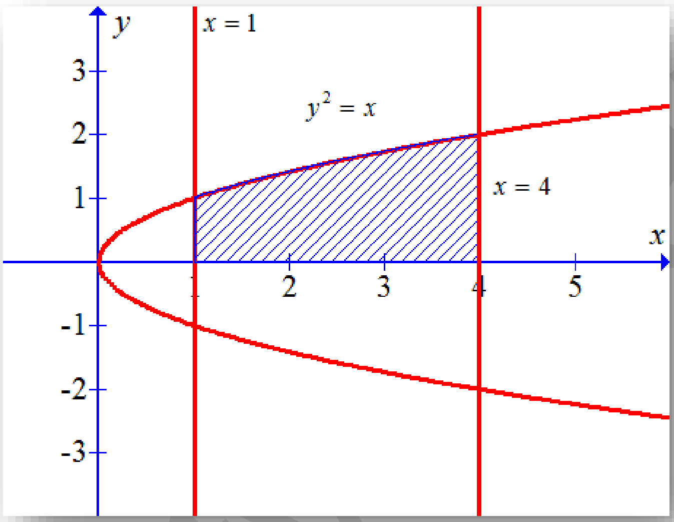
Chapter: 8. Applications of Integrals

Exercise 8.1

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the $x-$ axis in the first quadrant.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The area of the shaded region is the region bounded by the curve $y^2 = x$, the lines $x = 1, x = 4$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_1^4 \sqrt{x}dx$

$$A=\int_1^4 \sqrt{x}dx$$

$$=\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 \qquad \bullet \int x^n dx=\frac{x^{n+1}}{n+1}; n\neq -1$$

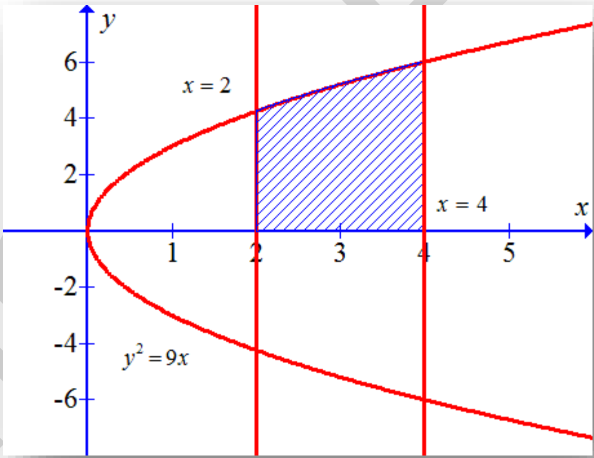
$$=\frac{2}{3}(8-1)$$

$$=\frac{14}{3}\text{square units}$$

Therefore, the area of the region bounded by the curve $y^2=x$ and the lines $x=1, x=4$ and the $x-$ axis in the first quadrant is $\frac{14}{3}$ square units

2. Find the area of the region bounded by $y^2=9x, x=2, x=4$ and the $x-$axis in the first quadrant.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The area of the shaded region is the region bounded by the curve $y^2=9x$, the lines $x=2, x=4$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A=\int_2^4 3\sqrt{x}dx$

$$A=3\int_{2}^{4}\sqrt{x}dx$$

$$=3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

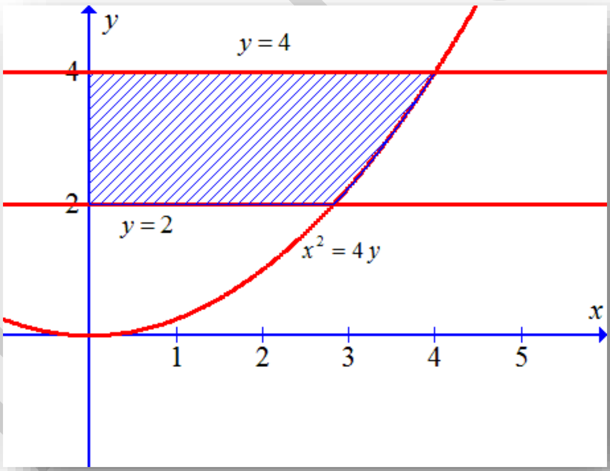
$$=3\left(\frac{2}{3}\right)\left(8-2^{\frac{3}{2}}\right)$$

$$=\left(16-4\sqrt{2}\right)\text{square units}$$

Therefore, the area of the region bounded by the curve $y^2=9x$ and the lines $x=1, x=4$ and the $x-$ axis in the first quadrant is $\left(16-4\sqrt{2}\right)$square units

3. Find the area of the region bounded by $x^2=4y, y=2, y=4$ and the $y-$axis in the first quadrant.

Solution: The required area of the region is the shaded region in the following figure



The area of the shaded region is the region bounded by the curve $x^2=4y$, the lines $y=2, y=4$ and the $y-$ axis in the first quadrant.

