Chapter: 9. Differential Equations

Exercise: 9.5

1. Show that, differential equation $\left(x^2+xy\right)dy=\left(x^2+y^2\right)dx$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy}$$

Checking for homogeneity

$$F(x,y)=\frac{x^2+y^2}{x^2+xy}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)^2+(\lambda x)(\lambda y)}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda^2\left(x^2+y^2\right)}{\lambda^2+\left(x^2+xy\right)}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{x^2+y^2}{x^2+xy}$$

$$\Rightarrow F(\lambda x,\lambda y)=F(\lambda x,\lambda y)$$

Thus it is an homogeneous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{x^2+(vx)^2}{x^2+x(vx)}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x^2\left(1+v^2\right)}{x^2+(1+v)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1+v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{1+v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2-v-v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v}{1+v}$$

Separate the differentials

$$\frac{1+v}{1-v}dv=\frac{dx}{x}$$

Integrate both side

$$\int\frac{1+v}{1-v}dv=\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{2-(1-v)}{1-v}dv=\log x-\log k$$

$$\Rightarrow \int\frac{2}{1-v}dv-\int\frac{1-v}{1-v}dv=\log\frac{x}{k}$$

$$\Rightarrow -2\log(1-v)-\int dv=\log\frac{x}{k}$$

$$\Rightarrow -2\log(1-v)-v=\log\frac{x}{k}$$

$$\Rightarrow v=-\log\frac{x}{k}-2\log(1-v)$$

$$\Rightarrow v=\log\left(\frac{k}{x(1-v)^2}\right)$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \frac{y}{x} = \log\left(\frac{k}{x\left(1-\frac{y}{x}\right)^2}\right)$$

$$\Rightarrow \frac{y}{x} = \log\left(\frac{kx}{(x-y)^2}\right)$$

$$\Rightarrow e^{\frac{y}{x}} = \frac{kx}{(x-y)^2}$$

$$\Rightarrow (x-y)^2 = kxe^{-\frac{y}{x}}$$

The solution of the given differential equation $(x-y)^2 = kxe^{-\frac{y}{x}}$

2. Show that, differential equation $y' = \frac{x+y}{x}$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Checking for homogeneity

$$F(x,y) = \frac{x+y}{x}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda(x+y)}{\lambda(x)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{x+y}{x}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{x+(vx)}{x}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x(1+v)}{x}$$

$$\Rightarrow v+x\frac{dv}{dx}=1+v$$

$$\Rightarrow x\frac{dv}{dx}=1$$

Separate the differentials

$$dv=\frac{dx}{x}$$

Integrate both side

$$\int dv=\int\frac{dx}{x}$$

$$\Rightarrow \int dv=\log x+\log k$$

$$\Rightarrow v=\log kx$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \frac{y}{x}=\log kx$$

$$\Rightarrow y=x\log kx$$

The solution of the given differential equation $y=x\log kx$

3. Show that, differential equation $(x-y)dy-(x+y)dx=0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

Checking for homogeneity

$$F(x,y)=\frac{x+y}{x-y}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda x+\lambda y}{\lambda x-\lambda y}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda(x+y)}{\lambda(x-y)}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{x+y}{x-y}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogeneous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{x+(vx)}{x-vx}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x(1+v)}{x(1-v)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1+v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v}{1-v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v-v+v^2}{1-v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1+v^2}{1-v}$$

Separate the differentials

$$\frac{1-v}{1+v^2}dv=\frac{dx}{x}$$

Integrate both side

$$\int\frac{1-v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow\int\frac{1-v}{1+v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow\int\frac{1}{1+v^2}dv-\int\frac{v}{1+v^2}dv=\log x+C$$

$$\Rightarrow\tan^{-1}v-\frac{1}{2}\int\frac{2v}{1+v^2}dv=\log x+C$$

$$\Rightarrow\tan^{-1}v-\frac{1}{2}\log\left(1+v^2\right)=\log x+C$$

$$\Rightarrow\tan^{-1}v=\log x+\frac{1}{2}\log\left(1+v^2\right)+C$$

$$\Rightarrow\tan^{-1}v=\frac{1}{2}\log x^2+\frac{1}{2}\log\left(1+v^2\right)+C$$

$$\Rightarrow\tan^{-1}v=\frac{1}{2}\log\left[x^2\left(1+v^2\right)\right]+C$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log\left[x^2\left(1+\frac{y^2}{x^2}\right)\right]+C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log\left(x^2+y^2\right)+C$$

