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Chapter: 9. Differential Equations

Exercise: 9.4

1. Find the general solution for $\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$

Solution:

The given differential equation is $\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$

Use trigonometric half-angle identities to simplify

$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\tan^2\frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx}=\sec^2\frac{x}{2}-1$$

Separate the differential and integrate

$$\int dy=\int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$\Rightarrow y=\int\sec^2\frac{x}{2}dx-\int dx$$

$$\Rightarrow y=2\tan\frac{x}{2}-x+c$$

Thus the general solution of given differential equation is $y=2\tan\frac{x}{2}-x+c$

2. Find the general solution for $\frac{dy}{dx}=\sqrt{4-y^2}\left(-2<y<2\right)$

Solution:

The given differential equation is $\frac{dy}{dx}=\sqrt{4-y^2}\left(-2<y<2\right)$

Simplify the expression

$$\frac{dy}{dx}=\sqrt{4-y^2}$$

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}}=dx$$

Use standard integration

$$\int \frac{dy}{\sqrt{4-y^2}}=\int dx$$

$$\Rightarrow \sin^{-1}\frac{y}{2}=x+c$$

$$\Rightarrow \frac{y}{2}=\sin\left(x+c\right)$$

$$\Rightarrow y=2\sin\left(x+c\right)$$

Thus the general solution of given differentia equation is $y=2\sin\left(x+c\right)$

3. Find the general solutions for $\Rightarrow \frac{dy}{dx}+y=1\left(y\neq 1\right)$

Solution:

The given differential equation is $\Rightarrow \frac{dy}{dx}+y=1\left(y\neq 1\right)$

Simplify the expression

$$\frac{dy}{dx}+y=1$$

$$\Rightarrow \frac{dy}{dx}=1-y$$

$$\Rightarrow \frac{dy}{1-y}=dx$$

Use standard integration

$$\Rightarrow \int \frac{dy}{1-y}=\int dx$$

$$\Rightarrow -\log(1-y)=x+c$$

$$\Rightarrow \log(1-y)=-(x+c)$$

$$\Rightarrow 1-y=e^{-(x+c)}$$

$$y=1-Ae^{-x}\left(A=e^{-c}\right)$$

Thus the general solution of given differential equation is $y=1-Ae^{-x}$

4. Find the general solution for $\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$

Solution:

The given differential equation is $\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$

Divide both side by $\tan x\tan y$

$$\frac{\sec^2 x\tan y dx+\sec^2 y\tan x dy}{\tan x\tan y}=0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x}dx+\frac{\sec^2 y}{\tan y}dy=0$$

Integrate both sides

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = 0$$

$$\int \frac{\sec^2 y}{\tan y} dy = -\int \frac{\sec^2 x}{\tan x} dx.........(1)$$

Use a substitution method for integration. Substitute $\tan x = u$

For integral on RHS

$$\Rightarrow \tan x = u$$

$$\Rightarrow \sec^2 x\, dx = du$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = \int \frac{du}{u}$$

$$\Rightarrow \int \frac{\sec^2 x}{\tan x} dx = \log u$$

$$\Rightarrow \int \frac{\sec^2 x}{\tan x} dx = \log(\tan x)$$

Thus evaluating result from (1)

$$\Rightarrow \log(\tan y) = -\log(\tan x) + \log(c)$$

$$\Rightarrow \log(\tan y) = \log\left(\frac{c}{\tan x}\right)$$

$$\Rightarrow \tan x \tan y = c$$

Thus the general solution of given differentia equation is $\tan x \tan y = c$

5. Find the general solution for $\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$

Solution:

The given differential equation is $\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$

Simplify the expression

$$dy=\left[\frac{e^x-e^{-x}}{e^x+e^{-x}}\right]dx$$

Integrate both sides

$$\int dy=\int\left[\frac{e^x-e^{-x}}{e^x+e^{-x}}\right]dx..........(1)$$

Use a substitution method for integration. Substitute $e^x+e^{-x}=t$

For integral on RHS

$$\Rightarrow e^x+e^{-x}=t$$

$$\Rightarrow\left(e^x+e^{-x}\right)dx=dt$$

$$\Rightarrow\int\left[\frac{e^x-e^{-x}}{e^x+e^{-x}}\right]dx=\int\frac{dt}{t}s$$

$$\Rightarrow\int\left[\frac{e^x-e^{-x}}{e^x+e^{-x}}\right]dx=\ln t+c$$

$$\Rightarrow\int\left[\frac{e^x-e^{-x}}{e^x+e^{-x}}\right]dx=\left(\log e^x+e^{-x}\right)+c$$

