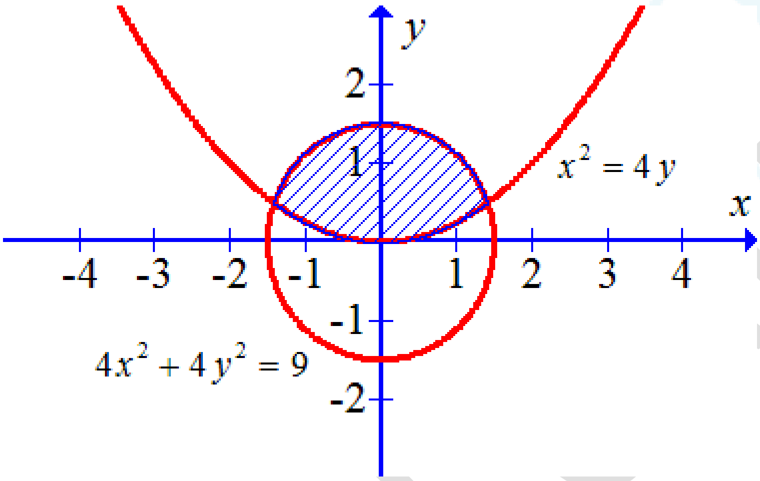
Chapter: 8. Applications of Integrals

Exercise 8.2

1. Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)dx\right|$

The point of intersection of the circle $4x^2+4y^2=9$ and the curve $x^2=4y$

$$4x^2+\left(\frac{x^2}{2}\right)^2=9$$

$$16x^2+x^4=36$$

$$x^4+16x^2-36=0$$

$$x^4+18x^2-2x^2-36=0$$

$$x^2\left(x^2+18\right)-2\left(x^2+18\right)=0$$

$$\left(x^2-2\right)\left(x^2+18\right)=0$$

Hence, $x=-\sqrt{2},\sqrt{2}$

The area of the shaded region is the region bounded by the curve

$y=\frac{1}{2}\sqrt{9-4x^2}, y=\frac{x^2}{4}$, the lines $x=-\sqrt{2}, x=\sqrt{2}$ and the $x-$ axis

Hence, the required area is $A=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\frac{1}{2}\sqrt{9-4x^2}-\frac{x^2}{4}\right)dx$

Hence,

$$A=\int_{-\sqrt{2}}^{\sqrt{2}}\left(\frac{1}{2}\sqrt{9-4x^2}-\frac{x^2}{4}\right)dx$$

$$=2\left(\frac{1}{4}\left(\frac{2x}{2}\sqrt{9-4x^2}+\frac{9}{2}\sin^{-1}\frac{2x}{3}\right)-\frac{x^3}{12}\right)_0^{\sqrt{2}}$$

$$=2\left(\frac{1}{4}\left(\frac{2\sqrt{2}}{2}+\frac{9}{2}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right)-\frac{2\sqrt{2}}{12}\right)$$

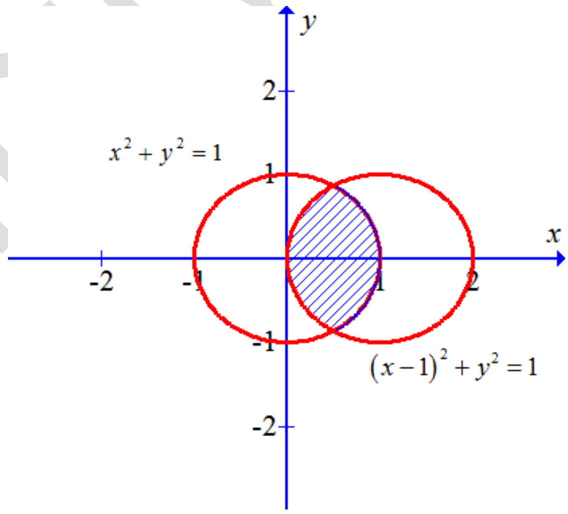
$$=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{3}$$

$$=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\frac{\sqrt{2}}{6}$$

Therefore area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$ is $\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\frac{\sqrt{2}}{6}$ square units

2. Find the area bounded by the curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y=f(x)$, the lines $x=a, x=b$ and $x-$ axis is $\left|\int_a^b f(x)\,dx\right|$.

