Chapter: 9. Differential Equations

Exercise: Miscellaneous

1. For each of the differential equations given below, indicate its order and degree (if defined)

i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

Solution:

The given differential equation is $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

The highest order derivative in the equation is of the term $\frac{d^2y}{dx^2}$, thus the order of the equation is 2 and its highest power is 1. Therefore its degree is 1

ii) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

Solution:

The given differential equation is $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y-\sin x=0$

The highest order derivative in the equation is of the term $\left(\frac{dy}{dx}\right)^3$, thus the order of the equation is 1 and its highest power is 3. Therefore its degree is 3

iii) $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Solution:

The given differential equation is $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

The highest order derivative in the equation is of the term $\frac{d^4y}{dx^4}$, thus the order of the equation is 4.

As the differential equation is not polynomial in its derivative, therefore its degree is not

defined.

2. For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

i) $xy = ae^{x} + be^{-x} + x^{2} : x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx} - xy + x^{2} - 2 = 0$

Solution:

The given function is $xy = ae^{x} + be^{-x} + x^{2}$

Take derivative on both side:

$$\Rightarrow y + x\frac{dy}{dx} = ae^{x} - be^{-x} + 2x$$

Take derivative on both side:

$$\Rightarrow \frac{dy}{dx} + x\frac{d^{2}y}{dx^{2}} + \frac{dy}{dx} = ae^{x} + be^{-x} + 2$$

$$\Rightarrow x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx} = ae^{x} + be^{-x} + 2..........(1)$$

The given differential equation is

$$x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx} - xy + x^{2} - 2 = 0$$

Solving LHS

Substitute $x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx}$ from the result (1) and xy

$$\Rightarrow \left(x\frac{d^{2}y}{dx^{2}} + 2\frac{dy}{dx}\right) - xy + x^{2} - 2$$

$\Rightarrow \left(ae^{x}+be^{-x}+2\right)-\left(ae^{x}+be^{-x}+x^{2}\right)+x^{2}-2$

$\Rightarrow 2-x^{2}+x^{2}-2$

$\Rightarrow 0$

Thus LHS = RHS the given function is the solution of the given differential equation

ii) $y=e^{x}\left(a\cos x+b\sin x\right):\frac{d^{2}y}{dx^{2}}-2\frac{dy}{dx}+2y=0$

Solution:

The given function is $y=e^{x}\left(a\cos x+b\sin x\right)$

Take derivative on both sides

$\Rightarrow \frac{dy}{dx}=e^{x}\left(a\cos x+b\sin x\right)+e^{x}\left(-a\sin x+b\cos x\right)$

$\Rightarrow \frac{dy}{dx}=e^{x}\left(\left(a+b\right)\cos x+\left(b-a\right)\sin x\right)$

Take derivative on both side

$\Rightarrow \frac{d^{2}y}{dx^{2}}=e^{x}\left(\left(a+b\right)\cos x+\left(b-a\right)\sin x\right)+e^{x}\left(-\left(a+b\right)\sin x+\left(b-a\right)\cos x\right)$

$\Rightarrow \frac{d^{2}y}{dx^{2}}=e^{x}\left(\left(a+b+b-a\right)\cos x+\left(b-a-a-b\right)\sin x\right)$

$\Rightarrow \frac{d^{2}y}{dx^{2}}=e^{x}\left(2b\cos x-2a\sin x\right)$

The given differential equation is

$\frac{d^{2}y}{dx^{2}}-2\frac{dy}{dx}+2y=0$

Solving LHS

$\Rightarrow e^{x}\left(2b\cos x-2a\sin x\right)-2e^{x}\left(\left(a+b\right)\cos x+\left(b-a\right)\sin x\right)+2y$

$$\Rightarrow e^x\left(\left(2b-2a-2b\right)\cos x+\left(-2a-2b+2a\right)\sin x\right)-2y$$

$$\Rightarrow e^x\left(-2a\cos x-2b\sin x\right)-2y$$

$$\Rightarrow -2e^x\left(a\cos x+b\sin x\right)-2y$$

$$\Rightarrow 0$$

Thus LHS = RHS, the given function is the solution of the given differential equation

iii) $y=x\sin 3x:\frac{d^2y}{dx^2}+9y-6\cos 3x=0$

Solution:

The given function is $y=x\sin 3x$

Take derivative on both side

$$\Rightarrow \frac{dy}{dx}=\sin 3x+3x\cos 3x$$

Take derivative on both side

$$\Rightarrow \frac{d^2y}{dx^2}=3\cos 3x+3\left(\cos 3x+x\left(-3\sin 3x\right)\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}=3\cos 3x+3\cos 3x-9x\sin 3x$$

$$\Rightarrow \frac{d^2y}{dx^2}=6\cos 3x-9x\sin 3x$$

The given differential equation is

$$\frac{d^2y}{dx^2}+9y-6\cos 3x=0$$

Solving LHS

$$\Rightarrow \frac{d^2y}{dx^2}+9y-6\cos 3x$$

$$\Rightarrow (6\cos 3x - 9x\sin 3x) + 9(x\sin 3x) - 6\cos 3x$$

$$\Rightarrow 6\cos 3x - 9x\sin 3x + 9x\sin 3x - 6\cos 3x$$

Thus LHS = RHS, the given function is the solution of the given differential equation.

iv) $x^2 = 2y^2 \log y : (x^2 + y^2)\frac{dy}{dx} - xy = 0$

Solution:

The given function is $x^2 = 2y^2 \log y$

Take derivative on both side

$$\Rightarrow 2x = 2\left(2y\log y + y^2\left(\frac{1}{y}\right)\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{(2y\log y + y)}$$

Multiply numerator and denominator by y

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{(2y^2\log y + y^2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{(x^2 + y^2)}$$

The given differential equation is

$$(x^2 + y^2)\frac{dy}{dx} - xy = 0$$

Solving LHS

$$\Rightarrow (x^2 + y^2)\frac{dy}{dx} - xy$$

$$\Rightarrow (x^2 + y^2)\left(\frac{xy}{x^2 + y^2}\right) - xy$$

$$\Rightarrow xy - xy$$

$\Rightarrow 0$

Thus LHS = RHS, the given function is the solution of the given differential equation

3. Form the differential equation representing the family of curves given by $(x-a)^2 + 2y^2 = a^2$, where a is an arbitrary constant

Solution:

Given family of curve $(x-a)^2 + 2y^2 = a^2$

$$\Rightarrow x^2 - 2ax + a^2 + 2y^2 = a^2$$

$$\Rightarrow x^2 + 2y^2 = 2ax ....(1)$$

Differentiate both side with respect to x

$$\Rightarrow 2(x-a) + 4y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2(x-a)}{4y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a-2x}{4y}$$

Multiply numerator and denominator by x

$$\Rightarrow \frac{dy}{dx} = \frac{2ax-2x^2}{4xy}$$

Using expression (1) back substitute 2ax

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+2y^2-2x^2}{4xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2-x^2}{4xy}$$

Thus the differential equation for given family of curve is $\frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$

4. Prove that $x^2-y^2=c\left(x^2+y^2\right)^2$ is the general solution of differential equation $\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$, where c is a parameter

Solution:

Given differential equation

$$\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y}$$

As it can be seen that this is an homogenous equation. Substitute $y=vx$

$$\Rightarrow \frac{d(vx)}{dx}=\frac{x^3-3x(vx)^2}{(vx)^3-3x^2(vx)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{x^3\left(1-3v^2\right)}{x^3\left(v^3-3v\right)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2-v^4+3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v^4}{v^3-3v}$$

Separate the differential

$$\frac{v^3-3v}{1-v^4}dv=\frac{dx}{x}$$

Integrate both side

$$\int\frac{v^3-3v}{1-v^4}dv=\int\frac{dx}{x}$$

$$\Rightarrow\int\frac{v^3-3v}{1-v^4}dv=\log x+\log C$$

$$\Rightarrow I=\log x+\log C\left(I=\int\frac{v^3-3v}{1-v^4}dv\right)............(1)$$