The solution of the given differential equation $\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log\left(x^2+y^2\right)+C$

4. Show that, differential equation $\left(x^2-y^2\right)dx+2xy\,dy=0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx}=\frac{x^2-y^2}{2xy}$$

Checking for homogeneity

$$F(x,y)=-\frac{x^2-y^2}{2xy}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{(\lambda x)^2-(\lambda y)^2}{2(\lambda x)(\lambda y)}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda^2x^2-\lambda^2y^2}{2\lambda^2xy}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda^2\left(x^2-y^2\right)}{\lambda^2(2xy)}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{x^2-y^2}{2xy}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=-\frac{x^2-(vx)^2}{2x(vx)}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x^2(v^2-1)}{x^2(2v)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v^2-1}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1}{2v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx}=-\frac{1+v^2}{2v}$$

Separate the differentials

$$\frac{2v}{1+v^2}dv=-\frac{dx}{x}$$

Integrate both sides

$$\int\frac{2v}{1+v^2}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \log(1+v^2)=-\log x+C$$

$$\Rightarrow \log(1+v^2)+\log x=C$$

$$\Rightarrow \log\left[x(1+v^2)\right]=C$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \log\left[x\left(1+\frac{y^2}{x^2}\right)\right]=C$$

$$\Rightarrow\left(\frac{x^2+y^2}{x}\right)=k\; k=e^c$$

$$\Rightarrow x^2+y^2=kx$$

The solution of the given differential equations $x^2+y^2=kx$

5. Show that, differential equation $x^2\frac{dy}{dx}=x^2-2y^2+xy$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\frac{dy}{dx}=\frac{x^2-2y^2+xy}{x^2}$$

Checking for homogeneity

$$F(x,y)=\frac{x^2-2y^2+xy}{x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda^2x^2-2\lambda^2y^2+(\lambda x)(\lambda y)}{\lambda^2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda^2\left(x^2-2y^2+xy\right)}{\lambda^2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{x^2-2y^2+xy}{x^2}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{x^2-2(vx)^2+x(vx)}{x^2}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{x^2\left(1-2v^2+v\right)}{x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=1-2V^2+V$$

$$\Rightarrow x\frac{dv}{dx}=1-2V^2$$

Separate the differentials

$$\frac{1}{1-2v^2}dv=\frac{dx}{x}$$

Integrate both side

$$\int\frac{1}{1-2v^2}dv=\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int\frac{1}{\frac{1}{2}-v^2}dv=\log x+C$$

$$\Rightarrow \frac{1}{2}\int\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2-v^2}dv=\log x+C$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{2\times\frac{1}{\sqrt{2}}}\right)\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right|=\log x+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right|=\log x+C$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\left(\frac{y}{x}\right)}{1-\sqrt{2}\left(\frac{y}{x}\right)}\right|=\log x+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\frac{\left|x+\sqrt{2}y\right|}{\left|x-\sqrt{2}y\right|}=\log x+C$$

The solution of the given differential equation $\frac{1}{2\sqrt{2}}\log\frac{\left|x+\sqrt{2}y\right|}{\left|x-\sqrt{2}y\right|}=\log x+C$

6. Show that, differential equation $xdy-ydx=\sqrt{x^2+y^2}dx$ is homogenous and solves it

Solution:

Rewrite the equations in standard form

$$xdy=\sqrt{x^2+y^2}dx+ydx$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x^2+y^2}+y}{x}$$

Checking for homogeneity

$$F(x,y)=\frac{\sqrt{x^2+y^2}+y}{x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\sqrt{\lambda^2x^2+\lambda^2y^2}+\lambda y}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\sqrt{\lambda^2\left(x^2+y^2\right)}+\lambda y}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda\left(\sqrt{x^2+y^2}+y\right)}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\sqrt{x^2+y^2}+y}{x}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{\sqrt{x^2 + (vx)^2} + vx}{x}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{x\left(\sqrt{1+v^2} + v\right)}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$$