Thus evaluating result from (1)

$$y=\log\left(e^x+e^{-x}\right)+c$$

Thus the general solution of given differential equation is $y=\log\left(e^x+e^{-x}\right)+c$

6. Find the general solution for $\frac{dy}{dx}=\left(1+x^2\right)\left(1+y^2\right)$

Solution:

The given differential equation is $\frac{dy}{dx}=\left(1+x^2\right)\left(1+y^2\right)$

Simplify the expression

$$\frac{dy}{1+y^2}=\left(1+x^2\right)dx$$

Integrate both side

$$\int\frac{dy}{1+y^2}=\int\left(1+x^2\right)dx$$

Use standard integration

$$\tan^{-1}y=\int dx+\int x^2dx$$

$$\Rightarrow \tan^{-1}y=x+\frac{x^3}{3}+c$$

Thus the general solution of given differential equation is $\tan^{-1}y=x+\frac{x^3}{3}+c$

7. Find the general solution for $y\log y\,dx-xdy=0$

Solution:

The given differential equation is $y\log y\,dx-xdy=0$

Simplify the expression

$$y\log y\,dx=xdy$$

$$\Rightarrow \frac{dx}{x}=\frac{dy}{y\log y}$$

Integrate both sides

$$\int\frac{dx}{x}=\int\frac{dy}{y\log y}..........(1)$$

Use substitution method for integration on LHS. Substitute $\log y=t$

$$\log y=t$$

$$\Rightarrow \frac{1}{y} dy = dt$$

$$\Rightarrow \int \frac{dy}{y \log y} = \int \frac{dt}{t}$$

$$\Rightarrow \int \frac{dy}{y \log y} = \log t$$

$$\Rightarrow \int \frac{dy}{y \log y} = \log(\log y)$$

Evaluating expression (1)

$$\log(\log y) = \log x + \log c$$

$$\log(\log y) = \log c x$$

$$\log y = cx$$

$$\Rightarrow y = e^{cx}$$

Thus the general solution of given differential equation is $\Rightarrow y = e^{cx}$

8. Find the general solution for $x^5 \frac{dy}{dx} = -y^5$

Solution:

The given differential equation is $x^5 \frac{dy}{dx} = -y^5$

Simplify the expression

$$\frac{dy}{y^5} = -\frac{dx}{x^5}$$

Integrate both sides

$$\int \frac{dy}{y^5} = -\int \frac{dx}{x^{-5}}$$

$$\Rightarrow \int y^{-5} dy = -\int x^{-5} dx$$

$$\Rightarrow \frac{y^{-5+1}}{-5+1} = -\frac{x^{-5+1}}{-5+1} + c$$

$$\Rightarrow \frac{y^{-4}}{-4} = -\frac{x^{-4}}{-4} + c$$

$$\Rightarrow x^{-4} + y^{-4} = -4c$$

$$\Rightarrow x^{-4} + y^{-4} = A(A = -4c)$$

Thus the general solution of given differential equation is $x^{-4} + y^{-4} = A$

9. Find the general solution for $\frac{dy}{dx} = \sin^{-1} x$

Solution:

The given differential equation is $\frac{dy}{dx} = \sin^{-1} x$

Simplify the expression

$$dy = \sin^{-1} x\, dx$$

Integrate both side

$$\int dy = \int \sin^{-1} x\, dx$$

$$\Rightarrow y = \int 1 \times \sin^{-1} x\, dx$$

Use product rule of integration

$$\int \sin^{-1} x dx = \sin^{-1} x \int dx - \int \left(\frac{1}{\sqrt{1-x^2}} \int dx \right) dx$$

$$\Rightarrow \int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Substitute $1-x^2=t^2$

