Chapter: 9. Differential Equations

Exercise: 9.3

1. Form a differential equation representing the family of the curve $\frac{x}{a}+\frac{y}{b}=1$ by eliminating arbitrary constants

Solution:

The given differential equation is $\frac{x}{a}+\frac{y}{b}=1$

Take the derivative on both sides

$$\frac{d}{dx}\left(\frac{x}{a}+\frac{y}{b}\right)=0$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}y'=0$$

Take the derivative of the above equation to eliminate the constants

$$\Rightarrow 0+\frac{1}{b}y''=0$$

$$\Rightarrow y''=0$$

The curve for given differential equations is $y''=0$

2. Form a differential equation representing the family of the curve $y^2=a\left(b^2-x^2\right)$ by eliminating arbitrary constants

Solution:

The given differential equation is $y^2=a\left(b^2-x^2\right)$

Take the derivative on both sides

$$2y\frac{dy}{dx}=a(-2x)$$

$$\Rightarrow 2yy'\frac{dy}{dx} = -2ax$$

$$\Rightarrow yy'\frac{dy}{dx} = -ax..........(1)$$

Take the derivative of the above equation to eliminate the constants

$$\Rightarrow yy'' + y'y' = -a$$

$$\Rightarrow (y')^2 + yy'' = -a$$

Substitute this in result (1)

$$yy' = \left((y')^2 + yy''\right)x$$

$$\Rightarrow xyy'' + x(y')^2 - yy' = 0$$

The curve for given differential equation is $xyy'' + x(y')^2 - yy' = 0$

3. Form a differential equation representing the family of the curve $y = ae^{3x} + be^{-2x}$ by eliminating arbitrary constants

Solution:

The given differential equation is $y = ae^{3x} + be^{-2x}$

Take the derivative on both sides

$$y' = 3ae^{3x} - 2be^{-2x}$$

Take derivative of above equation

$$y'' = 9ae^{3x} + 4be^{-2x}$$

Compute $y' + 2y$

$$y' + 2y = 3ae^{3x} - 2be^{-2x} + 2ae^{3x} + 2be^{-2x}$$

$$\Rightarrow 5a^{3x} = y' + 2y$$

$$\Rightarrow a^{3x} = \frac{y' + 2y}{5}$$

Compute $3y - y'$

$$3y - y' = 3ae^{3x} + 3be^{-2x} - \left(3ae^{3x} - 2be^{-2x}\right)$$

$$\Rightarrow 5b^{-2x} = 3y - y'$$

$$\Rightarrow b^{-2x} = \frac{3y - y'}{5}$$

Substitute the above results in y''

$$\Rightarrow y'' = 9\left(\frac{y' + 2y}{5}\right) + 4\left(\frac{3y - y'}{5}\right)$$

$$\Rightarrow y'' = \frac{5y' + 30y}{5}$$

$$\Rightarrow y'' = y' + 6y$$

$$\Rightarrow y'' - y' - 6y = 0$$

The curve for given differential equation is $y'' - y' - 6y = 0$

4. Form a differential equation representing the family of the curve $y = e^{2x}\left(a + bx\right)$ by eliminating arbitrary constants

Solution:

The given differential equation is $y = e^{2x}\left(a + bx\right)$

Take the derivative on both sides

$$y' = 2e^{2x}\left(a + bx\right) + e^{2x}\left(b\right)$$

$$\Rightarrow y' = e^{2x}\left(2a + 2bx + b\right)$$

Compute $y'-2y$

$$y'-2y=e^{2x}(2a+2bx+b)-2e^{2x}(a+bx)$$

$$\Rightarrow y'-2y=be^{2x}\text{..........}(1)$$

Take derivative of the above equation

$$\Rightarrow y''-2y'=2be^{2x}$$

Substitute the above results using result from equation (1)

$$\Rightarrow y''-2y'=2(y'-2y)$$

$$\Rightarrow y''-4y'+4\mathrm{y}=0$$

The curve for given differential equation is $y''-4y'+4\mathrm{y}=0$

5. Form a differential equation representing the family of the curve $y=e^x(a\cos x+b\sin x)$ by eliminating arbitrary constants.

Solution:

The given differential equation is $y=e^x(a\cos x+b\sin x)$

Take the derivative on both sides

$$y'=e^x(a\cos x+b\sin x)+e^x(-a\sin x+b\cos x)$$

$$\Rightarrow y'=e^x\left[(a+b)\cos x-(a-b)\sin x\right]$$

Take the derivative of the above equation

$$y''=e^x\left[(a+b)\cos x-(a-b)\sin x\right]+e^x\left[-(a+b)\sin x-(a-b)\cos x\right]$$

$$y''=e^x\left[(a+b-a+b)\cos x-(a-b+a+b)\sin x\right]$$

$$\Rightarrow y''=e^x\left[2b\cos x-2a\sin x\right]$$

$$\Rightarrow y''=2e^{x}\left[b\cos x-a\sin x\right]$$

$$\Rightarrow \frac{y''}{2}=e^{x}\left[b\cos x-a\sin x\right]$$

Add y on both side

$$y+\frac{y''}{2}=e^{x}\left(a\cos x+b\sin x\right)+e^{x}\left(b\cos x-a\sin x\right)$$

$$\Rightarrow y+\frac{y''}{2}=e^{x}\left(\left(a+b\right)\cos x-\left(a-b\right)\sin x\right)$$