Separate the differentials

$$\frac{1}{\sqrt{1+v^2}}dv = \frac{dx}{x}$$

Integrate both sides

$$\int \frac{1}{\sqrt{1+v^2}}dv = \int \frac{dx}{x}$$

$$\Rightarrow \log\left|v + \sqrt{1+v^2}\right| = \log x + \log C$$

Backs substitute $v = \frac{y}{x} : s$

$$\Rightarrow \log\left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| = \log Cx$$

$$\Rightarrow \log\left|\frac{y + \sqrt{1+x^2}}{x}\right| = \log Cx$$

$$y + \sqrt{1+x^2} = Cx^2$$

The solution of the given differential equation $y + \sqrt{1+x^2} = Cx^2$

7. Show that, differential equation $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdy$$

$$\frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

Checking for homogeneity

$$F(x,y)=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)+y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda y}{\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\lambda^2\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\lambda^2\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{\left\{x\cos\left(\frac{xv}{x}\right) + vx\sin\left(\frac{vx}{x}\right)\right\}vx}{\left\{vx\sin\left(\frac{vx}{x}\right) - x\cos\left(\frac{vx}{x}\right)\right\}x}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{x^2\{\cos v + v\sin v\}v}{x^2\{v\sin v - \cos v\}}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

Separate the differentials

$$\frac{v\sin v - \cos v}{v\cos v}dv = 2\frac{dx}{x}$$

Integrate both side

$$\int\frac{v\sin v - \cos v}{v\cos v}dv = 2\int\frac{dx}{x}$$

$$\int\frac{v\sin v}{v\cos v}dv - \int\frac{\cos v}{v\cos v} = 2\int\frac{dx}{x}$$

$$\Rightarrow \int\tan v dv - \int\frac{1}{v}dv = 2\log|x| + \log C$$

$$\Rightarrow \log|\sec v| - \log|v| = \log C|x|^2$$

$$\Rightarrow \log \frac{|\sec v|}{|v|} = \log C|x|^2$$

$$\Rightarrow \sec v = Cvx^2$$

Back substitute $v = \frac{y}{x} : s$

$$\Rightarrow \sec \frac{y}{x} = C\left(\frac{y}{x}\right)x^2$$

$$\Rightarrow \cos \frac{y}{x} = \frac{k}{xy} k = \frac{1}{C}$$

The solution of the given differential equation is $\cos \frac{y}{x} = \frac{k}{xy}$

8. Show that, differential equation $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

$$\frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

Checking for homogeneity

$$F(x,y) = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

$$F(\lambda x, \lambda y) = \frac{\lambda y - \lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda\left(y - x\sin\left(\frac{y}{x}\right)\right)}{\lambda x}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{y - x\sin\left(\frac{y}{x}\right)}{x}$$

$$\Rightarrow F(x, y) = F(\lambda x, \lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx - x\sin\left(\frac{vx}{x}\right)}{x}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{x(v - \sin v)}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow x\frac{dv}{dx} = -\sin v$$

Separate the differentials

$$\frac{1}{\sin v}dv = -\frac{dx}{x}$$

Integrate both side

$$\int \operatorname{cosec} v\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \log\left|\operatorname{cos}ec\, v - \cot v\right| = -\log x + \log C$$

$$\Rightarrow \log\left|\operatorname{cos}ec\, v - \cot v\right| = \log\frac{C}{x}$$

$$\Rightarrow \operatorname{cosec} v - \cot v = \frac{C}{x}$$

$$\Rightarrow \frac{1}{\sin v} - \frac{\cos v}{\sin v} = \frac{C}{x}$$

$$\Rightarrow 1 - \cos v = \frac{C}{x} \sin v$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow 1 - \cos \frac{y}{x} = \frac{C}{x} \sin \frac{y}{x}$$

$$\Rightarrow x\left(1 - \cos \frac{y}{x}\right) = C \sin\left(\frac{y}{x}\right)$$

The solution of the given differentia equation $x\left(1 - \cos \frac{y}{x}\right) = C \sin\left(\frac{y}{x}\right)$

9. Show that, differential equation $ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$ydx = 2xdy - x\log\left(\frac{y}{x}\right)dy$$