$$1-x^2=t^2$$

$$\Rightarrow -2xdx=2tdt$$

$$\Rightarrow -xdx=tdt$$

Evaluating the integral

$$\frac{2x^2+x}{(x+1)(x^2+1)}=\frac{A}{x+1}=\frac{Bx+C}{x^2+1}$$

$$\int \sin^{-1}xdx=x\sin^{-1}x+\int\frac{tdt}{\sqrt{t^2}}$$

$$\Rightarrow \int \sin^{-1}xdx=x\sin^{-1}x+t+c$$

$$\Rightarrow \int \sin^{-1}xdx=x\sin^{-1}x+\sqrt{1-x^2}+c$$

$$\Rightarrow y=x\sin^{-1}x+\sqrt{1-x^2}+c$$

Thus the general solution of given differential equation is $y=x\sin^{-1}x+\sqrt{1-x^2}+c$

10. Find the general solution for $e^x\tan ydx+(1-e^x)\sec^2 ydy=0$

Solution:

The given differential equation is $e^x\tan ydx+(1-e^x)\sec^2 ydy=0$

Simplify the expression

$$(1-e^x)\sec^2 y\,dy=-e^x\tan y\,dy$$

$$\Rightarrow \frac{\sec^2 y}{\tan y}dy=-\frac{e^x}{(1-e^x)}dx$$

Integrate both sides

$$\int \frac{\sec^2 y}{\tan y} dy = -\int \frac{e^x}{\left(1-e^x\right)} dx \ldots\ldots\ldots.(1)$$

Substitute $\tan y = u$

$\tan y = u$

$\Rightarrow \sec^2 y = du$

Evaluating the LHS integral of (1)

$$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy = \int \frac{du}{u}$$

$$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy = \log u$$

$$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy = \log(\tan y)$$

Substitute $1-e^x = v$

$1-e^x = v$

$\Rightarrow -e^x dx = dv$

Evaluating the RHS integral of (1)

$$\Rightarrow -\int \frac{e^x}{\left(1-e^x\right)} dx = \int \frac{dv}{v}$$

$$\Rightarrow -\int \frac{e^x}{\left(1-e^x\right)} dx = \log v$$

$$\Rightarrow -\int \frac{e^x}{\left(1-e^x\right)} dx = \log\left(1-e^x\right)$$

Therefore the integral (1) will be

$$\log(\tan y) = \log\left(1-e^x\right) + \log c$$

$$\Rightarrow \log(\tan y) = \log c(1-e^x)$$

$$\Rightarrow \tan y = \log c(1-e^x)$$

Thus the general solution of given differential equation is $\tan y = \log c(1-e^x)$

11. Find the particular solution of $(x^3+x^2+x+1)\frac{dy}{dx} = 2x^2+x; y=1, x=0$ to satisfy the given condition

Solution:

The given differential equation is $(x^3+x^2+x+1)\frac{dy}{dx} = 2x^2+x; y=1, x=0$

Simplify the expression

$$\frac{dy}{dx} = \frac{2x^2+x}{(x^3+x^2+x+1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2+x}{(x+1)(x^2+1)}$$

$$\Rightarrow dy = \frac{2x^2+x}{(x+1)(x^2+1)}dx$$

Integrate both side

$$\int dy = \int \frac{2x^2+x}{(x+1)(x^2+1)}dx...........(1)$$

Use partial fraction method to simplify the RHS

$$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 2x^2+x = A(x^2+1)+(Bx+C)(x+1)$$

$$\Rightarrow 2x^2+x=(A+B)x^2+(B+C)x+(A+C)$$

By comparing coefficients

$$A+B=2$$
$$B+C=1$$
$$A+C=0$$

Solving this we get

$$\frac{2x^2+x}{(x+1)(x^2+1)}=\frac{\left(\frac{1}{2}\right)}{x+1}+\frac{\left(\frac{3}{2}\right)x+\left(-\frac{1}{2}\right)}{x^2+1}$$

$$\Rightarrow \frac{2x^2+x}{(x+1)(x^2+1)}=\frac{1}{2}\left(\frac{1}{x+1}+\frac{3x-1}{x^2+1}\right)$$

Rewriting the integral (1)

$$y=\frac{1}{2}\int\left(\frac{1}{x+1}+\frac{3x-1}{x^2+1}\right)dx$$

$$\Rightarrow y=\frac{1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{3x-1}{x^2+1}dx$$

$$\Rightarrow y=\frac{1}{2}\log(x+1)+\frac{3}{2}\int\frac{x}{x^2+1}dx-\frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$\Rightarrow y=\frac{1}{2}\log(x+1)+\frac{3}{4}\int\frac{2x}{x^2+1}dx-\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow y=\frac{1}{2}\log(x+1)+\frac{3}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+c$$

