-1$ in equation (1)

$$2=C(4)$$
$$C=\frac{1}{2}$$

Compare the coefficient of x^2 in the equation (1)

$$0=A+C$$
$$A=-C$$
$$=-\frac{1}{2}$$

Consider the integral

$$\int \frac{3x+5}{(x-1)^2(x+1)}dx = -\frac{1}{2}\int \frac{1}{x-1}dx + 4\int \frac{1}{(x-1)^2}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$

$$= -\frac{1}{2}\log|x-1| + 4\frac{1}{(x-1)} + \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) - \frac{4}{(x-1)} + C$$

Therefore, $\int \frac{3x+5}{(x-1)^2(x+1)}dx = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) - \frac{4}{(x-1)} + C$

10. Find the integral of the function $\frac{2x-3}{(x^2-1)(2x+3)}$

Solution:

The integrand is $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)}$

- It has three linear factors in the denominator
- It is proper fraction

Suppose that $\frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$

It implies

$$2x-3 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1) \quad \text{….. (1)}$$

Plug in $x = 1$ in equation (1)

$$-1 = A(10)$$

$$A = -\frac{1}{10}$$

Plug in $x = -1$ in equation (1)

$$-5 = B(-2)$$

$$B = \frac{5}{2}$$

Plug in $x=-\frac{3}{2}$ in equation (1)

$$-6=C\left(\frac{5}{4}\right)$$
$$C=-\frac{24}{5}$$

Consider the integral

$$\int\frac{2x-3}{(x-1)(x+1)(2x+3)}dx=-\frac{1}{10}\int\frac{1}{x-1}dx+\frac{5}{2}\int\frac{1}{(x+1)}dx-\frac{24}{5}\int\frac{1}{2x+3}dx$$
$$=-\frac{1}{10}\log|x-1|+\frac{5}{2}\log|x+1|-\frac{24}{5}\log|2x+3|\left(\frac{1}{2}\right)+C$$
$$=\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+C$$

Therefore, $\int\frac{2x-3}{(x-1)(x+1)(2x+3)}dx=\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+C$

11. Find the integral of the function $\frac{5x}{(x+1)(x^2-4)}$

Solution:

The integrand is $\frac{5x}{(x+1)(x^2-4)}=\frac{5x}{(x+1)(x-2)(x+2)}$

- It has three linear factors in the denominator
- It is proper fraction

Suppose that $\frac{5x}{(x+1)(x-2)(x+2)}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x+2}$

It implies

$$5x=A(x-2)(x+2)+B(x+1)(x+2)+C(x+1)(x-2) \quad \text{….. (1)}$$

Plug in $x=-1$ in equation (1)

$$-5 = A(-3)$$
$$A = \frac{5}{3}$$

Plug in $x = 2$ in equation (1)

$$10 = B(12)$$
$$B = \frac{5}{6}$$

Plug in $x = -2$ in equation (1)

$$-10 = C(-1)(-4)$$
$$C = -\frac{10}{4}$$
$$= -\frac{5}{2}$$

Consider the integral

$$\int \frac{5x}{(x+1)(x^2-4)}dx = \int \frac{5x}{(x+1)(x-2)(x+2)}dx$$
$$= \frac{5}{3}\int \frac{1}{x+1}dx + \frac{5}{6}\int \frac{1}{(x-2)}dx - \frac{5}{2}\int \frac{1}{x+2}dx$$
$$= \frac{5}{3}\log|x+1| + \frac{5}{6}\log|x-2| - \frac{5}{2}\log|x+2| + C$$

Therefore, $\int \frac{5x}{(x+1)(x^2-4)}dx = \frac{5}{3}\log|x+1| + \frac{5}{6}\log|x-2| - \frac{5}{2}\log|x+2| + C$

12. Find the integrand of the function $\frac{x^3+x+1}{x^2-1}$

Solution:

The integrand is $\frac{x^3+x+1}{x^2-1}$

- It has reducible quadratic factor in the denominator
- It is improper fraction

$$\frac{x^3+x+1}{x^2-1}=\frac{x\left(x^2+1\right)+1}{x^2-1}$$
$$=\frac{x\left(x^2-1\right)+2x+1}{x^2-1}$$
$$=x+\frac{2x+1}{(x-1)(x+1)}$$

