Chapter: 7. Integrals

Exercise: 7.8

1. Find the value of $\int_a^b x\,dx$

Solution:

The given integral is $\int_a^b x\,dx$

Use the limit as sum formula

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\dfrac{b-a}{n}$

Plug in $f(x)=x$

Hence,

$$\begin{aligned}\int_a^b x\,dx&=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+a+h+....+a+(n-1)h\right]\\&=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h(1+2+3+...+n-1)\right]\\&=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(\frac{(n-1)(n)}{2}\right)\right]\\&=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{(b-a)}{n}\left(\frac{(n-1)(n)}{2}\right)\right]\\&=(b-a)\lim_{n\to\infty}\left[a+\left(\frac{(n-1)(b-a)}{2n}\right)\right]\\&=(b-a)\left(a+\left(\frac{b-a}{2}\right)\right)\\&=\frac{1}{2}(b-a)(b+a)\\&=\frac{b^2-a^2}{2}\end{aligned}$$

Therefore, $\int_a^b x\,dx=\dfrac{b^2-a^2}{2}$

2. Find the value the definite integral $\int_0^b (x+1)\,dx$ using the limit as sum concept.

Solution:

Consider the integral $\int_0^5 (x+1)\,dx$

Use the limit as sum formula

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{b-a}{n}=\frac{5}{n}, a=0, b=5$

Plug in $f(x)=x+1$

$$\int_0^5 (x+1)\,dx = (5)\lim_{n\to\infty}\frac{1}{n}\left[1+1+h+....+1+(n-1)h\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+h(1+2+3+...+n-1)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{5}{n}\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$=5\lim_{n\to\infty}\left[1+\frac{5}{n^2}\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$=5\left(1+\frac{5}{2}\right)$$

$$=\frac{35}{2}$$

Therefore, $\int_0^5 (x+1)\,dx = \frac{35}{2}$

3. Find the value of $\int_2^3 x^2 dx$ using limit as sum concept

Solution:

Consider the integral $\int_2^3 x^2 dx$

Use the limit as sum formula

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{3-2}{n}=\frac{1}{n}, a=2, b=3$

Plug in $f(x)=x^2$

$$\int_2^3 x^2dx=\lim_{n\to\infty}\frac{1}{n}\left[2^2+(2+h)^2+....+(2+(n-1)h)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n\left(2^2\right)+h^2\left(1^2+2^2...+(n-1)^2\right)+4h\left(1+2+...+(n-1)\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n2^2+h^2\left(\frac{(n-1)(n)(2n-1)}{6}\right)+4h\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$=\lim_{n\to\infty}\left[4+\frac{1}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right)+4\left(\frac{(n-1)(n)}{2n^2}\right)\right]$$

$$=4+\frac{2}{6}+4\left(\frac{1}{2}\right)$$

$$=\frac{19}{3}$$

Therefore, $\int_2^3 x^2dx=\frac{19}{3}$

4. Find the value of $\int_1^4\left(x^2-x\right)dx$ using limit as sum concept

Solution:

Consider the integral $\int_1^4\left(x^2-x\right)dx$

Use the limit as sum formula

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{4-1}{n}=\frac{3}{n}, a=1, b=4$

Plug in $f(x)=x^2-x$

$$\int_1^4 \left(x^2 - x\right)dx = \lim_{n\to\infty}\frac{3}{n}\left[\begin{array}{r}1^2 + (1+h)^2 + ... + (1+(n-1)h)^2 \\ -(1+1+h+1+2h+...+1+(n-1)h)\end{array}\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\begin{array}{r}n(1^2) + h^2\left(1^2 + ... + (n-1)^2\right) + 2h(1+2+...+(n-1)) \\ -n-h(1+2..+(n-1))\end{array}\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[n + h^2\left(\frac{(n-1)(n)(2n-1)}{6}\right) + 2h\left(\frac{(n-1)(n)}{2}\right) - n - h\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[h^2\left(\frac{(n-1)(n)(2n-1)}{6}\right) + h\left(\frac{(n-1)(n)}{2}\right)\right]$$

$$\lim_{n\to\infty}\left[\frac{27}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right) + \frac{9}{n}\left(\frac{(n-1)(n)}{2n^2}\right)\right]$$

$$= 27\left(\frac{1}{3}\right) + \frac{9}{2}$$

$$= \frac{27}{2}$$

Therefore, $\int_1^4 \left(x^2 - x\right)dx = \frac{27}{2}$

5. Find the value of $\int_{-1}^{1} e^x dx$ using limit as sum concept.

Solution:

Consider the integral $\int_{-1}^{1} e^x dx$

Use the limit as sum formula

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + ... + f(a+(n-1)h)\right]$$

Here $h = \frac{1+1}{n} = \frac{2}{n}, a = -1, b = 1$

Plug in $f(x) = e^x$

$$\int_{-1}^{1} e^x dx = 2\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2\cdot\frac{2}{n}\right)+....+f\left(-1+\frac{(n-1)\cdot 2}{n}\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2.\frac{2}{n}\right)}+....e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}-1}}{e^{\frac{2}{n}-1}}\right]$$

$$=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{2}-1}{e^{\frac{2}{n}-1}}\right]$$

$$==\frac{e^{-1}\times 2\left(e^{2}-1\right)}{\lim\limits_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}-1}}{\frac{2}{n}}\right)\times 2}$$

$$=e^{-1}\left[\frac{2\left(e^{2}-1\right)}{2}\right]$$

$$=\frac{e^{2}-1}{e}$$

$$=e-\frac{1}{e}$$

Therefore, $\int_{-1}^{1} e^x dx = e-\frac{1}{e}$

6. Find the value of $\int_{0}^{4}\left(x+e^{2x}\right)dx$ using limit as sum

Solution:

Consider the integral $\int_{0}^{4}\left(x+e^{2x}\right)dx$

Use the limit as sum formula

$$\int_{a}^{b} f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$$

Here $h=\frac{4-0}{n}=\frac{4}{n}, a=0, b=4$

Plug in $f(x)=x+e^{2x}$

$$\int_0^4 \left(x+e^{2x}\right)dx=(4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+....+f\left((n-1)h\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^{0}\right)+\left(h+e^{2h}+\left(2h+e^{2\cdot 2h}\right)+.....+\left\{(n-1)h+e^{2(n-1)h}\right\}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+.....+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+....+e^{2(n-1)h}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+....+(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=\Rightarrow 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}\cdot\frac{(n-1)n}{2}+\left(\frac{e^{8}-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$=4(2)+4\lim_{n\to\infty}\frac{\left(e^{8}-1\right)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$=8+\frac{e^{8}-1}{2}$$

$$=\frac{e^{8}+15}{2}$$

Therefore, $\int_0^4 \left(x+e^{2x}\right)dx=\frac{e^{8}+15}{2}$