Points of intersection of the both curves

$$(x-1)^2+1-x^2=1$$
$$(x-1)^2=x^2$$
$$x-1=-x$$
$$2x=1$$
$$x=\frac{1}{2}$$

Both circles intersect at $x=\frac{1}{2}$

The area of the shaded region is sum of the following two regions

(i) Two times the area of the region bounded by $y=\sqrt{1-(x-1)^2}$, the lines $x=0, x=\frac{1}{2}$ and $x-$ axis in the first quadrant.

(ii) Two times the area of the region bounded by the $y=\sqrt{1-x^2}$, the lines $x=\frac{1}{2}, x=1$ and $x-$ axis in the first quadrant.

Hence, the required area is $A=2\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx\right]$

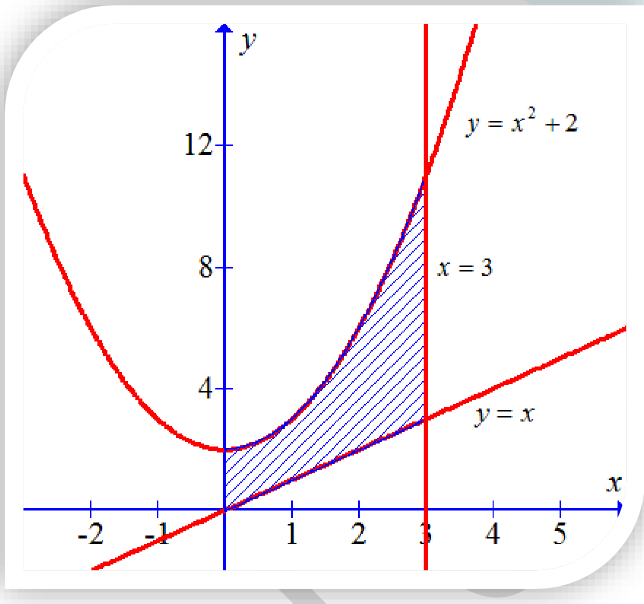
$$A=2\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx\right]$$
$$=2\left[\left[\frac{x-1}{2}\sqrt{1-(x-1)^2}+\frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}}+\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right]_{\frac{1}{2}}^{1}\right]$$
$$=2\left[\left[-\frac{1}{4}\frac{\sqrt{3}}{2}-\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)-\frac{1}{2}\left(\sin^{-1}(-1)\right)\right]+\left[\frac{1}{2}\sin^{-1}(1)-\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]\right]$$
$$=4\left[-\frac{1}{4}\frac{\sqrt{3}}{2}-\frac{\pi}{12}+\frac{\pi}{4}\right]$$
$$=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

Therefore, the area bounded by the curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$

$\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$square units

3. Find the area of the region bounded by the curve $y = x^2 + 2,\ y = x, x = 0$ and $x = 3$.

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded by two curves $y = f(x)$ and $y = g(x)$ the lines $x = a, x = b$ is $\int_a^b |f(x) - g(x)|\,dx$.

The area of the shaded region is the region bounded by the curve $y = x^2 + 2$, the lines $y = x, x = 0, x = 3$ and the $x-$ axis in the first quadrant.