Solving integral I

$$\Rightarrow I=\int\frac{v^3-3v}{1-v^4}dv$$

$$\Rightarrow\frac{v^3-3v}{1-v^4}=\frac{v^3-3v}{\left(1-v^2\right)\left(1+v^2\right)}$$

$$\Rightarrow\frac{v^3-3v}{1-v^4}=\frac{v^3-3v}{(1-v)(1+v)\left(1+v^2\right)}$$

Using partial fraction

$$\Rightarrow\frac{v^3-3v}{1-v^4}=\frac{A}{1-v}+\frac{B}{1+v}+\frac{Cv+D}{1+v^2}$$

Solving for A, B, C and D

$$A=-\frac{1}{2}$$

$$B=\frac{1}{2}$$

$$C=-2$$

$$D=0$$

$$\Rightarrow\frac{v^3-3v}{1-v^4}=\frac{-\frac{1}{2}}{1-v}+\frac{\frac{1}{2}}{1+v}+\frac{-2v+0}{1+v^2}$$

$$I=-\frac{1}{2}\int\frac{1}{1-v}dv+\frac{1}{2}\int\frac{1}{1+v}dv-\int\frac{2v}{1+v^2}dv$$

$$I=-\frac{1}{2}\left(-\log\left(1-v\right)\right)+\frac{1}{2}\left(\log\left(1+v\right)\right)-\log\left(1+v^2\right)$$

$$I=\frac{1}{2}\left(\log\left(1-v^2\right)\right)-\frac{2}{2}\log\left(1+v^2\right)$$

$$I=\frac{1}{2}\left(\log\frac{\left(1-v^2\right)}{\left(1+v^2\right)^2}\right)$$

$$\Rightarrow I=\frac{1}{2}\left(\log\frac{\left(1-\frac{y^2}{x^2}\right)}{\left(1+\frac{y^2}{x^2}\right)^2}\right)$$

$$\Rightarrow I=\frac{1}{2}\left(\log\frac{x^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right)$$

$$\Rightarrow I=\frac{1}{2}\left(\log\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right)+\frac{1}{2}\log x^2$$

$$\Rightarrow I=\frac{1}{2}\left(\log\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right)+\log x$$

Back substitute I in expression (1)

$$\Rightarrow\frac{1}{2}\left(\log\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right)+\log x=\log x+\log C$$

$$\Rightarrow\log\frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}=2\log C$$

$$\Rightarrow \frac{x^2-y^2}{\left(x^2+y^2\right)^2}=C^2$$

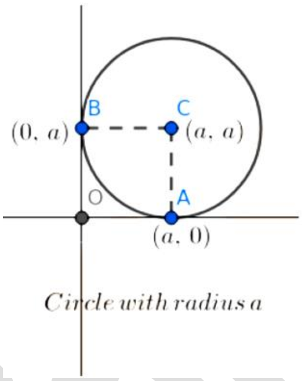
$$\Rightarrow x^2-y^2=c\left(x^2+y^2\right)^2=\left(c=C^2\right)$$

Thus for given differential equation, its general solution is $x^2-y^2=c\left(x^2+y^2\right)^2$

5. Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes

Solution:

Draw a circle according to question



The equation of the given circle will be

$$(x-a)^2+(y-a)^2=a^2$$

Differentiate both side with respect to x

$$\Rightarrow 2(x-a)+2(y-a)\frac{dy}{dx}=0$$

$$\Rightarrow x-a+yy'-ay'=0$$

$$\Rightarrow a=\frac{x+yy'}{1+y'}$$

Using equation of circle

$$\Rightarrow (x-a)^2+(y-a)^2=a^2$$

$$\Rightarrow \left(x-\left(\frac{x+yy'}{1+y'}\right)\right)^2+\left(y-\left(\frac{x+yy'}{1+y'}\right)\right)^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow \left(\frac{x+xy'-x-yy'}{1+y'}\right)^2+\left(\frac{y+yy'-x-yy'}{1+y'}\right)^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow \left(\frac{y'(x-y)}{1+y'}\right)^2+\left(\frac{y-x}{1+y'}\right)^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow (x-y)^2\left(1+y'^2\right)=(x+yy')^2$$