$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

Checking for homogeneity

$$F(x, y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda y}{2\lambda x - \lambda x \log\left(\frac{\lambda y}{\lambda x}\right)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda\left(2x - x \log\left(\frac{y}{x}\right)\right)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{y}{2x - x \log\left(\frac{y}{x}\right)}$$

$$\Rightarrow F(x, y) = F(\lambda x, \lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx}{2x - x \log\left(\frac{vx}{x}\right)}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{xv}{x(2 - \log(v))}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2 - \log(v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\log v - v}{2 - \log v}$$

Separate the differentials

$$\frac{2 - \log v}{v\log v - v}dv = \frac{dx}{x}$$

Integrate both side

$$\int \frac{2-\log v}{v\log v-v}dv=\int \frac{dx}{x}$$

$$\int \frac{1+1-\log v}{v\left(\log v-v\right)}dv=\log x+\log C$$

$$\Rightarrow \int \frac{1}{v\left(\log v-1\right)}dv+\int \frac{1-\log v}{v\left(\log v-1\right)}dv=\log x+\log C$$

$$\Rightarrow \int \frac{1}{v\left(\log v-1\right)}dv-\int \frac{1}{v}dv=\log C\,x..........(1)$$

Solving:

$$\int \frac{1}{v\left(\log v-1\right)}dv$$

Substituting $\log v-1=t$

$\log v-1=t$

$$\Rightarrow \frac{1}{v}dv=dt$$

Thus the integral will be

$$\Rightarrow \int \frac{1}{v\left(\log v-1\right)}dv=\int \frac{dt}{t}$$

$$\Rightarrow \int \frac{1}{v\left(\log v-1\right)}dv=\log t$$

$$\Rightarrow \int \frac{1}{v\left(\log v-1\right)}dv=\log\left(\log v-1\right)$$

Using above result for solving (1)

$$\Rightarrow \log\left(\log v-1\right)-\log v=\log Cx$$

$$\Rightarrow \log \frac{\log v - 1}{v} = \log Cx$$

$$\Rightarrow \log \frac{\log v - 1}{v} = Cx$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow \frac{\log \frac{y}{x} - 1}{\frac{y}{x}} = Cx$$

$$\Rightarrow \log \frac{y}{x} - 1 = Cx\left(\frac{y}{x}\right)$$

$$\Rightarrow \log \frac{y}{x} - 1 = Cy$$

The solution of the given differential equation $\log \frac{y}{x} - 1 = Cy$

10. Show that, differential equation $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$ is homogenous and solves it

Solution:

Rewrite the equation in standard form

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\frac{dx}{dy} = \frac{e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{\left(1 + e^{\frac{x}{y}}\right)}$$

Checking for homogeneity

$$F(x,y)=\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

$$F(\lambda x,\lambda y)=\frac{e^{\frac{\lambda x}{\lambda y}}\left(1-\frac{\lambda x}{\lambda y}\right)}{\left(1+e^{\frac{\lambda x}{\lambda y}}\right)}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $x=vy$

$$\frac{d(vy)}{dy}=-\frac{e^{\frac{vy}{y}}\left(1-\frac{vy}{y}\right)}{1+e^{\frac{vy}{y}}}$$

$$\Rightarrow v\frac{dy}{dy}+y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{1+e^{v}}$$

$$\Rightarrow v+y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{e^{v}(1-v)}{1+e^{v}}-v$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{-e^{v}+ve^{v}-v-ve^{v}}{1+e^{v}}$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{-\left(e^{v}+v\right)}{1+e^{v}}$$