Hence, the required area is $A = \int_0^3 (x^2 + 2 - x)\,dx$

$$A = \int_0^3 (x^2 + 2 - x)\,dx$$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$$

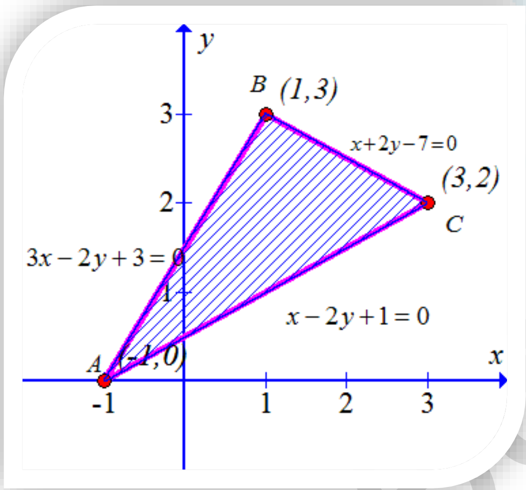
$$= 9 + 6 - \frac{9}{2}$$

$$= \frac{21}{2}$$

Therefore, the area of the region bounded by the curve $y = x^2 + 2,\ y = x, x = 0$ and $x = 3$. $\frac{21}{2}$ square units

4. Using the integration, find the area of the region bounded by the triangle whose vertices are $(-1,0),(1,3),(3,2)$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y = f(x)$ and $y = g(x)$ between the Lines $x = a, x = b$ is $\left|\int_a^b (f(x) - g(x))\,dx\right|$.

The equations of the sides of the triangle are as below

Equation of AB is $y = \frac{3}{2}(x+1)$

Equation of BC is $y = \frac{-x+7}{2}$

Equation of CA is $y = \frac{1}{2}(x+1)$

The area of the triangle is sum of the following areas

(i) The area of the region between the lines $AB, AC, x = -1, x = 1$

(ii) The area of the region between the lines $BC, AC, x = 1, x = 3$

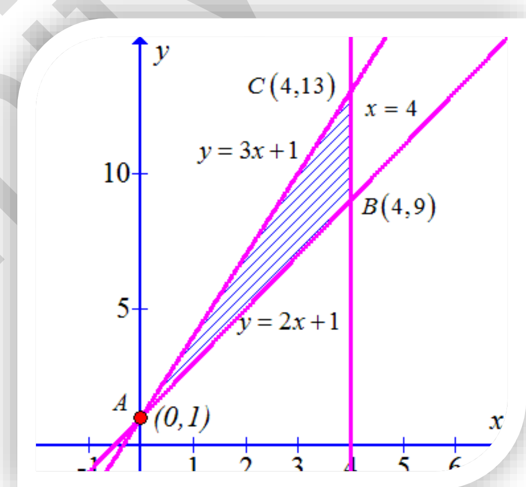
Hence, the area of the triangle is

$$A=\int_{-1}^{1}(AB-AC)\,dx+\int_{1}^{3}(BC-AC)\,dx$$
$$=\int_{-1}^{1}\frac{3}{2}(x+1)-\frac{1}{2}(x+1)\,dx+\int_{1}^{3}\left(\frac{1}{2}(-x+7)-\frac{1}{2}(x+1)\right)dx$$
$$=\int_{-1}^{1}(x+1)\,dx+\frac{1}{2}\int_{1}^{3}(-2x+6)\,dx$$
$$=\left[\frac{x^2}{2}+x\right]_{-1}^{1}+\frac{1}{2}\left[-x^2+6x\right]_1^3$$
$$=\frac{1}{2}+1-\frac{1}{2}+1+\frac{1}{2}(9+1-6)$$
$$=2+2$$
$$=4$$

Therefore, the area of the triangle formed by the points $(-1,0),(1,3),(3,2)$ is 4 square units.

5. Use the integration find the area of the triangular region whose sides have the equations $y=2x+1, y=3x+1$ and $x=4$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y=f(x)$ and $y=g(x)$ between the Lines $x=a, x=b$ is $\left|\int_a^b (f(x)-g(x))\,dx\right|$.