Thus the differential equation for given family of curves is

$$(x-y)^2\left(1+y'^2\right)=(x+yy')^2$$

6. Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

Solution:

The given differential equation is $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$

$$\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=-\frac{dx}{\sqrt{1-x^2}}$$

Integrate both side

$$\Rightarrow \int\frac{dy}{\sqrt{1-y^2}}=-\int\frac{dx}{\sqrt{1-x^2}}$$

$\Rightarrow \sin^{-1} y = -\sin^{-1} x + C$

$\Rightarrow \sin^{-1} y + \sin^{-1} x = C$

Thus the general solution of given differential equation is $\sin^{-1} y + \sin^{-1} x = C$

7. Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ is given by $(x+y+1) = A(1-x-y-2xy)$ where A is a parameter

Solution:

The given differential equation is $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+2\left(\frac{1}{2}\right)y+\frac{1}{4}-\frac{1}{4}+1}{x^2+2\left(\frac{1}{2}\right)x+\frac{1}{4}-\frac{1}{4}+1}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$\Rightarrow \frac{dy}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}} = -\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Integrate both side

$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{4}\right)^2} = -\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{4}\right)^2}$$

$$\Rightarrow \frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]=-\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+C$$

$$\Rightarrow \frac{2}{\sqrt{3}}\left(\tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]\right)=C$$

$$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]=\frac{\sqrt{3}}{2}C$$

Thus the general solution for given differential equation is

$$\tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]=\frac{\sqrt{3}}{2}C$$

8. Find the equation of the curve passing through the point $\left(0,\frac{\pi}{4}\right)$ whose differential equation is $\sin x\cos ydx+\cos x\sin ydy=0$

Solution:

Given differential equation is

$$\sin x\cos ydx+\cos x\sin ydy=0$$

$$\Rightarrow \sin x\cos ydx+\cos x\sin ydy=0$$

Divide both side by $\cos x\cos y$

$$\Rightarrow \frac{\sin x\cos ydx+\cos x\sin ydy}{\cos x\cos y}=0$$

$$\Rightarrow \tan xdx+\tan ydy=0$$

$$\Rightarrow \tan y\, dy-\tan x\, dx$$

Integrate both side

$$\Rightarrow \int \tan y\, dy-\int \tan x\, dx$$

$\Rightarrow \log(\sec y) = -\log(\sec x) + C$

$\Rightarrow \log(\sec y) + \log(\sec x) = C$

$\Rightarrow \log(\sec x \sec y) = C$

$\Rightarrow \sec x \sec y = k \left(k = e^C\right)$

As curve passes through $\left(0, \frac{\pi}{4}\right)$

$\sec 0 \sec\left(\frac{\pi}{4}\right) = k$

$\Rightarrow k = \sqrt{2}$

$\Rightarrow \sec x \sec y = \sqrt{2}$

Thus the equation of required curve is $\sec x \sec y = \sqrt{2}$

9. Find the particular solution of the differential equation $\left(1+e^{2x}\right)dy + \left(1+y^2\right)e^x dx = 0$ given that $y = 1$ when $x = 0$

Solution:

The given differential equation is $\left(1+e^{2x}\right)dy + \left(1+y^2\right)e^x dx = 0$

Divide both side $\left(1+e^{2x}\right)\left(1+y^2\right)$

$$\frac{dy}{\left(1+y^2\right)} + \frac{e^x}{\left(1+e^{2x}\right)}dx = 0$$

$$\int \frac{dy}{\left(1+y^2\right)} = -\int \frac{e^x}{\left(1+e^{2x}\right)}dx$$

$$\tan^{-1} y = -\int \frac{e^x}{\left(1+\left(e^x\right)^2\right)}dx$$

Substitute $t = e^x$

$dt = e^x dx$

$\Rightarrow \tan^{-1} y = -\int \frac{1}{(1+t^2)} dt$

$\Rightarrow \tan^{-1} y = -\tan^{-1} t + C$

$\Rightarrow \tan^{-1} y = -\tan^{-1} e^x + C$

$\Rightarrow \tan^{-1} y + \tan^{-1} e^x = C$

As $y = 1$ when $x = 0$

$\tan^{-1}(1) + \tan^{-1}(e^0) = C$

$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = C$

$\Rightarrow C = \frac{\pi}{2}$

$\Rightarrow \tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{2}$

Thus the required particular solution is $\tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{2}$

