Chapter: 9. Differential Equations

Exercise: 9.6

1. Find the general solution for the differential equation $\frac{dy}{dx}+2y=\sin x$

Solution:

The given differential equation is $\frac{dy}{dx}+2y=\sin x$

It is a linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=2$$
$$Q=\sin x$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int 2dx}$$

$$\Rightarrow I.F=e^{2x}$$

General solution is of the form

$$y(I.F)=\int (Q\times I.F)dx+C$$

$$ye^{2x}=\int \sin x\left(e^{2x}\right)dx+C$$

$$\Rightarrow ye^{2x}=I+C\left(I=\int \sin x\left(e^{2x}\right)dx\right)..........(1)$$

$$I=\int \sin x\left(e^{2x}\right)dx$$

$$\Rightarrow I=(\sin x)\int e^{2x}\,dx-\int\left((\sin x)'\int e^{2x}dx\right)dx$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin x-\int\left(\cos x\left(\frac{e^{2x}}{2}\right)\right)dx$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin x-\frac{1}{2}\int\left[\cos x\int e^{2x}dx-\int\left((\cos x)'\left(\int e^{2x}dx\right)\right)\right]dx$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin x-\frac{1}{2}\left[\frac{e^{2x}}{2}\cos x+\frac{1}{2}\int e^{2x}(\sin x)dx\right]$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin x-\frac{1}{2}\left[\frac{e^{2x}}{2}\cos x+\frac{1}{2}I\right]$$

$$\Rightarrow I=\frac{e^{2x}}{2}\sin x-\frac{e^{2x}}{4}\cos x-\frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I=\frac{e^{2x}}{2}\sin x-\frac{e^{2x}}{4}\cos x$$

$$\Rightarrow I=\frac{2e^{2x}}{5}\sin x-\frac{e^{2x}}{5}\cos x$$

$$\Rightarrow I=\frac{e^{2x}}{5}[2\sin x-\cos x]$$

Back substituting I in expression (1)

$$\Rightarrow ye^{2x}=\frac{e^{2x}}{5}[2\sin x-\cos x]+C$$

$$\Rightarrow y=\frac{1}{5}(2\sin x-\cos x)+Ce^{-2x}$$

The general solution for given differential equation is $y=\frac{1}{5}(2\sin x-\cos x)+Ce^{-2x}$

2. Find the general solution for the differential equation $\frac{dy}{dx}+3y=e^{-2x}$

Solution:

The given differential equation is $\frac{dy}{dx}+3y=e^{-2x}$

It is a linear differential equation of the form $\frac{dy}{dx}+px=Q$, with

$$p=3$$
$$Q=e^{-2x}$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int 3dx}$$

$$\Rightarrow I.F=e^{3x}$$

General solution is of the form

$$\Rightarrow y(I.F)=\int (Q\times I.F)dx+C$$

$$\Rightarrow ye^{3x}=\int e^{-2x}\left(e^{3x}\right)dx+C$$

$$\Rightarrow ye^{3x}=\int e^{-2x+3x}\,dx+C$$

$$\Rightarrow ye^{3x}=\int e^{x}\,dx+C$$

$$\Rightarrow ye^{3x}=e^{x}+C$$

$$\Rightarrow y=e^{-2x}+Ce^{-3x}$$

The general solution for given differential equation is $y=e^{-2x}+Ce^{-3x}$

3. Find the general solution for the differential equation $\frac{dy}{dx}+\frac{y}{x}=x^2$

Solution:

The given differential equation is $\frac{dy}{dx}+\frac{y}{x}=x^2$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p = \frac{1}{x}$$
$$Q = x^2$$