Separate the differentials

$$\frac{1+e^{v}}{e^{v}+v}dv=-\frac{dy}{y}$$

Integrate both side

$$\int\frac{1+e^{v}}{e^{v}+v}dv=-\int\frac{dy}{y}$$

$$\int\frac{e^{v}+1}{e^{v}+v}dv=-\log y+\log C..........(1)$$

Solving the LHS integral. Substitute $e^{v}+v=t$

$$e^{v}+v=t$$

$$\Rightarrow\left(e^{v}+1\right)dv=dt$$

Solving the expression (1)

$$\Rightarrow\int\frac{1}{t}dt=\log\frac{C}{y}$$

$$\Rightarrow\log(t)=\log\frac{C}{y}$$

$$\Rightarrow\log\left(e^{v}+v\right)=\log\frac{C}{y}$$

$$\Rightarrow e^{v}+v=\frac{C}{y}$$

Back substitute $v=\frac{x}{y}$

$$\Rightarrow e^{\frac{x}{y}}+\frac{x}{y}=\frac{C}{y}$$

$$\Rightarrow ye^{\frac{x}{y}} + x = C$$

The solution of the given differential equation $ye^{\frac{x}{y}} + x = C$

11. For the differential equation $(x+y)dy + (x-y)dx = 0$. Find the particular solution for the condition $y = 1$ when $x = 1$

Solution:

Given differential equation is

$$(x+y)dy + (x-y)dx = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x-y}{x+y}$$

Checking for homogeneity

$$F(x,y) = -\frac{x-y}{x+y}$$

$$\Rightarrow F(\lambda x, \lambda y) = -\frac{\lambda(x-y)}{\lambda(x+y)}$$

$$\Rightarrow F(\lambda x, \lambda y) = -\frac{(x-y)}{(x+y)}$$

$$\Rightarrow F(x,y) = F(\lambda x, \lambda y)$$

Thus it is an homogeneous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = -\frac{x-(vx)}{x+(vx)}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{x(v-1)}{x(v+1)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1-v^2-v}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{v+1}$$

Separate the differentials

$$\frac{v+1}{1+v^2}dv = -\frac{dx}{x}$$

Integrate both side

$$\int \frac{v+1}{1+v^2}dv = -\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{v}{1+v^2}dv + \int \frac{1}{1+v^2}dv = -\log x + k$$

$$\Rightarrow \frac{1}{2}\log\left(1+v^2\right) + \tan^{-1} v + \log x = k$$

$$\Rightarrow \frac{1}{2}\log\left[x\left(1+v^2\right)\right] + \tan^{-1} v = k$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow \frac{1}{2}\log\left[x\left(1+\frac{y^2}{x^2}\right)\right] + \tan^{-1}\frac{y}{x} = k$$

$$\Rightarrow \frac{1}{2}\log\left[\frac{x^2+y^2}{x}\right] + \tan^{-1}\frac{y}{x} = k$$

Now $y = 1$ and $x = 1$

$$\Rightarrow \frac{1}{2}\log\left[\frac{1^2+1^2}{1}\right]+\tan^{-1}\frac{1}{1}=k$$

$$k=\frac{1}{2}\log 2+\frac{\pi}{4}$$

The required particular solution

$$\Rightarrow \frac{1}{2}\log\left[\frac{x^2+y^2}{x}\right]+\tan^{-1}\frac{y}{x}=\frac{1}{2}\log 2+\frac{\pi}{4}$$

12. For the differential equation $x^2dy+\left(xy+y^2\right)dx=0$. Find the particular solution for the condition $y=1$ when $x=1$

Solution:

Given differential equation is $x^2dy+\left(xy+y^2\right)dx=0$

$$\Rightarrow \frac{dy}{dx}=-\frac{xy+y^2}{x^2}$$

Checking for homogeneity

$$F(x,y)=-\frac{xy+y^2}{x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{(\lambda x)(\lambda y)+\lambda^2y^2}{\lambda^2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda^2\left(xy+y^2\right)}{\lambda^2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{xy+y^2}{x^2}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y = vx$

$$\frac{d(vx)}{dx} = -\frac{-x(vx)+(vx)^2}{x^2}$$

$$\Rightarrow v\frac{dx}{dx} + x\frac{dv}{dx} = -\frac{vx^2 + v^2x^2}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{x^2(v+v^2)}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = -v - v^2$$

$$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v$$

Separate the differentials

$$\frac{1}{v^2+2v}dv = -\frac{dx}{x}$$

Integrate both side

$$\int \frac{1}{v^2+2v}dv = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int \frac{v+2-v}{v(v+2)}dv = -\log x + \log C$$

$$\Rightarrow \frac{1}{2}\int \frac{v+2}{v(v+2)}dv - \frac{1}{2}\int \frac{v}{v(v+2)}dv = -\log x + \log C$$

$$\Rightarrow \frac{1}{2}\int \frac{1}{v}dv - \frac{1}{2}\int \frac{1}{v+2}dv = -\log x + \log C$$

$$\Rightarrow \frac{1}{2}\log v - \frac{1}{2}\log(v+2) = \log\frac{C}{x}$$

$$\Rightarrow \frac{1}{2}\log\frac{v}{v+2} = \log\frac{C}{x}$$

$$\Rightarrow \frac{v}{v+2}=\left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{v}{v+2}=\frac{C^2}{x^2}$$