The vertices of triangle are

Equation of AB is $y=2x+1$

Equation of BC is $x = 4$

Equation of CA is $y = 3x + 1$

The area of the triangle is the area of the region between the lines $AB, AC, x = 0, x = 4$

Hence, the area of the triangle is

$$A = \int_0^4 (AC - AB)\,dx$$

$$= \int_0^4 (3x+1) - (2x+1)\,dx$$

$$= \int_0^4 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4$$

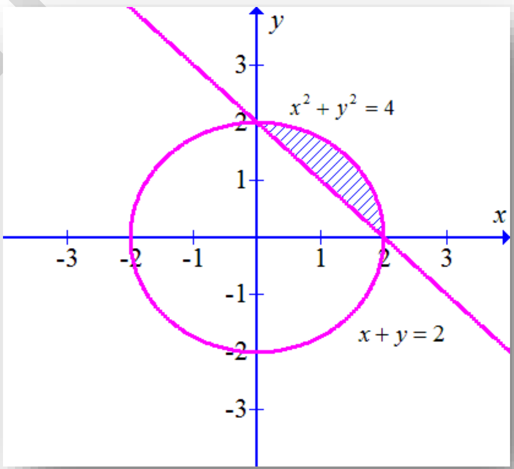
$$= 8 \text{ square units}$$

Therefore, the area of the region bounded by the lines $y = 2x+1, y = 3x+1$ and $x = 4$ is 8 square units

6. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

1) $2(\pi - 2)$ 2) $(\pi - 2)$ 3) $2\pi - 1$ 4) $2(\pi + 2)$

Solution: The required area of the region is the shaded region in the following figure



The area of the region bounded between two curves $y = f(x)$ and $y = g(x)$ between the Lines $x = a, x = b$ is $\left|\int_a^b \left(f(x) - g(x)\right)dx\right|$.

The area of the shaded region is area of the region bounded by the curve $y = \sqrt{4 - x^2}$ and $y = 2 - x$, lines $x = 0, x = 2$ in the first quadrant.

Hence, the required area is

$$A = \int_0^2 \sqrt{4 - x^2} - (2 - x)\,dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 2\sin^{-1}(1) - 4 + 2$$

$$= \pi - 2$$

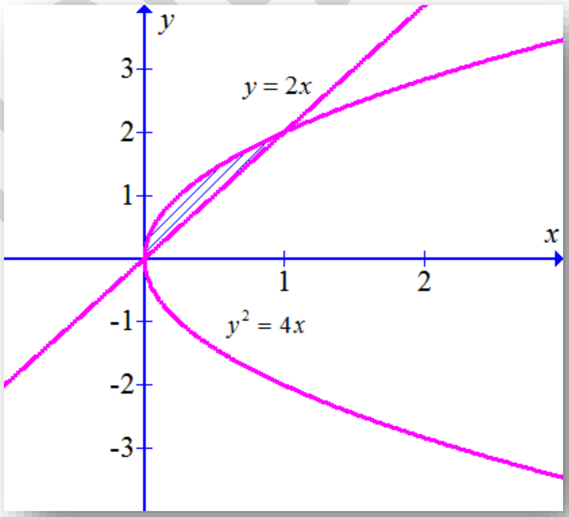
Therefore, option 2 is correct

7. Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

1) $\frac{2}{3}$ 2) $\frac{1}{3}$ 3) $\frac{1}{4}$ 4) $\frac{3}{4}$

Solution:

The required area of the region is the shaded region in the following figure



The area of the region bounded by the curve $y = f(x)$, the lines $x = a, x = b$ and $x-$ axis is $\left| \int_a^b f(x)\,dx \right|$.

The point of intersection of the curve $y^2 = 4x$ and the line $y = 2x$

$$(2x)^2 = 4x$$

$$4x^2 = 4x$$

$$x(x-1) = 0$$

$$x = 0 \text{ or } x = 1$$

The area of the shaded region is the area of the region bounded by the curves $y^2 = 4x$ and the line $y = 2x$, lines $x = 0, x = 1$ and $x-$ axis in the first quadrant.

Hence, the required area is

$$A = \int_0^1 \left(2\sqrt{x} - 2x\right) dx$$

$$= 2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2}\right)_0^1$$

$$= 2\left(\frac{2}{3} - \frac{1}{2}\right)$$

$$= 2\left(\frac{1}{6}\right)$$

$$= \frac{1}{3}$$

Therefore, option 2 is correct