10. Solve the differential equation $ye^{\frac{x}{y}} dx = \left(xe^{\frac{x}{y}} + y^2 \right) dy \, (y \neq 0)$

Solution:

The given differential equation is $ye^{\frac{x}{y}} dx = \left(xe^{\frac{x}{y}} + y^2 \right) dy$

$ye^{\frac{x}{y}} \frac{dx}{dy} = xe^{\frac{x}{y}} + y^2$

$$\Rightarrow ye^{\frac{x}{y}}\frac{dx}{dy}-xe^{\frac{x}{y}}=y^2$$

$$\Rightarrow e^{\frac{x}{y}}\frac{\left[y\frac{dx}{dy}-x\right]}{y^2}=1$$

Substitute $z=e^{\frac{x}{y}}$

$$z=e^{\frac{x}{y}}$$

$$\frac{d}{dz}z=\frac{d}{dy}e^{\frac{x}{y}}$$

$$\Rightarrow \frac{dz}{dy}=\frac{d}{dy}\left(e^{\frac{x}{y}}\right)$$

$$\Rightarrow \frac{dz}{dy}=e^{\frac{x}{y}}\frac{d}{dy}\left(\frac{x}{y}\right)$$

$$\Rightarrow \frac{dz}{dy}=e^{\frac{x}{y}}\left[\left(\frac{1}{y}\right)\frac{dx}{dy}-\frac{x}{y^2}\right]$$

$$\Rightarrow \frac{dz}{dy}=e^{\frac{x}{y}}\left[\frac{y\frac{dx}{dy}-x}{y^2}\right]$$

$$\Rightarrow \frac{dz}{dy}=1$$

$$\Rightarrow dz=dy$$

$$\Rightarrow \int dz=\int dy$$

$$\Rightarrow z=y+C$$

$$\Rightarrow e^{\frac{x}{y}}=y+C$$

Thus the required general solution is $e^{\frac{x}{y}} = y + C$

11. Find a particular solution of the differential equation $(x-y)(dx+dy) = dx - dy$ given that $y = -1$ when $x = 0$. Hint (put $x - y = t$)

Solution:

Given differential equation is $(x-y)(dx+dy) = dx - dy$

$$\Rightarrow (x-y)dx - dx = (y-x)dy - dy$$

$$\Rightarrow (x-y+1)dy = (1-x+y)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-x+y}{x-y+1}$$

Put $x - y = t$

$$\Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 1 - \frac{dt}{dx} = \frac{dy}{dx}$$

$$\Rightarrow 1 - \frac{dt}{dx} = \frac{1-t}{1+t}$$

$$\Rightarrow \frac{dt}{dx} = 1 - \frac{1-t}{1+t}$$

$$\Rightarrow \frac{dt}{dx} = \frac{1+t-1+t}{1+t}$$

$$\Rightarrow \frac{dt}{dx} = \frac{2t}{1+t}$$

$$\Rightarrow \frac{1+t}{t}dt = 2dx$$

Integrating both side

$$\Rightarrow \int \frac{1+t}{t} dt = 2\int dx$$

$$\Rightarrow \int \frac{1}{t} dt + \int dt = 2x + C$$

$$\Rightarrow \log|t| + t = 2x + C$$

$$\Rightarrow \log|x-y| + x - y = 2x + C$$

$$\Rightarrow \log|x-y| - y = x + C$$

As y = -1 when x = 0

$$\Rightarrow \log|0-(-1)| - (-1) = 0 + C$$

$$\Rightarrow \log 1 + 1 = C$$

$$\Rightarrow C = 1$$

Thus the required particular solution is $\log|x-y| - y = x + 1$

12. Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1 \left(x \neq 0\right)$

Solution:

Given differential equation is $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is linear differential equation of the form $\Rightarrow \frac{dy}{dx} + py = Q$

$$p=\frac{1}{\sqrt{x}}$$

$$Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Calculating integrating factor

$$I.F=e^{\int pdx}$$

$$I.F=e^{\int\frac{1}{\sqrt{x}}dx}$$

$$I.F=e^{2\sqrt{x}}$$

The general solution is given by

$$y\times I.F=\int(Q\times IF)dx+C$$

$$y\times\left(e^{2\sqrt{x}}\right)=\int\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\times e^{2\sqrt{x}}\right)dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=\int\left(\frac{e^{-2\sqrt{x}+2\sqrt{x}}}{\sqrt{x}}\right)dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=2\sqrt{x}+C$$

Thus the general solution for the given differential equation is

$$ye^{2\sqrt{x}}=2\sqrt{x}+C$$

13. Find a particular solution of the differential equation $\frac{dy}{dx}+y\cot x=4x\cos ecx\,(x\neq 0)$ given that $y=0$ when $x=\frac{\pi}{2}$

Solution:

The given differential equation is

$$\frac{dy}{dx}+y\cot x=4x\operatorname{cos}ecx$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$

$$p=\cot x$$
$$Q=4x\operatorname{cos}ec\,x$$

Calculating integrating factor

$$I.F=e^{\int pdx}$$

$$I.F=e^{\int \cot xdx}$$

$$I.F=e^{\log|\sin x|}$$

$$I.F=\sin x$$

The general solution is given by

$$y\times I.F=\int(Q\times I.F)dx+C$$

$$\Rightarrow y\times\sin x=\int(4x\operatorname{cos}ecx)\sin x\,dx+C$$

$$\Rightarrow y\sin x=4\int x\,dx+C$$

$$\Rightarrow y\sin x=4\left(\frac{x^2}{2}\right)+C$$

$$\Rightarrow y\sin x=2x^2+C$$

As $y=0$ when $x=\frac{\pi}{2}$

$$0\times\sin\left(\frac{\pi}{2}\right)=2\left(\frac{\pi}{2}\right)^2+C$$

$C=-2\left(\frac{\pi^2}{4}\right)$

$C=-\frac{\pi^2}{2}$

Thus the required particular solution is

$y\sin x=2x^2-\frac{\pi^2}{2}$

14. Find a particular solution of the differential equation $(x+1)\frac{dy}{dx}=2e^{-y}-1$ given that $y=0$ when $x=0$. $(x+1)\frac{dy}{dx}=2e^{-y}-1$

Solution:

The given differential equation is $(x+1)\frac{dy}{dx}=2e^{-y}-1$

$\Rightarrow \frac{dy}{2e^{-y}-1}=\frac{dx}{x+1}$

Integrate both side

$\Rightarrow \int \frac{dy}{2e^{-y}-1}=\int \frac{dx}{x+1}$

$\Rightarrow \int \frac{dy}{2e^{-y}-1}=\log(x+1)+\log C.........(1)$

Evaluating LHS integral

$\Rightarrow \int \frac{dy}{2e^{-y}-1}=\int \frac{e^y dy}{2-e^y}$

Put $t=2-e^y$

$t=2-e^y$

$dt = e^{y} dy$

$$\Rightarrow \int \frac{dy}{2e^{-y}-1} = -\int \frac{dt}{t}$$

$$\Rightarrow \int \frac{dy}{2e^{-y}-1} = \log \frac{1}{t}$$

$$\Rightarrow \int \frac{dy}{2e^{-y}-1} = \log \frac{1}{2-e^{y}}$$

Back substituting in expression (1)

$$\Rightarrow \int \frac{dy}{2e^{-y}-1} = \log(x+1) + \log C$$

$$\Rightarrow \log\left(\frac{1}{2-e^{y}}\right) = \log C(x+1)$$

$$\Rightarrow 2-e^{y} = \frac{1}{C(x+1)}$$

As $y = 0$ when $x = 0$

$$\Rightarrow 2-e^{0} = \frac{1}{C(0+1)}$$

$$\Rightarrow 2-1 = \frac{1}{C}$$

$$\Rightarrow C = 1$$

Thus the required particular solution is

$$\Rightarrow 2-e^{y} = \frac{1}{(x+1)}$$

$$\Rightarrow e^{y} = 2-\frac{1}{(x+1)}$$

$$\Rightarrow e^{y} = \frac{2x+2-1}{(x+1)}$$

$$\Rightarrow e^y = \frac{2x+1}{(x+1)}$$

$$\Rightarrow y = \log\left(\frac{2x+1}{x+1}\right)$$

Thus for given conditions the particular solution is $y = \log\left(\frac{2x+1}{x+1}\right)$