Calculate the integrating factor

$$\Rightarrow I.F = e^{\int \frac{1}{x} dx}$$

$$\Rightarrow I.F = e^{\log x}$$

$$\Rightarrow I.F = x$$

General solution is of the form

$$y(I.F) = \int (Q \times I.F) dx + C$$

$$\Rightarrow yx = \int x^2 (x) dx + C$$

$$\Rightarrow xy = \int x^3 dx + C$$

$$\Rightarrow xy = \frac{x^{3+1}}{3+1} + C$$

$$\Rightarrow xy = \frac{x^4}{4} + C$$

The general solution for given differential equation is $xy = \frac{x^4}{4} + C$

4. Find the general solution for the differential equation

$$\frac{dy}{dx} + (\sec x) y = \tan x \left(0 \le x \le \frac{\pi}{2} \right)$$

Solution:

The given differential equation is $\frac{dy}{dx} + (\sec x) y = \tan x$

It is a linear differential equation of the form $\frac{dy}{dx} + py = Q$, with

$p = \sec x$

$Q = \tan x$

Calculate the integrating factor

$$I.F = e^{\int p dx}$$

$$\Rightarrow I.F = e^{\int \sec x dx}$$

$$\Rightarrow I.F = e^{\log(\sec x + \tan x)}$$

$$\Rightarrow I.F = (\sec x + \tan x)$$

General solution is of form

$$y(I.F) = \int (Q \times I.F) dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x) dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int \tan x \sec x \, dx + \int \tan^2 x \, dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \int (\sec^2 x - 1) dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \int \sec^2 x \, dx - \int dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \tan x - x + C$$

The general solution for given differential equation is

$$y(\sec x + \tan x) = \sec x + \tan x - x + C$$

5. Find the general solution for the differential equation

$$\cos^2 x \frac{dy}{dx} + y = \tan x \left(0 \le x \le \frac{\pi}{2}\right)$$

Solution:

The given differential equation is $\cos^2 x\frac{dy}{dx}+y=\tan x$

$$\frac{dy}{dx}+\left(\sec^2 x\right)y=\sec^2 x\tan x$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\sec^2 x$$
$$Q=\sec^2 x\tan x$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int \sec^2 xdx}$$

$$\Rightarrow I.F=e^{\tan x}$$

General solution is of the form

$$y\left(I.F\right)=\int\left(Q\times I.F\right)dx+C$$

$$\Rightarrow ye^{\tan x}=\int e^{\tan x}\left(\sec^2 x\tan x\right)dx+C$$

$$\Rightarrow ye^{\tan x}=I+C\left(I=\int e^{\tan x}\left(\sec^2 x\tan x\right)dx\right)............(1)$$

Solving the integral I

$$I=\int e^{\tan x}\left(\sec^2 x\tan x\right)dx$$

Substitute $\tan x=t$

$$\tan x=t$$

$$\Rightarrow \sec^2 xdx=dt$$

$$\Rightarrow I=\int e^t t\,dt$$

$$\Rightarrow I=t\int e^t\,dt-\int\left(\left(t\right)'\int e^t dt\right)dt$$

$$\Rightarrow I = te^t - I!\left(e^t\right)dt$$

$$\Rightarrow I = te^t - e^t$$

Back substitute t:

$$I = \tan x e^{\tan x} - e^{\tan x}$$

Back substitute I in expression (1)

$$\Rightarrow ye^{\tan x} = I + C$$

$$\Rightarrow ye^{\tan x} = \tan x e^{\tan x} - e^{\tan x} + C$$

The general solution for given differential equation is $ye^{\tan x} = \tan x e^{\tan x} - e^{\tan x} + C$

6. Find the general solution for the differential equation $x\frac{dy}{dx} + 2y = x^2 \log x$

Solution:

The given differential equation is $x\frac{dy}{dx} + 2y = x^2 \log x$

$$\frac{dy}{dx} + \frac{2}{x}y = x\log x$$

It is linear differential equation of the form $\frac{dy}{dx} + py = Q$, with

$$p = \frac{2}{x}$$
$$Q = x\log x$$

Calculate the integrating factor

$$I.F = e^{\int pdx}$$

$$\Rightarrow I.F = e^{\int \frac{2}{x}dx}$$

$$\Rightarrow I.F = e^{2\log x}$$

$$\Rightarrow I.F = e^{\log x^2}$$

$$\Rightarrow I.F = x^2$$

General solution is of the form

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

$$\Rightarrow yx^2 = \int x^2 (x \log x)\,dx + C$$

$$\Rightarrow yx^2 = I + C\left(I = \int x^3 \log x dx\right) \ldots\ldots..(1)$$