Hence, the required area is $A=\int_{2}^{4}2\sqrt{y}dy$

$$A = 2\int_2^4 \sqrt{y}\,dy$$

$$= 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

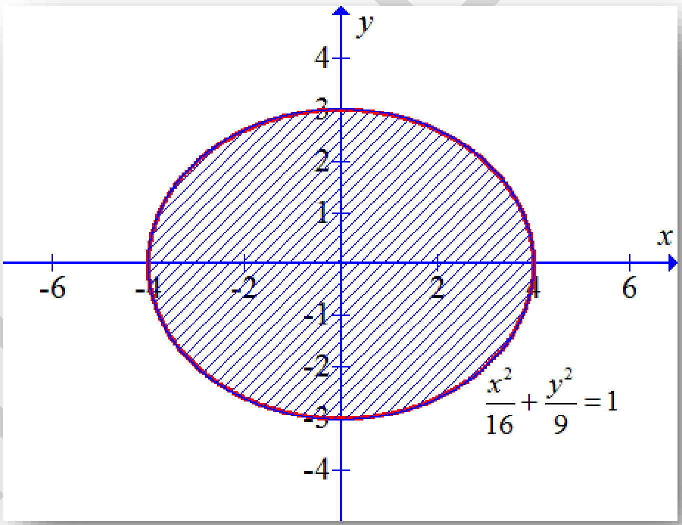
$$= 2\left(\frac{2}{3}\right)\left(8 - 2^{\frac{3}{2}}\right)$$

$$= \frac{32 - 8\sqrt{2}}{3} \text{square units}$$

Therefore, the area of the region bounded by the curve $x^2 = 4y$, the lines $y = 2, y = 4$ and the $y-$ axis in the first quadrant is $\frac{32 - 8\sqrt{2}}{3}$ square units

4. Find the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The given equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$ can be rewrite as below

$$\frac{y^2}{9}=1-\frac{x^2}{16}$$

$$y^2=9\left(1-\frac{x^2}{16}\right)$$

$$y=\frac{3}{4}\sqrt{16-x^2}$$

The area of the shaded region is 4 times the region bounded by the curve $y=\frac{3}{4}\sqrt{(16-x^2)}$, the lines $x=0, x=4$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A=4\left(\frac{3}{4}\int_0^4\sqrt{16-x^2}dx\right)$

$$A=3\int_0^4\sqrt{16-x^2}dx$$

$$=3\left\{\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right\}_0^4$$

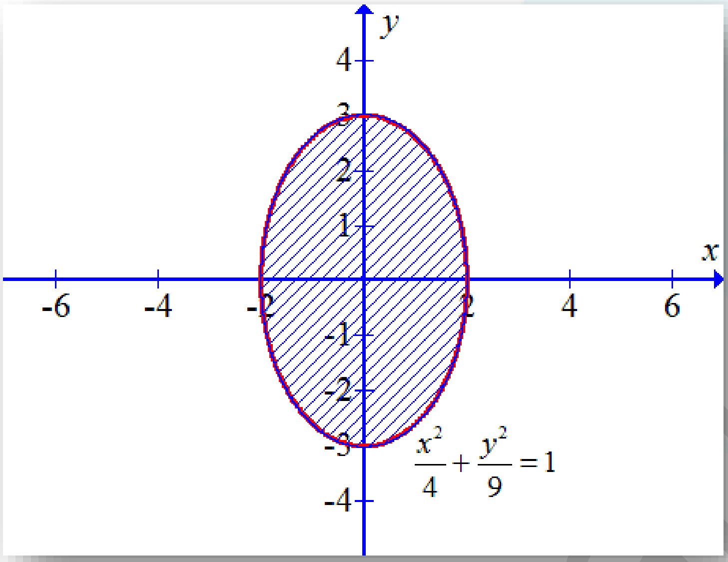
$$=3\left(8\cdot\frac{\pi}{2}\right)$$

$$=12\pi$$

Therefore, the area of the region bounded by the curve $\frac{x^2}{16}+\frac{y^2}{9}=1$ is 12π square units