Back substitute y'

$$y+\frac{y''}{2}=y'$$

$$\Rightarrow 2y+y''=2y'$$

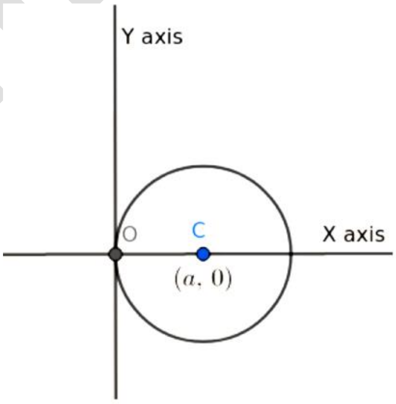
$$\Rightarrow y''-2y'+2y=0$$

The curve for given differential equation is $y''-2y'+2y=0$

6. Form the differential equation of the family of circles touching the y-axis at the origin

Solution:

Let a circle with radius a touches y-axis at origin



Thus the given circle will have the center at $(a,0)$. So it equation will be

$(x-a)^2+y^2=a^2$

$\Rightarrow x^2+y^2=2ax........(1)$

Take the derivative of the above equation

$2x+2yy'=2a$

$\Rightarrow x+yy'=a$

Back substitute a in equation (1)

$x^2+y^2=2(x+yy')x$

$\Rightarrow 2x^2+2yy'x=x^2+y^2$

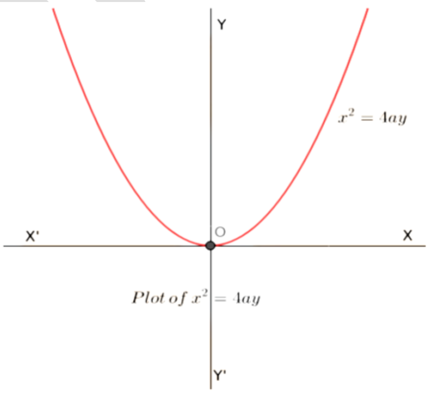
$\Rightarrow x^2+2yy'x=y^2$

Thus the differential equation for the given family of the circle is $x^2+2yy'x=y^2$

7. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis

Solution:

Draw a general parabola with given properties



$x^2=4ay..........(1)$

Take the derivative of the above equation

$2x = 4ay'$

$$\Rightarrow a = \frac{x}{2y'}$$

Back substitute a in equation (1)

$$x^2 = 4\left(\frac{x}{2y'}\right)y$$

$$\Rightarrow x^2 = 2\left(\frac{x}{y'}\right)y$$

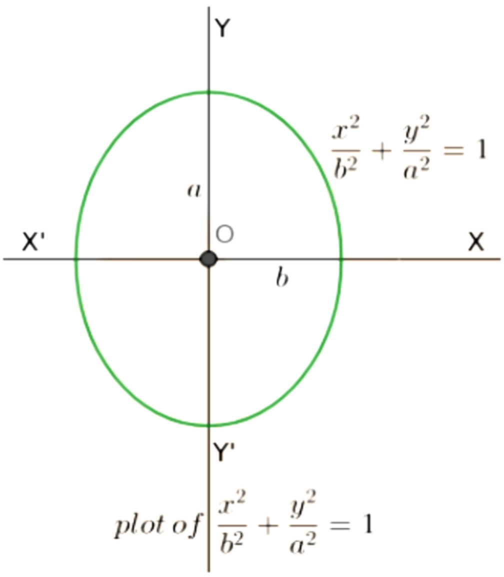
$$\Rightarrow x^2 y' - 2xy = 0$$

Thus the differential equation for the given family of the parabolas is $x^2 y' - 2xy = 0$

8. Form the differential equation of the family of ellipses having foci on y-axis and centre at origin

Solution:

Draw a standard eclipse with given properties



$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Take the derivative of the above equation

$$\frac{2x}{b^2}+\frac{2yy^1}{a^2}=0$$

$$\Rightarrow \frac{x}{b^2}+\frac{yy^1}{a^2}=0......(1)$$

Take the derivative of the above equation

$$\frac{1}{b^2}+\frac{y'y'+yy''}{a^2}=0$$

$$\frac{1}{b^2}=-\frac{1}{a^2}\left(y'^2+yy''\right)$$

Back substitute in equation (1)

$$x\left[-\frac{1}{a^2}\left(y'^2+yy''\right)\right]+\frac{yy'}{a^2}=0$$

$$\Rightarrow -xy'^2-xyy''+yy'=0$$

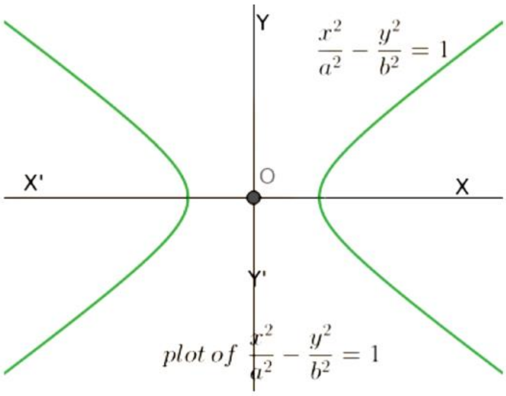
Thus the differential equation for the given family of the parabolas is

$$-xy'^2-xyy''+yy'=0$$

9. From the differential equation of the family of hyperbolas having foci on x-axis and centre at origin

Solution:

Draw a standard hyperbola with given properties



$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Take the derivative of the above equation

$$\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$$

$$\Rightarrow \frac{x}{b^2}+\frac{yy'}{a^2}=0..........(1)$$

Take the derivative of the above equation

$$\frac{1}{b^2}+\frac{y'y'+yy''}{a^2}=0$$

$$\frac{1}{b^2}=-\frac{1}{a^2}\left(y'^2+yy''\right)$$

Back substitute in equation (1)

$$x\left[-\frac{1}{a^2}\left(y'^2+yy''\right)\right]+\frac{yy'}{a^2}=0$$

$$\Rightarrow -xy'^2-xyy''+yy'=0$$

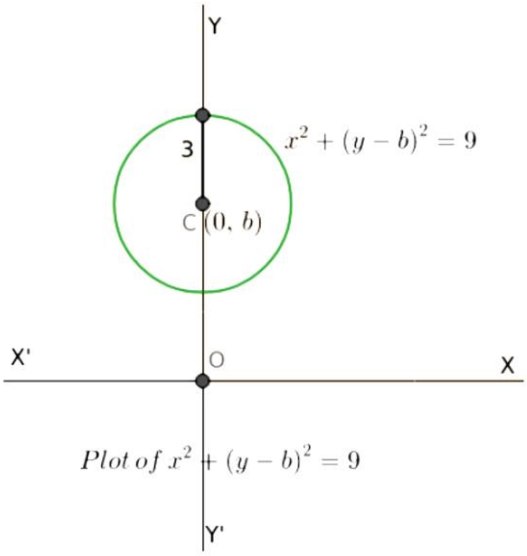
$$\Rightarrow -xyy''-xy'^2-yy'=0$$

Thus the differential equation for the given family of the parabola is $xyy''+xy'^2-yy'=0$

10. Form the differential equation of the family of circles having center on y-axis and radius 3 units

Solution:

Draw the circle with given properties



$$x^2 + (y-b)^2 = 9$$

Take the derivative of the above equation

$$2x + 2(y-b)y' = 0$$

$$\Rightarrow x + (y-b)y' = 0$$

$$\Rightarrow (y-b) = -\frac{x}{y'}$$

Back substitute in the equation circle

$$x^2 + \left(\frac{-x}{y'}\right)^2 = 9$$

$$\Rightarrow x^2 + y'^2 + x^2 = 9y'^2$$

$$\Rightarrow (x^2 - 9)y'^2 + x^2 = 0$$

Thus the differential equation for the given family of the parabolas is $\left(x^2-9\right)y'^2+x^2=0$

11. Which of the following differential equation has $y=c_1e^x+c_2e^{-x}$ as the general solution?

A) $\frac{d^2y}{dx^2}+y=0$ B) $\frac{d^2y}{dx^2}-y=0$ C) $\frac{d^2y}{dx^2}+1=0$ D) $\frac{d^2y}{dx^2}-1=0$

Solution:

The given equation is $y=c_1e^x+c_2e^{-x}$

Differentiate the equation

$$\frac{dy}{dx}=c_1e^x-c_2e^{-x}$$

Differentiate the above equation

$$\frac{d^2y}{dx^2}=c_1e^x+c_2e^{-x}$$

Back substitute the y

$$\frac{d^2y}{dx^2}=y$$

$$\Rightarrow\frac{d^2y}{dx^2}-y=0$$

Thus the correct answer is option B

12. Which of the following differential equation has $y=x$ as one of its particular solutions?

A) $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=x$ B) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+xy=x$

C) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$ D) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

Solution:

The given equation is $y = x$

Differentiate the equation

$$\frac{dy}{dx} = 1$$

Differentiate the above equation

$$\frac{d^2y}{dx^2} = 0$$

Deducing for the option

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0 - x^2(1) + x(x)$$

$$\Rightarrow \frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = -x^2 + x^2$$

$$\Rightarrow \frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy0$$

Thus the correct answer is option C