For $y=1$ when $x=0$

$$1=\frac{1}{2}\log(0+1)+\frac{3}{4}\log(0+1)-\frac{1}{2}\tan^{-1}0+c$$

$$\Rightarrow c=1$$

Thus the required particular solution is

$$\Rightarrow y=\frac{1}{2}\log(x+1)+\frac{3}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+1$$

12. Find the particular solution of $x(x^2-1)\frac{dy}{dx}=1; y=0$ when $x=2$ to satisfy the given condition

Solution:

The given differential equation is $x(x^2-1)\frac{dy}{dx}=1; y=0$ when x = 2

Simplify the expression

$$x(x^2-1)\frac{dy}{dx}=1$$

$$\Rightarrow dy=\frac{dx}{x(x^2-1)}$$

$$\Rightarrow dy=\frac{dx}{x(x-1)(x+1)}$$

Integrate both sides

$$\int dy=\int\frac{dx}{x(x-1)(x+1)}............(1)$$

Use partial fraction method to simplify the RHS

$$\frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

$$\Rightarrow 1=A(x^2-1)+Bx(x+1)+Cx(x-1)$$

$$\Rightarrow 1=(A+B+C)x^2+(B-C)x-A$$

By comparing coefficients

$$A+B+C=0$$

$B-C=0$

$-A=1$

Solving this we get

$$\frac{1}{x(x-1)(x+1)}=\frac{(-1)}{x}+\frac{\left(\frac{1}{2}\right)}{x-1}+\frac{\left(\frac{1}{2}\right)}{x+1}$$

$$\Rightarrow \frac{1}{x(x-1)(x+1)}=-\frac{1}{x}+\frac{1}{2}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)$$

Rewriting the integral (1)

$$y=\int\left(-\frac{1}{x}+\frac{1}{2}\left(\frac{1}{x-1}+\frac{1}{x+1}\right)\right)dx$$

$$\Rightarrow y=\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{x-1}dx+\frac{1}{2}\int\frac{1}{x+1}dx$$

$$\Rightarrow y=-\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+\log c$$

$$\Rightarrow y=-\frac{2}{2}\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+\frac{2}{2}\log c$$

$$\Rightarrow y=\frac{1}{2}\left(-\log x^2+\log(x-1)+\log(x+1)+\log c^2\right)$$

$$\Rightarrow y=\frac{1}{2}\log\left[\frac{c^2\left(x^2-1\right)}{x^2}\right]$$

For y = 0 when x = 2

$$0=\frac{1}{2}\log\left[\frac{c^2\left(2^2-1\right)}{2^2}\right]$$

$$0=\log\left[\frac{3c^2}{4}\right]$$

$$\Rightarrow \frac{3c^2}{4} = 1$$

$$\Rightarrow c^2 = \frac{4}{3}$$

Thus the required particular solution is $y = \frac{1}{2}\log\left[\frac{4\left(x^2 - 1\right)}{3x^2}\right]$

13. Find the particular solution of $\cos\left(\frac{dy}{dx}\right) = a\left(a \in R\right)$; $y = 1$ when $x = 0$ to satisfy the given equation

Solution:

The given differential equation is $\cos\left(\frac{dy}{dx}\right) = a\left(a \in R\right)$; $y = 1$ when $x = 0$

Simplify the expression

$$\cos\left(\frac{dy}{dx}\right) = a$$

$$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$$

$$\Rightarrow dy = \cos^{-1} a\, dx$$

Integrate both sides

$$\int dy = \int \cos^{-1} a\, dx$$

$$\Rightarrow y = \cos^{-1} a \int dx$$

$$\Rightarrow y = x\cos^{-1} a + c$$

For y =1 when x = 0

$$1 = 0\cos^{-1} a + c$$

$\Rightarrow c = 1$

Thus the required particular solution is

$y = x\cos^{-1} a + 1$

$\Rightarrow \frac{y-1}{x} = \cos^{-1} a$

$\Rightarrow \cos\left(\frac{y-1}{x}\right) = a$

14. Find the particular solution of $\frac{dy}{dx} = y\tan x; y = 1$ when $x = 0$ to satisfy the given condition

Solution:

The given differential equation is $\frac{dy}{dx} = y\tan x; y = 1$ when $x = 0$

Simplify the expression

$\frac{dy}{dx} = y\tan x$

$\Rightarrow \frac{dy}{dx} = \tan x\, dx$

Integrate both sides

$\Rightarrow \int \frac{dy}{dx} = \int \tan x\, dx$

$\Rightarrow \int \log y = \log(\sec x) + \log c$

$\Rightarrow \log y = \log(c\sec x)$

$\Rightarrow y = c\sec x$

For y = 1 when x = 0

$1 = c\sec 0$

$\Rightarrow c = 1$

Thus the required particular solution is $y = \sec x$

15. Find the equation of a curve passing through the point $(0,0)$ and whose differential equation is $y' = e^x \sin x$

Solution:

The given differential equation is $y' = e^x \sin x$

The curve passes through $(0,0)$

Simplify the expression

$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$

$$\Rightarrow dy = e^x \sin x\, dx$$

Integrate both sides

$$\int dy = \int e^x \sin x\, dx$$

Use product rules for integration of RHS. Let

$$I = \int e^x \sin x\, dx$$

$$\Rightarrow I = \sin x \int e^x\, dx - \int\left(\cos x \int e^x dx\right)dx$$

$$\Rightarrow I = e^x \sin x - \left(\cos x \int e^x dx + \int\left(\sin x \int e^x dx\right)dx\right)$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - \int\left(e^x \sin x\right)dx$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$

$$\Rightarrow I=\frac{e^{x}}{2}(\sin x-\cos x)$$

Thus integral will be

$$y=\frac{e^{x}}{2}(\sin x-\cos x)+c$$

Thus as the curve passes through $(0,0)$

$$0=\frac{e^{0}}{2}(\sin 0-\cos 0)+c$$

$$0=\frac{1}{2}(0-1)+c$$

$$\Rightarrow c=\frac{1}{2}$$

Thus the equation of the curve will be

$$y=\frac{e^{x}}{2}(\sin x-\cos x)+\frac{1}{2}$$

$$\Rightarrow y=\frac{e^{x}}{2}(\sin x-\cos x+1)$$

16. For the differential equation $xy\frac{dy}{dx}=(x+2)(y+2)$ find the solution curve passing through the point $(1,-1)$

Solution:

The given differential equation is $xy\frac{dy}{dx}=(x+2)(y+2)$

The curve passes through $(1,-1)$

Simplify the expression

$$\Rightarrow\left(\frac{y}{y+2}\right)dy=\frac{(x+2)}{x}dx$$

$$\Rightarrow\left(1-\frac{2}{y+2}\right)dy=\frac{(x+2)}{x}dx$$

Integrate both side

$$\int\left(1-\frac{2}{y+2}\right)dy=\int\frac{(x+2)}{x}dx$$

$$\Rightarrow\int dy-2\int\frac{1}{y+2}dy=\int\frac{x}{x}dx+\int\frac{2}{x}dx$$

$$\Rightarrow y-2\log(y+2)=x+2\log x+c$$

$$\Rightarrow y-x=2\log(y+2)+2\log x+c$$

$$\Rightarrow y-x=2\log\left[x(y+2)\right]+c$$

$$\Rightarrow y-x=\log\left[x^2(y+2)^2\right]+c$$

Thus as the curve passes through $(1,-1)$

$$\Rightarrow -1-1=\log\left[(1)^2(-1+2)^2\right]+c$$

$$\Rightarrow -2=\log 1+c$$

$$\Rightarrow c=-2$$

Thus the equation of the curve will be

$$y-x=\log\left[x^2(y+2)^2\right]-2$$

$$\Rightarrow y-x+2=\log\left(x^2(y+2)^2\right)$$

17. Find the equation of a curve passing through the point $(0,-2)$ given that at any point (x,y) on the curve, the product of the slope of its tangent and -coordinate of the point is equal to the -coordinate of the point y x

Solution:

According to question, the equation is given by

$$y\frac{dy}{dx}=x$$

The curve passes through $(0,-2)$

Simplify the expression

$$\Rightarrow ydy=xdx$$

Integrate both sides

$$\int y\,dy=\int x\,dx$$

$$\Rightarrow \frac{y^2}{2}=\frac{x^2}{2}+c$$

$$\Rightarrow y^2-x^2=2c$$

Thus as the curve passes through $(0,-2)$

$$\Rightarrow (-2)^2-0^2=2C$$

$$\Rightarrow 4=2c$$

$$\Rightarrow c=2$$

Thus the equation of the curve will be

$$y^2-x^2=2(2)$$

$$\Rightarrow y^2-x^2=4$$

18. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4,-3)$. Find the equation of the curve given that it passes through $(-2,1)$.

Solution:

Let the point of contact of the tangent be (x, y). Then the slope of the segment joining point of contact and $(-4,-3)$

$$m=\frac{y+3}{x+4}$$

According to question the for the slope of tangent $\frac{dy}{dx}$ if follows

$$\frac{dy}{dx}=2m$$

$$\Rightarrow \frac{dy}{dx}=2\left(\frac{y+3}{x+4}\right)$$

Simplify the expression

$$\frac{dy}{dx}=2\left(\frac{y+3}{x+4}\right)$$

$$\frac{dy}{y+3}=\frac{2}{x+4}dx$$

Integrate both sides

$$\int\frac{dy}{y+3}=\int\frac{2}{x+4}dx$$

$$\Rightarrow \log(y+3)=2\log(x+4)+\log c$$

$$\Rightarrow \log(y+3)=\log(x+4)^2+\log c$$

$$\Rightarrow \log(y+3)=\log c(x+4)^2$$

$$\Rightarrow y+3=c(x+4)^2$$

Thus as the curve passes through $(-2,1)$

$$1+3=c(-2+4)^2$$

$$\Rightarrow 4=4C$$

$$\Rightarrow c=1$$

Thus the equation of the curve will be $y+3=(x+4)^2$

19. The volume of spherical balloons being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds

Solution:

Let the volume of spherical balloon be V and its radius r. Let the rate of change of volume be k.

$$\frac{dV}{dt}=k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)=k$$

$$\Rightarrow \frac{4}{3}\pi\frac{d}{dt}\left(r^3\right)=k$$

$$\Rightarrow \frac{4}{3}\pi\left(3r^2\right)\frac{d}{dt}=k$$

$$\Rightarrow 4\pi r^2 dr=k\,dt$$

Integrate both sides

$$\int 4\pi r^2 dr=\int k\,dt$$

$$\Rightarrow 4\pi\int r^2 dr = kt + C$$

$$\Rightarrow \frac{4}{3}\pi r^3 = kt + C$$

At initial time, $t = 0$ and $r = 3$

$$\frac{4}{3}\pi 3^3 = k(0) + C$$

$$\Rightarrow C = 36\pi$$

At t = 3 the radius r = 6

$$\frac{4}{6}\pi\left(6^3\right) = k(3) + 36\pi$$

$$\Rightarrow 3k = 288\pi - 36\pi$$

$$\Rightarrow k = 84\pi$$

Thus the radius-time relation can be given by

$$\frac{4}{3}\pi r^3 = 84\pi t + 36\pi$$

$$\Rightarrow r^3 = 63t + 27$$

$$\Rightarrow r = (63t + 27)^{\frac{1}{3}}$$

The radius of balloon after t seconds given by $r = (63t + 27)^{\frac{1}{3}}$

20. In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs. 100 doubles itself in 10 years $(\log_e 2 = 0.6931)$

Solution:

Let the principal be p, according to question

$$\frac{dp}{dt} = \left(\frac{r}{100}\right)p$$

Simplify the expression

$$\frac{dp}{p}=\left(\frac{r}{100}\right)dt$$

Integrate both side

$$\int\frac{dp}{p}=\int\left(\frac{r}{100}\right)dt$$

$$\Rightarrow \log p=\frac{rt}{100}+c$$

$$\Rightarrow p=e^{\frac{rt}{100}+c}$$

$$\Rightarrow p=Ae^{\frac{rt}{100}}\left(A=e^{c}\right)$$

At t = 0, p = 100

$$100=Ae^{\frac{r(0)}{100}}$$

$$\Rightarrow A=100$$

Thus the principle and rate of interest relation

$$p=100e^{\frac{rt}{100}}$$

At $t=10, p=2\times100=200$

$$200=100e^{\frac{r(10)}{100}}$$

$$\Rightarrow 2=e^{\frac{r}{10}}$$

Take logarithm on both side

$$\log\left(e^{\frac{r}{10}}\right)=\log(2)$$

$$\Rightarrow \frac{r}{10}=0.6931$$

$\Rightarrow r = 6.931$

Thus the rate of intersect $r = 6.931\%$

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years $\left(e^{0.5} = 1.648\right)$