Suppose that $\frac{2x+1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$

It implies

$$2x+1=A(x+1)+B(x-1) \qquad \ldots\ldots (1)$$

Plug in $x=-1$ in equation (1)

$$-1=B(-2)$$
$$B=\frac{1}{2}$$

Plug in $x=1$ in equation (1)

$$3=A(2)$$
$$A=\frac{3}{2}$$

Consider the integral

$$\int\frac{x^3+x+1}{x^2-1}dx=\int x+\frac{2x+1}{(x-1)(x+1)}dx$$
$$=\frac{x^2}{2}+\int\frac{\frac{3}{2}}{x-1}dx+\frac{1}{2}\int\frac{1}{x+1}dx$$
$$=\frac{x^2}{2}+\frac{3}{2}\log|x-1|+\frac{1}{2}\log|x+1|+C$$

Therefore, $\int\frac{x^3+x+1}{x^2-1}dx=\frac{x^2}{2}+\frac{3}{2}\log|x-1|+\frac{1}{2}\log|x+1|+C$

13. Find the integral of the function $\frac{2}{(1-x)\left(1+x^2\right)}$

Solution:

The given integrand is $\frac{2}{(1-x)(1+x^2)}$

- It has irreducible quadratic factor and one linear factor in the denominator
- It is proper fraction

Suppose that $\frac{2}{(1-x)(1+x^2)}=\frac{A}{(1-x)}+\frac{Bx+C}{(1+x^2)}$

It implies that

$$2=A(1+x^2)+(Bx+C)(1-x)$$

Plug in $x=1$

$$2=A(2)$$
$$A=1$$

Compare the coefficient of x^2

$$0=A-B$$
$$B=A$$
$$=1$$

Compare the constant term

$$2=A+C$$
$$C=2-1$$
$$=1$$

Hence,

$$\frac{2}{(1-x)(1+x^2)}=\frac{1}{(1-x)}+\frac{x+1}{(1+x^2)}$$

Consider the integral

$$\int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{(1-x)}dx + \int \frac{x+1}{(1+x^2)}dx$$

$$= -\log|1-x| + \frac{1}{2}\int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$$= -\log|1-x| + \frac{1}{2}\log|x^2+1| + \tan^{-1}(x) + C$$

Therefore, $\int \frac{2}{(1-x)(1+x^2)}dx = -\log|1-x| + \frac{1}{2}\log|x^2+1| + \tan^{-1}(x) + C$

14. Find the integral of the function $\frac{3x-1}{(x+2)^2}$

Solution:

The integrand is $\frac{3x-1}{(x+2)^2}$

- It has repeated linear factors in the denominator
- It is proper fraction

Suppose that $\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$

It implies that

$$3x-1 = A(x+2) + B$$

Plug in $x = -2$

$$-7 = B$$

Compare the coefficient of x

$$3 = A$$

Hence,

$$\frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

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Consider the integral

$$\int \frac{3x-1}{(x+2)^2}dx = \int \frac{3}{(x+2)}dx - \int \frac{7}{(x+2)^2}dx$$
$$= 3\log|x+2| + 7\frac{1}{x+2} + C$$
$$= 3\log|x+2| + \frac{7}{x+2} + C$$

Therefore, $\int \frac{3x-1}{(x+2)^2}dx = 3\log|x+2| + \frac{7}{x+2} + C$

15. Find the integral of the function $\frac{1}{x^4-1}$

Solution:

The integrand is $\frac{1}{x^4-1}$

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)}$$
$$= \frac{1}{(x+1)(x-1)(1+x^2)}$$

- The integrand has two linear factors and one irreducible quadratic factor in the denominator
- It is proper fraction.

Suppose that $\frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$

Simplifying

1 $$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2-1) \quad \ldots\ldots (1)$$

Plug in $x=-1$ in (1)

$$1 = A(2)(-2)$$
$$A = -\frac{1}{4}$$

Plug in $x=1$ in (1)

$$1=B(1+1)(1+1)$$
$$B=\frac{1}{4}$$

Comparing x^2 coefficient in the equation (1)

$$A+B+C=0$$
$$-\frac{1}{4}+\frac{1}{4}+C=0$$
$$C=0$$

Plug in $x=0$ in (1)

$$1=A(-1)+B(1)+D(-1)$$
$$1=-A+B-D$$
$$1=\frac{1}{4}+\frac{1}{4}-D$$
$$D=-\frac{1}{2}$$