Solving the integral I

$$I = \int x^3 \log x dx$$

$$\Rightarrow I = \log x \int x^3 dx - \int \left((\log x)' \int x^3 dx\right) dx$$

$$\Rightarrow I = \frac{x^4}{4} \log x - \int \left(\frac{1}{x}\left(\frac{x^4}{4}\right)\right) dx$$

$$\Rightarrow I = \frac{x^4}{4} \log x - \frac{1}{4} \int x^3\, dx$$

$$\Rightarrow I = \frac{x^4}{4} \log x - \frac{1}{4} \int \left(\frac{x^4}{4}\right)$$

$$\Rightarrow I = \frac{x^4}{4} \log x - \frac{x^4}{16}$$

Back substitute I in expression (1)

$$\Rightarrow yx^2 = I + C$$

$$\Rightarrow yx^2 = \frac{x^4}{4} \log x - \frac{x^4}{16} + C$$

$$\Rightarrow y = \frac{x^4}{16}(4 \log x - 1) + Cx^{-2}$$

The general solution for given differential equation is $y=\frac{x^4}{16}(4\log x-1)+Cx^{-2}$

7. Find the general solution for the differential equation $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$

Solution:

The given differential equation is $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$

$$\frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x^2}$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\frac{1}{x\log x}$$

$$Q=\frac{2}{x^2}$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int \frac{1}{x\log x}dx}$$

Substitute $\log x=t$

$$\log x=t$$

$$\Rightarrow \frac{1}{x}dx=dt$$

$$\Rightarrow I.F=e^{\int \frac{1}{t}dt}$$

$$\Rightarrow I.F=e^{\log t}$$

$$\Rightarrow I.F=t$$

$$\Rightarrow I.F=\log x$$

General solution is of the form

$$\Rightarrow y(I.F)=\int(Q\times I.F)dx+C$$

$$\Rightarrow y\log x=\int\frac{2}{x^2}(\log x)dx+C$$

$$\Rightarrow y\log x=I+C\left(I=\int\frac{2}{x^2}(\log x)dx\right)............(1)$$

Solving the integral I

$$I=\int\frac{2}{x^2}(\log x)dx$$

$$I=2\left[\log x\int\frac{1}{x^2}dx-\int\left((\log x)'\int\frac{1}{x^2}dx\right)dx\right]$$

$$I=2\left[\log x\left(\frac{-1}{x}\right)-\int\left(\frac{1}{x}\left(\frac{-1}{x}\right)\right)dx\right]$$

$$I=2\left[-\frac{\log x}{x}+\int\left(\frac{1}{x^2}\right)dx\right]$$

$$I=2\left[-\frac{\log x}{x}-\frac{1}{x}\right]$$

Back substitute I in expression (1)

$$y\log x=I+C$$

$$\Rightarrow y\log x=2\left[-\frac{\log x}{x}-\frac{1}{x}\right]+C$$

$$\Rightarrow y\log x=-\frac{2}{x}(\log x+1)+C$$

The general solution for given differential equation is $y\log x=-\frac{2}{x}(\log x+1)+C$

8. Find the general solution for the differential equation

$$\left(1+x^2\right)dy+2xydx=\cot xdx\left(x\neq 0\right)$$

Solution:

The given differential equation is

$$\left(1+x^2\right)dy+2xydx=\cot xdx\left(x\neq 0\right)$$

$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\cot x}{1+x^2}$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\frac{2x}{1+x^2}$$

$$Q=\frac{\cot x}{1+x^2}$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int \frac{2x}{1+x^2}dx}$$