5. Find the area of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The given equation $\frac{x^2}{4}+\frac{y^2}{9}=1$ can be rewrite as below

$$\frac{y^2}{9}=1-\frac{x^2}{4}$$

$$y^2=9\left(1-\frac{x^2}{4}\right)$$

$$y=\frac{3}{2}\sqrt{4-x^2}$$

The area of the shaded region is 4 times the region bounded by the curve $y=\frac{3}{2}\sqrt{\left(4-x^2\right)}$, the lines $x=0, x=4$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A=4\left(\frac{3}{2}\int_0^4 \sqrt{4-x^2}dx\right)$

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$$A = 6\int_0^4 \sqrt{4 - x^2}\,dx$$

$$= 6\left\{\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right\}_0^2$$

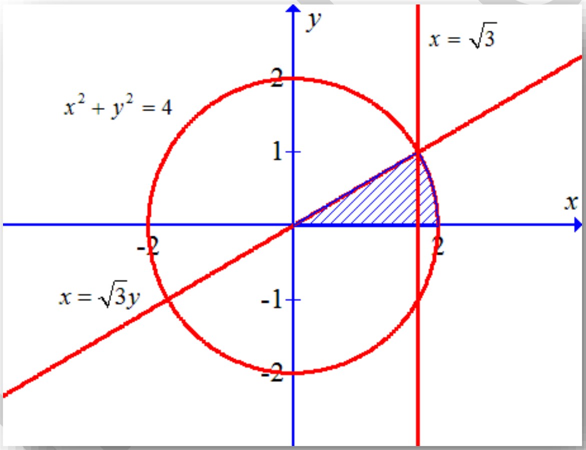
$$= 6\left(2 \cdot \frac{\pi}{2}\right)$$

$$= 6\pi$$

Therefore, the area of the region bounded by the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is 6π square units

6. Find the area of the region in the first quadrant enclosed by $x-$ axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is sum of the following regions

(i) The area of the shaded region bounded by the line $x = \sqrt{3}y$, lines $x = 0, x = \sqrt{3}$ and the $x-$ axis in the first quadrant, and it is equal to $\frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} x\,dx$

(ii) The area of the shaded region bounded by the curve $y=\sqrt{4-x^2}$, lines $x=\sqrt{3}, x=2$ and the $x-$ axis in the first quadrant, and it is equal to $\int_{\sqrt{3}}^{2}\sqrt{4-x^2}dx$

Hence, the required area is

$$A=\frac{1}{\sqrt{3}}\int_{0}^{\sqrt{3}}xdx+\int_{\sqrt{3}}^{2}\sqrt{4-x^2}dx$$

$$=\frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_0^{\sqrt{3}}+\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)_{\sqrt{3}}^{2}$$

$$=\frac{3}{2\sqrt{3}}+2\left(\frac{\pi}{2}\right)-\frac{\sqrt{3}}{2}-2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$=\frac{\sqrt{3}}{2}+\pi-\frac{\sqrt{3}}{2}-2\left(\frac{\pi}{3}\right)$$

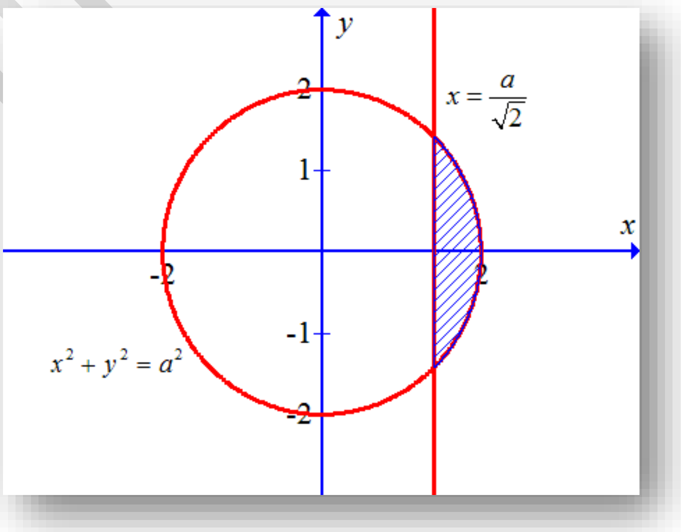
$$=\frac{\pi}{3}$$

Therefore, the area of the region in the first quadrant enclosed by $x-$ axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4$ is $\frac{\pi}{3}$ square units

7. Find the area of the smaller part of the circle $x^2+y^2=a^2$ cut off by the line $x=\frac{a}{\sqrt{2}}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is two times the area of the region bounded by the curve $y = \sqrt{a^2 - x^2}$, the lines $x = \frac{a}{\sqrt{2}}, x = a$ and $x-$ axis in the first quadrant.