Hence, $\frac{1}{(x+1)(x-1)(1+x^2)}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}+\frac{-\frac{1}{2}}{x^2+1}$

Integrate both sides

$$\int\frac{1}{(x+1)(x-1)(1+x^2)}dx=-\frac{1}{4}\int\left(\frac{1}{x+1}\right)dx+\frac{1}{4}\int\left(\frac{1}{x-1}\right)dx+\int\frac{-1}{2(x^2+1)}dx$$
$$=-\frac{1}{4}\log(x+1)+\frac{1}{4}\log(x-1)-\frac{1}{2}\tan^{-1}x$$
$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

Therefore, $\int\frac{1}{(x+1)(x-1)(1+x^2)}dx=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$

16. Find the integration of the function $\frac{1}{x(x^n+1)}$

Solution:

The given integrand can be rewrite as

$$\frac{1}{x\left(x^n+1\right)}=\frac{x^{n-1}}{x^n\left(x^n+1\right)}$$

Apply integration on both sides

$$\int\frac{1}{x\left(x^n+1\right)}dx=\int\frac{x^{n-1}}{x^n\left(x^n+1\right)}dx$$

Put $x^n=t$, so that $dt=nx^{n-1}dx$

$$\begin{aligned}\int\frac{1}{x\left(x^n+1\right)}dx&=\frac{1}{n}\int\frac{dt}{t(t+1)}\\&=\frac{1}{n}\int\frac{1}{t}dt-\int\frac{1}{t+1}dt\\&=\frac{1}{n}\left|\log|t|-\log|t+1|\right|+C\\&=\frac{1}{n}\log\left|\frac{t}{t+1}\right|+C\\&=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C\end{aligned}$$

Therefore, $\int\frac{1}{x\left(x^n+1\right)}dx=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$

17. Find the integral of the function $\frac{\cos x}{(1-\sin x)(2-\sin x)}$

Solution:

The given integral is $\int\frac{\cos x}{(1-\sin x)(2-\sin x)}dx$

Put $\sin x=t$ so that $\cos xdx=dt$

$$\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \int \frac{1}{(1-t)(2-t)}dt$$
$$= \int \frac{1}{1-t} - \frac{1}{2-t}dt$$
$$= -\log(1-t) + \log(2-t) + C$$
$$= \log\left|\frac{2-t}{1-t}\right| + C$$
$$= \log\left|\frac{2-\sin x}{1-\sin x}\right|$$

Therefore, $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx = \log\left|\frac{2-\sin x}{1-\sin x}\right|$

18. Find the integral of the function $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Solution:

Consider the integrand $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Suppose that $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{A}{x^2+3} + \frac{B}{x^2+4}$

It implies that

$$(x^2+1)(x^2+2) = (x^2+3)(x^2+4) + A(x^2+4) + B(x^2+3)$$

Substitute $x^2 = -4$

$$(-3)(-2) = 0 + A(0) + B(-1)$$
$$B = -6$$

Substitute $x^2 = -3$

$$(-2)(-1) = A(1)$$
$$A = 2$$

Hence,

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=1+\frac{2}{x^2+3}-\frac{6}{x^2+4}$$

Integrate both sides

$$\int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=\int 1dx+\int\frac{2}{x^2+3}dx-\int\frac{6}{x^2+4}dx$$

$$=x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+C$$

Therefore, $\int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+C$

19. Find the integral of the function $\frac{2x}{(x^2+1)(x^2+3)}$

Solution:

Consider the integral $\int\frac{2x}{(x^2+1)(x^2+3)}dx$

We know that $\frac{1}{x^2+1}-\frac{1}{x^2+3}=\frac{2}{(x^2+1)(x^2+3)}$

$$\int\frac{2x}{(x^2+1)(x^2+3)}dx=\frac{1}{2}\int\frac{2x}{(x^2+1)}-\frac{2x}{x^2+3}dx$$

$$=\frac{1}{2}\left(\log(x^2+1)-\log(x^2+3)\right)+C$$

$$=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$$

Therefore, $\int\frac{2x}{(x^2+1)(x^2+3)}dx=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$

20. Find the integral of the function $\frac{1}{x(x^4-1)}$

Solution:

The given integral is $\int \frac{1}{x\left(x^4-1\right)}dx$

$$\int \frac{1}{x\left(x^4-1\right)}dx = \int \frac{x^3}{x^4\left(x^4-1\right)}dx$$
$$= \frac{1}{4}\int \frac{4x^3}{x^4\left(x^4-1\right)}dx$$

Put $x^4 = t$, so that $4x^3dx = dt$

Hence,

$$\int \frac{1}{x\left(x^4-1\right)}dx = \frac{1}{4}\int \frac{4x^3}{x^4\left(x^4-1\right)}dx$$
$$= \frac{1}{4}\int \frac{dt}{t(t-1)}$$
$$= \frac{1}{4}\int \frac{1}{t} - \frac{1}{t-1}dt$$
$$= \frac{1}{4}\left[\log|t| - \log|t-1|\right] + C$$
$$= \frac{1}{4}\log\left|\frac{t}{t-1}\right| + C$$
$$= \frac{1}{4}\log\left|\frac{x^4}{x^4-1}\right| + C$$

Therefore, $\int \frac{1}{x\left(x^4-1\right)}dx = \frac{1}{4}\log\left|\frac{x^4}{x^4-1}\right| + C$

21. Find the integral of the function $\frac{1}{\left(e^x-1\right)}$

Solution:

Substitute $e^x = t$, so that $e^x dx = dt$

Hence,

$$\int \frac{1}{e^x-1}dx=\int\left(\frac{1}{t-1}\right)\frac{dt}{t}$$

$$=\int \frac{1}{t(t-1)}dt$$

Use the partial fractions

$$\int \frac{1}{t(t-1)}dt=\int \frac{1}{t-1}-\frac{1}{t}dt$$

$$=\log|t-1|-\log|t|+C$$

$$=\log\left|\frac{t-1}{t}\right|+C$$

Substitute $t=e^x$

$$\int \frac{1}{e^x-1}dx=\log\left|\frac{e^x-1}{e^x}\right|+C$$

Therefore, $\int \frac{1}{e^x-1}dx=\log\left|\frac{e^x-1}{e^x}\right|+C$

22. $\int \frac{xdx}{(x-1)(x-2)}=$

A) $\log\left|\frac{(x-1)^2}{x-2}\right|+C$

B) $\log\left|\frac{(x-2)^2}{x-1}\right|+C$

C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|+C$

D) $\log|(x-1)(x-2)|+C$

Solution:

The given integral is $\int \frac{xdx}{(x-1)(x-2)}$

Suppose that $\frac{x}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$

This can be written as

$$x = A(x-2) + B(x-1) \qquad \ldots\ldots (1)$$

Plug in $x = 2$ in the equation (1)

$$2 = B$$

Plug in $x = 1$ in the equation (1)

$$1 = -A \Rightarrow A = -1$$

Hence, $\dfrac{x}{(x-1)(x-2)} = -\dfrac{1}{x-1} + \dfrac{2}{x-2}$

Hence, the integral becomes

$$\begin{aligned}\int \frac{x}{(x-1)(x-2)}dx &= -\int \frac{1}{x-1}dx + \int \frac{2}{x-2}dx \\ &= -\log|x-1| + 2\log|x-2| + C \\ &= \log\left|\frac{(x-2)^2}{x-1}\right| + C\end{aligned}$$

Thus, the correct answer is (B)

23. $\displaystyle\int \frac{dx}{x(x^2+1)} =$

A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

B) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

C) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

D) $\frac{1}{2}\log|x| + \log(x^2+1) + C$

Solution:

The integrand is $\dfrac{1}{x(x^2+1)}$

- One linear and one irreducible quadratic factor in the denominator
- It is proper fraction

Suppose that $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

Simplifying

$$1 = A\left(x^2+1\right)+\left(Bx+C\right)x \qquad \ldots\ldots (1)$$

Plug in $x=0$ in the equation (1)

$$1 = A$$

Compare the coefficient of x^2 in the equation (1)

$$A+B=0 \Rightarrow B=-A \Rightarrow B=-1$$

Comparing the coefficient of x in the equation (1)

$$C=0$$

The integral can be written as

$$\int \frac{1}{x\left(x^2+1\right)}dx = \int \frac{1}{x}dx + \int \frac{-x}{x^2+1}dx$$

$$= \log|x| - \frac{1}{2}\log\left|x^2+1\right| + C$$

Therefore, the option (A) is correct.