Substitute $\log x=t$

$$1+x^2=t$$

$$\Rightarrow 2xdx=dt$$

$$\Rightarrow I.F=e^{\int \frac{1}{t}dt}$$

$$\Rightarrow I.F=e^{\log t}$$

$$\Rightarrow I.F=t$$

$$\Rightarrow I.F=1+x^2$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)\,dx+C$$

$$\Rightarrow y(1+x^2)=\int\frac{\cot x}{1+x^2}(1+x^2)\,dx+C$$

$$\Rightarrow y(1+x^2)=\int\cot x\,dx+C$$

$$\Rightarrow y(1+x^2)=\log|\sin x|+C$$

The general solution for given differential equation is $y(1+x^2)=\log|\sin x|+C$

9. Find the general solution for the differential equation

$$x\frac{dy}{dx}+y-x+xy\cot x=0\,(x\neq 0)$$

Solution:

The given differential equation is $x\frac{dy}{dx}+y-x+xy\cot x=0$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x}-1+y\cot x=0$$

$$\Rightarrow \frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$$

It is linear differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\left(\frac{1}{x}+\cot x\right)$$
$$Q=1$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int\left(\frac{1}{x}+\cot x\right)dx}$$

$\Rightarrow I.F = e^{\int \frac{1}{x}dx + \int \cot x dx}$

$\Rightarrow I.F = e^{\log x + \log(\sin x)}$

$\Rightarrow I.F = e^{\log(x \sin x)}$

$\Rightarrow I.F = x \sin x$

General solution is of the form

$y(I.F) = \int (Q \times I.F) dx + C$

$\Rightarrow y(x \sin x) = \int (1)(x \sin x) dx + C$

$\Rightarrow xy \sin x = I + C \left(I = \int x \sin x \, dx \right) ....(1)$

Solving the integral I

$I = \int x \sin x \, dx$

$\Rightarrow I = x \int \sin x \, dx - \int \left((x)' \int \sin x dx \right) dx$

$\Rightarrow I = x(-\cos x) + \int (\cos x) dx$

$\Rightarrow I = x \cos x + \sin x$

Back substitute I in expression (1)

$xy \sin x = I + C$

$\Rightarrow xy \sin x = -x \cos x + \sin x + C$

$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$

The general solution for given differential equation is $y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$

10. Find the general solution for the differential equation $(x+y)\frac{dy}{dx} = 1$

Solution:

The given differential equation is $(x+y)\frac{dy}{dx}=1$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+y}$$

$$\Rightarrow \frac{dx}{dy}=x+y$$

$$\Rightarrow \frac{dx}{dy}-x=y$$

It is differential equation of the form $\frac{dx}{dy}+px=Q$, with

$$p=-1$$
$$Q=y$$

Calculate the integrating factor

$$I.F=e^{\int pdy}$$

$$\Rightarrow I.F=e^{\int -1dy}$$

$$\Rightarrow I.F=e^{-y}$$

General solution is of the form

$$x(I.F)=\int (Q\times I.F)\,dy+C$$

$$\Rightarrow x\left(e^{-y}\right)=\int (y)\left(e^{-y}\right)dy+C$$

$$\Rightarrow xe^{-y}=I+C\left(I=\int ye^{-y}dy\right)\ldots\ldots\ldots\ldots(1)$$

Solving the integral I

$$I=\int ye^{-y}dy$$

$$\Rightarrow I=y\int e^{-y}dy-\int\left((y)'\int e^{-y}dy\right)dy$$

$$\Rightarrow I = -ye^{-y} + \int ((1)e^{-y})\,dy$$

$$\Rightarrow I = -ye^{-y} + \int e^{-y}\,dy$$

$$\Rightarrow I = -ye^{-y} - e^{-y}$$

Back substitute I in expression (1)

$$xe^{-y} = I + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$$

$$\Rightarrow x = -y - 1 + Ce^{y}$$

$$\Rightarrow x + y + 1 = Ce^{y}$$

The general solution for given differential equation is $x + y + 1 = Ce^{y}$

11. Find the general solution for the differential equation $ydx + (x - y^2)dy = 0$

Solution:

The given differential equation is $ydx + (x - y^2)dy = 0$

$$\Rightarrow \frac{dx}{dy} = \frac{y^2 - x}{y}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{y}\right)x = y$$