Hence, the required area is

$$A = 2\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx$$

$$= 2\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right)_{\frac{a}{\sqrt{2}}}^{a}$$

$$= 2\left(\frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{a^2}{2} - \frac{a^2}{2}\left(\frac{\pi}{4}\right)\right)$$

$$= 2\left(\left(\frac{\pi a^2}{8}\right) - \frac{a^2}{2}\right)$$

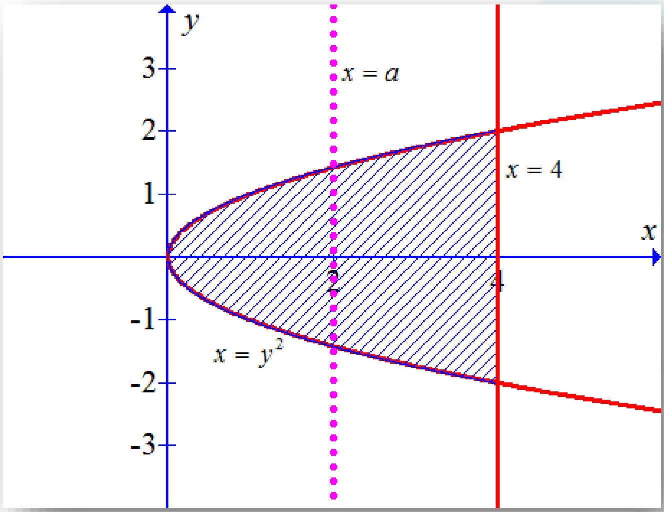
$$= \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$$

Therefore, the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$ is $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$ square units

8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$ then find the value of a

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

Suppose that the line $x = a$ divide the area between $x = y^2$ and $x = 4$

Hence,

$$\int_0^a \sqrt{x}dx = \frac{1}{2}\int_0^4 \sqrt{x}dx$$

$$\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a = \frac{1}{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$\frac{2}{3}a^{\frac{3}{2}} = \frac{1}{2}\left(\frac{2}{3}\right)(8)$$

$$a^{\frac{3}{2}} = 4$$

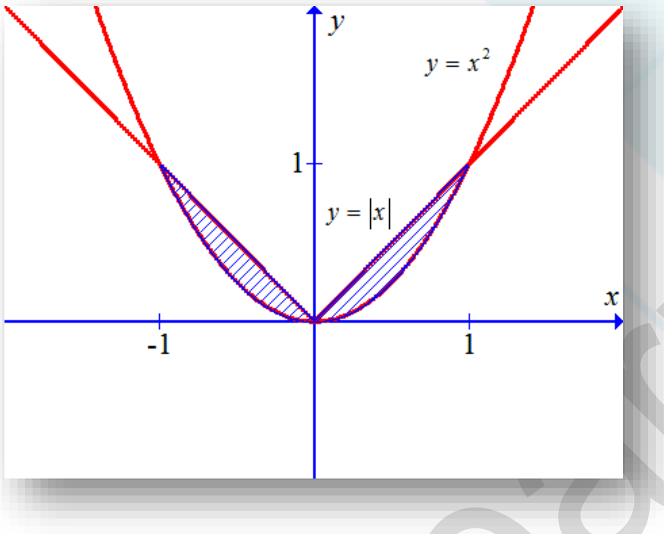
$$a = 4^{\frac{2}{3}}$$

Therefore, the value of a is $4^{\frac{2}{3}}$, such that the line $x = 4^{\frac{2}{3}}$ divide the region into two equal parts of the region bounded by $x = y^2$, the line $x = 4$

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

The area of the shaded region is two times the area of the region bounded by the curve $y = x^2$, the line $y = x$, the lines $x = 0, x = 1$ and $x-$ axis in the first quadrant.