Solution:

Let the principal be p, according to question principle increases at the rate of 5% per year

$$\frac{dp}{dt} = \left(\frac{5}{100}\right)p$$

Simplify the expression

$$\frac{dp}{dt} = \frac{p}{20}$$

$$\Rightarrow \frac{dp}{p} = \frac{1}{20}dt$$

Integrate both side

$$\int \frac{dp}{p} = \int \frac{1}{20}dt$$

$$\Rightarrow \log p = \frac{t}{20} + c$$

$$\Rightarrow p = e^{\frac{t}{20}+c}$$

$$\Rightarrow p = Ae^{\frac{t}{20}}\left(A = e^{c}\right)$$

At t = 0, p = 1000

$$1000 = Ae^{\frac{0}{20}}$$

$\Rightarrow A = 1000$

Thus the relation of principal and time relation

$\Rightarrow p = 1000e^{\frac{t}{20}}$

At t = 10

$p = 1000e^{\frac{10}{20}}$

$\Rightarrow p = 1000e^{0.5}$

$\Rightarrow p = 1000 \times 1.648$

$\Rightarrow p = 1648$

Thus after 10 this year the amount will become Rs. 1648

22. In a culture, the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present?

Solution:

Let the number of bacteria by y at time t. According to question

$\frac{dy}{dt} \propto y$

$\frac{dy}{dt} = cy$

Here c is constant

Simplify the expression

$\frac{dy}{y} = cdt$

Integrate both side

$$\int \frac{dy}{y} = \int c\,dt$$

$$\Rightarrow \log y = ct + D$$

$$\Rightarrow y = e^{ct+D}$$

$$\Rightarrow y = Ae^{ct}\left(A = e^{D}\right)$$

At t = 0, y = 100000

$$100000 = Ae^{c(0)}$$

$$\Rightarrow A = 100000$$

At $t = 2, y = \frac{11}{10}(100000) = 110000$

$$y = 100000\,e^{ct}$$

$$\Rightarrow 110000 = 100000e^{c(2)}$$

$$\Rightarrow e^{2c} = \frac{11}{10}$$

$$\Rightarrow 2c = \log\left(\frac{11}{10}\right)$$

$$\Rightarrow c = \frac{1}{2}\log\left(\frac{11}{10}\right) \text{...........}(1)$$

For $y = 200000$

$$200000 = 100000e^{ct}$$

$$\Rightarrow e^{ct} = 2$$

$$\Rightarrow ct = \log 2$$

$$\Rightarrow t = \frac{\log 2}{c}$$

Back substituting using expression (1)

$$t=\frac{\log 2}{\frac{1}{2}\log\left(\frac{11}{10}\right)}$$

$$\Rightarrow t=\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Thus time required for bacteria to reach 200000 is $\Rightarrow t=\frac{\log 2}{\frac{1}{2}\log\left(\frac{11}{10}\right)}$ hrs

23. Find the general solution of the differential equation $\frac{dy}{dx}=e^{x+y}$

Solution:

The given differential equation is $\frac{dy}{dx}=e^{x+y}$. Simplify the expression

$$\frac{dy}{dx}=e^{x}e^{y}$$

$$\Rightarrow \frac{dy}{e^{y}}=e^{x}dx$$

$$\Rightarrow e^{-y}dy=e^{x}dx$$

Integrate both side

$$\int e^{-y}dy=\int e^{x}dx$$

$$\Rightarrow -e^{-y}=e^{x}+D$$

$$\Rightarrow e^{x}+e^{-y}=-D$$

$$\Rightarrow e^{x}+e^{-y}=C\left(C=-D\right)$$

Thus the general solution of given differential equation is $e^{x}+e^{-y}=C$