Back substitute $v=\frac{y}{x}$

$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2}=\frac{C^2}{x^2}$$

$$\Rightarrow \frac{x^2y}{y+2x}=C^2$$

Now y = 1 and x = 1

$$\Rightarrow \frac{1^2(1)}{1+2(1)}=C^2$$

$$\Rightarrow C^2=\frac{1}{3}$$

The required particular solution

$$\Rightarrow \frac{x^2y}{y+2x}=\frac{1}{3}$$

$$\Rightarrow y+2x=3x^2y$$

13. For the differential equation $\left[x\sin^2\left(\frac{x}{y}\right)-y\right]dx+xdy=0$. Find the particular solution for the condition $y=\frac{\pi}{4}$ when x = 1

Solution:

Given differential equation is $\left[x\sin^2\left(\frac{x}{y}\right)-y\right]dx+xdy=0$

$$\Rightarrow \frac{dy}{dx}=-\frac{\left[x\sin^2\left(\frac{x}{y}\right)-y\right]}{x}$$

Checking for homogeneity

$$F(x,y)=-\frac{x\sin^2\left(\frac{x}{y}\right)-y}{x}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda x\sin^2\left(\frac{\lambda x}{\lambda y}\right)-\lambda y}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{\lambda\left(x\sin^2\left(\frac{y}{x}\right)-y\right)}{\lambda x}$$

$$\Rightarrow F(\lambda x,\lambda y)=-\frac{x\sin^2\left(\frac{y}{x}\right)-y}{x}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=-\frac{x\sin^2\left(\frac{vx}{x}\right)-vx}{x}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{-x\sin^2(v)+vx}{x}$$

$$\Rightarrow v+x\frac{dv}{dx}=-\sin^2 v+v$$

$$\Rightarrow x\frac{dv}{dx} = -\sin^2 v$$

Separate the differentials

$$\operatorname{cosec}^2 v\,dv = -\frac{dx}{x}$$

Integrate both sides

$$\int \operatorname{cosec}^2 v\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow -\cot v = -\log x - \log C$$

$$\Rightarrow \cot v = \log Cx$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow \cot\frac{y}{x} = \log Cx$$

Now $y = \frac{\pi}{4}$ and x = 1

$$\cot\frac{\frac{\pi}{4}}{1} = \log C(1)$$

$$\Rightarrow \log C = \cot\frac{\pi}{4}$$

$$\Rightarrow \log C = 1$$

$$\Rightarrow C = e$$

The required particular solution

$$\Rightarrow \cot\frac{y}{x} = \log|ex|$$

14. For the differential equation $\frac{dy}{dx}-\frac{y}{x}+\cos ec\left(\frac{y}{x}\right)=0$. Find the particular solution for the condition y = 0 and x = 1

Solution:

Given differential equation is $\frac{dy}{dx}-\frac{y}{x}+\cos ec\left(\frac{y}{x}\right)=0$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)$$

Checking for homogeneity

$$F(x,y)=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=v-\cos ec(v)$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=v-\cos ec(v)$$

$$\Rightarrow v+x\frac{dv}{dx}=v-\cos ec(v)$$

$$\Rightarrow x\frac{dv}{dx}=-\cos ec(v)$$

Separate the differentials

$$\sin v dv=-\frac{dx}{x}$$

Integrate both sides

$$\int \sin v dv = -\int \frac{dx}{x}$$

$$\Rightarrow \cos v = \log|Cx|$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow \cos\frac{y}{x} = \log|Cx|$$