Hence, the required area is $A = 2\int_0^1 \left(x - x^2\right)dx$

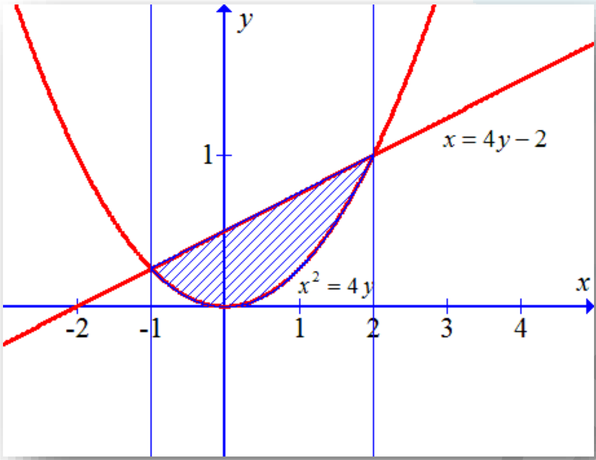
$$A = 2\int \left(x - x^2\right)dx$$
$$= 2\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right) + C$$
$$= \left(x^2 - \frac{2x^3}{3}\right)_0^1$$
$$= 1 - \frac{2}{3}$$
$$= \frac{1}{3}$$

Therefore, the area of the region bounded by the curve $y = x^2$ and the lines $y = |x|$ is $\frac{1}{3}$ square units

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$.

Points of intersection of the curve $x^2 = 4y$ and the line $x = 4y - 2$

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$x^2 - 2x + x - 2 = 0$$
$$x(x-2) + 1(x-2) = 0$$
$$(x+1)(x-2) = 0$$

Hence, the points of intersection of both curve and the line are $x = -1, x = 2$

The area of the shaded region is the area of the region bounded by the curve $x^2 = 4y$, the line $x = 4y - 2$, the lines $x = -1, x = 2$ and $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$

$$A=\int_{-1}^{2}\left(\frac{x+2}{4}-\frac{x^2}{4}\right)dx$$

$$=\frac{1}{4}\int_{-1}^{2}\left(x+2-x^2\right)dx$$

$$=\frac{1}{4}\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)_{-1}^{2}$$

$$=\frac{1}{4}\left(2+4-\frac{8}{3}-\frac{1}{2}+2-\frac{1}{3}\right)$$

$$=\frac{1}{4}\left(\frac{9}{2}\right)$$

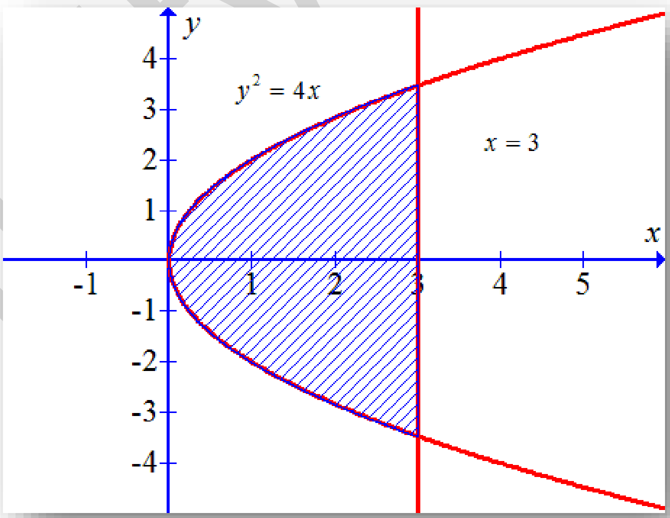
$$=\frac{9}{8}\text{square units}$$

Therefore, the area of the region bounded by the curve $x^2=4y$ and the line $x=4y-2$ is $\frac{9}{8}$ square units

11. Find the area of the region bounded by the curve $y^2=4x$ and the line $x=3$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $x=f(y)$, the lines $y=a, y=b$ and $y-$ axis is $\left|\int_{a}^{b} f(y)dy\right|$.

Points of intersection of the curve $y^2=4x$ and the line $x=3$

$y^2 = 12 \Rightarrow y = \pm 2\sqrt{3}$

The area of the shaded region is the area of the region bounded by the curve $x = \frac{y^2}{4}$,

the line $y = 2\sqrt{3}, y = -2\sqrt{3}$ and $y-$ axis.