15. The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Solution:

Let the population at time t be y. The according to question $\frac{dy}{dt} \propto y$

$$\Rightarrow \frac{dy}{dt} = ky$$

$$\Rightarrow \frac{dy}{y} = kdt$$

Integrate both side

$$\Rightarrow \int \frac{dy}{y} = k\int dt$$

$$\Rightarrow \log y = kt + C$$

In 1999 (taking reference year) t = 0 population was y = 20000

$$\Rightarrow \log(20000) = k(0) + C$$

$$\Rightarrow C = \log(20000)$$

In 2004, t =5 population was y = 25000

$$\Rightarrow \log(25000) = k(5) + \log(20000)$$

$$\Rightarrow 5k = \log(25000) - \log(20000)$$

$$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right)$$

$$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right)$$

Thus the population relation will be

$$\log y = \frac{1}{5}\log\left(\frac{5}{4}\right)t + \log(20000)$$

For 2009, t =10

$$\Rightarrow \log y = \frac{1}{5}\log\left(\frac{5}{4}\right)\times 10 + \log(20000)$$

$$\Rightarrow \log y = 2\log\left(\frac{5}{4}\right) + \log(20000)$$

$$\Rightarrow \log y = \log\left(\frac{5}{4}\right)^2 + \log(20000)$$

$$\Rightarrow \log y = \log\left(\frac{25}{16}\times 20000\right)$$

$$\Rightarrow y = 31250$$

Thus the population in the year 2009, is 31250

16. The general solution of the differential equation $\frac{ydx - xdy}{y} = 0$

Solution:

Given differential equation

$$\frac{ydx - xdy}{y} = 0$$

Divide both side by x

$$\Rightarrow \frac{ydx - xdy}{xy} = 0$$

$$\Rightarrow \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrate both side

$$\Rightarrow \int \frac{dx}{x} - \int \frac{dy}{y} = 0$$

$$\Rightarrow \log\left|\frac{x}{y}\right| = \log k$$

$$\Rightarrow \frac{x}{y} = k$$

$$\Rightarrow y = Cx\left(C = \frac{1}{k}\right)$$

Thus the correct option is (C)

17. Find the general solution of a differential equation of the type $\frac{dx}{dy} + P_1x = Q_1$

Solution:

The given differential equation is

$$\frac{dx}{dy} + P_1x = Q_1$$

It is linear differential equation and its general solution is

$$xe^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$$

With integrating factor $I.F = e^{\int P_1 dy}$

Thus the correct option is (C)

18. Find the general solution of the differential equation $e^x dy + \left(ye^x + 2x\right)dx = 0$

Solution:

The given differential equation is

$$e^x dy + \left(ye^x + 2x\right)dx = 0$$

$$\Rightarrow e^x \frac{dy}{dx} + ye^x = -2x$$

$$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$$

The given differential equation is of the form

$$\frac{dy}{dx} + Py = Q$$

$$\Rightarrow P = 1$$
$$\Rightarrow Q = -2xe^{-x}$$

Calculating integrating factor

$$I.F = e^{\int Pdx}$$

$$\Rightarrow I.F = e^{\int dx}$$

$$\Rightarrow I.F = e^x$$

It is a linear differential equation and its general solution is

$$y\left(I.F\right) = \int \left(Q \times I.F\right)dx + C$$

$$y\left(e^x\right) = \int \left(-2xe^{-x} \times e^x\right)dy + C$$

$$\Rightarrow ye^x = -2\int x\,dx + C$$

$$\Rightarrow ye^x = -2\left(\frac{x^2}{2}\right) + C$$

$$\Rightarrow ye^x + x^2 = C$$

Thus the correct answer is option (C)