Now Y = 0 and x = 1

$$\Rightarrow \cos\frac{0}{1} = \log|C1|$$

$$\Rightarrow \log C = 1$$

$$\Rightarrow C = 1$$

The required particular solution

$$\Rightarrow \cos\frac{y}{x} = \log|ex|$$

15. For the differential equation $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$. Find the particular solution for the condition y = 2 when x = 1

Solution:

Given differential equation is $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

Checking for homogeneity

$$F(x,y)=\frac{2xy+y^2}{2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{2(\lambda x)(\lambda y)+\lambda^2y^2}{2\lambda^2x^2}$$

$$\Rightarrow F(\lambda x,\lambda y)=\frac{2xy+y^2}{2x^2}$$

$$\Rightarrow F(x,y)=F(\lambda x,\lambda y)$$

Thus it is an homogenous equation

Let $y=vx$

$$\frac{d(vx)}{dx}=\frac{2x(vx)+(vx)^2}{2x^2}$$

$$\Rightarrow v\frac{dx}{dx}+x\frac{dv}{dx}=\frac{2v+v^2}{2}$$

$$\Rightarrow v+x\frac{dv}{dx}=v+\frac{v^2}{2}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2}{2}$$

Separate the differentials

$$\frac{dv}{v^2}=\frac{1}{2}\left(\frac{dx}{x}\right)$$

Integrate both sides

$$2\int\frac{dv}{v^2}=\int\left(\frac{dx}{x}\right)$$

$$\Rightarrow\frac{v^{-2+1}}{-2+1}=\log|x|+C$$

$$\Rightarrow-\frac{2}{v}=\log|x|+C$$

Back substitute $v = \frac{y}{x}$

$$\Rightarrow -\frac{2x}{y} = \log|x| + C$$

Now y = 2 and x = 1

$$\Rightarrow -\frac{2(1)}{2} = \log|1| + C$$

$$\Rightarrow C = -1$$

The required particular solution

$$\Rightarrow -\frac{2x}{y} = \log|x| - 1$$

$$\Rightarrow y = \frac{2x}{1 - \log|x|}(x \neq 0, e)$$

16. What substitution should be used for solving homogeneous differential equation $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$

Solution:

The required equation substitution will be

$$\frac{x}{y} = v$$

$$\Rightarrow x = vy$$

The correct answer is (C)

17. Which of the following equation is homogeneous

A) $(4x + 6y + 5)dy - (3y + 2x + 4)dx = 0$

B) $(xy)dx-(x^3+y^3)dy=0$

C) $(x^3+2y^2)dx+2xy\,dy=0$

D) $y^2dx+(x^2-xy-y^2)dy=0$

Solution:

For option (A)

$$F(x,y)=\frac{3y+2x+4}{4x+6y+5}$$

$$F(\lambda x,\lambda y)=\frac{3\lambda y+2\lambda x+4}{4\lambda x+6\lambda y+5}$$

$$F(\lambda x,\lambda y)\neq F(x,y)$$

For option (B)

$$F(x,y)=\frac{xy}{x^3+y^3}$$

$$F(\lambda x,\lambda y)=\frac{(\lambda x)(\lambda y)}{(\lambda x)^3+(\lambda y)^3}$$

$$F(\lambda x,\lambda y)=\frac{xy}{\lambda(x+y)}$$

$$F(\lambda x,\lambda y)\neq F(x,y)$$

For Option (C)

$$F(x,y)=-\frac{x^3+2y^2}{2xy}$$

$$F(\lambda x,\lambda y)=-\frac{\lambda^3x^3+2\lambda^2y^2}{2(\lambda x)(\lambda y)}$$

$$F(\lambda x,\lambda y)=-\frac{\lambda x^3+2y^2}{2xy}$$

$$F(\lambda x, \lambda y) \neq F(x, y)$$

For option (D)

$$F(x, y) = -\frac{y^2}{x^2 - xy - y^2}$$

$$F(\lambda x, \lambda y) = -\frac{\lambda^2 y^2}{\lambda^2 x^2 - (\lambda x)(\lambda y) - \lambda^2 y^2}$$

$$F(\lambda x, \lambda y) = -\frac{y^2}{x^2 - xy - y^2}$$

$$F(\lambda x, \lambda y) = F(x, y)$$

Thus the correct answer is option (D)