Hence, the required area is $A = \int_{-2\sqrt{3}}^{2\sqrt{3}} \left(3 - \frac{y^2}{4}\right) dx$

$$A = \int_{-2\sqrt{3}}^{2\sqrt{3}} \left(3 - \frac{y^2}{4}\right) dx$$

$$= \left(3y - \frac{y^3}{12}\right)_{-2\sqrt{3}}^{2\sqrt{3}}$$

$$= 6\sqrt{3} - \frac{24\sqrt{3}}{12} + 6\sqrt{3} - \frac{24\sqrt{3}}{12}$$

$$= 12\sqrt{3} - 4\sqrt{3}$$

$$= 8\sqrt{3}$$

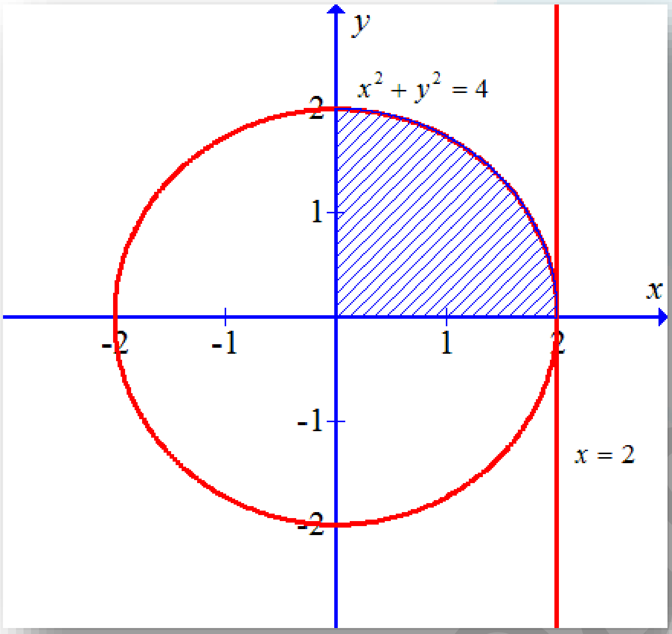
Therefore, the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$ is $8\sqrt{3}$ square units

12. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is in square units

1) π 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{4}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

The area of the shaded region is the area of the region bounded by the curve $y = \sqrt{4 - x^2}$, the line $x = 0, x = 2$ and $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_0^2 \sqrt{4 - x^2}\,dx$

$$A = \int_0^2 \sqrt{4 - x^2}\,dx$$

$$= \left(\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right)_0^2$$

$$= \left(2\sin^{-1}(1)\right)$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi$$

Therefore, the option 1 is correct

13. Area of the region bounded by the curve $y^2 = 4x$, $y-$ axis and the line $y = 3$in square units.

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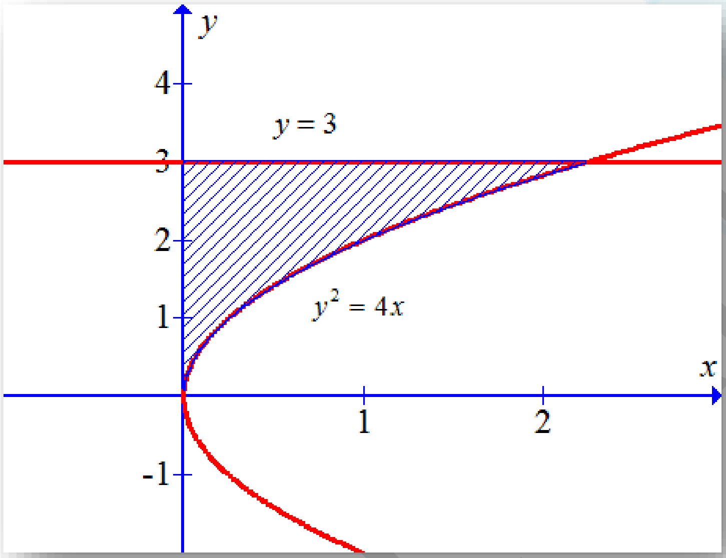
1) 2 2) $\frac{9}{4}$ 3) $\frac{9}{3}$ 4) $\frac{9}{2}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $x = f(y)$, the lines $y = a, y = b$ and $y-$ axis is $\left|\int_a^b f(y)dy\right|$.

The area of the shaded region is the area of the region bounded by the curve $x = \frac{y^2}{4}$, the line $y = 0, y = 3$ and $y-$ axis.

Hence, the required area is $A = \int_0^3 \left(\frac{y^2}{4}\right) dx$

$$A = \int_0^3 \left(\frac{y^2}{4}\right) dx$$

$$= \left(\frac{y^3}{12}\right)_0^3$$

$$= \frac{27}{12}$$

$$= \frac{9}{4} \text{square units}$$

Therefore, the option 2 is correct.

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