It is differential equation of the form $\frac{dx}{dy} + px = Q$, with

$$p = \frac{1}{y}$$

$$Q = y$$

Calculate the integrating factor

$I.F = e^{\int p dy}$

$\Rightarrow I.F = e^{\int \frac{1}{y} dy}$

$\Rightarrow I.F = e^{\log y}$

$\Rightarrow I.F = y$

General solution is of the form

$x(I.F) = \int (Q \times I.F) dy + C$

$\Rightarrow x(y) = \int (y)(y) dy + C$

$\Rightarrow xy = \int y^2 dy + C$

$\Rightarrow xy = \frac{y^3}{3} + C$

$\Rightarrow x = \frac{y^2}{3} + \frac{C}{y}$

The general solution for given differential equation is $x = \frac{y^2}{3} + \frac{C}{y}$

12. Find the general solution for the differential equation $\left(x + 3y^2\right)\frac{dy}{dx} = y\left(y > 0\right)$

Solution:

The given differential equation is $\left(x + 3y^2\right)\frac{dy}{dx} = y$

$\Rightarrow \frac{dx}{dy} = \frac{x + 3y^2}{y}$

$\Rightarrow \frac{dx}{dy} - \left(\frac{1}{y}\right)x = 3y$

It is differential equation of the form $\frac{dx}{dy}+px=Q$, with

$$p=-\frac{1}{y}$$
$$Q=3y$$

Calculate the integrating factor

$$I.F=e^{\int pdy}$$

$$\Rightarrow I.F=e^{-\int\frac{1}{y}dy}$$

$$\Rightarrow I.F=e^{-\log y}$$

$$\Rightarrow I.F=e^{\log y^{-1}}$$

$$\Rightarrow I.F=\frac{1}{y}$$

General solution is of the form

$$x(I.F)=\int(Q\times I.F)dy+C$$

$$\Rightarrow x\left(\frac{1}{y}\right)=\int(3y)\left(\frac{1}{y}\right)dy+C$$

$$\Rightarrow \frac{x}{y}=3\int dy+C$$

$$\Rightarrow \frac{x}{y}=3y+C$$

$$\Rightarrow x=3y^2+Cy$$

The general solution for given differential equation is $x=3y^2+Cy$

13. Find particular solution for $\frac{dy}{dx}+2y\tan x=\sin x$ satisfying $y=0$ when $x=\frac{\pi}{3}$

Solution:

The given differential equation is $\frac{dy}{dx}+2y\tan x=\sin x$

$$\frac{dy}{dx}+(2\tan x)y=\sin x$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=2\tan x$$
$$Q=\sin x$$

Calculate the integration factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{2\int \tan x dx}$$

$$\Rightarrow I.F=e^{2\log|\sec x|}$$

$$\Rightarrow I.F=e^{\log(\sec x)^2}$$

$$\Rightarrow I.F=\sec^2 x$$

General solution is of the form

$$\Rightarrow y(I.F)=\int(Q\times I.F)dx+C$$

$$\Rightarrow y\sec^2 x=\int(\sin x)(\sec^2 x)dx+C$$

$$\Rightarrow y\sec^2 x=\int \tan x\sec x\,dx+C$$

$$\Rightarrow y\sec^2 x=\sec x+C$$

$$\Rightarrow y=\cos x+C\cos^2 x$$

Given $y=0$ when $x=\frac{\pi}{3}$

$$0=\cos\left(\frac{\pi}{3}\right)+C\cos^2\left(\frac{\pi}{3}\right)$$

$$\Rightarrow 0=\frac{1}{2}+C\left(\frac{1}{2}\right)^2$$

$$\Rightarrow C=-2$$

Therefore the particular solution will be

$$\Rightarrow y=\cos x-2\cos^2 x$$

The particular solution for given differential equation satisfying the given conditions is

$$y=\cos x-2\cos^2 x$$

14. Find particular solution for $\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$ satisfying $y=0$ when $x=1$

Solution:

The given differential equation is $\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$

$$\frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y=\frac{1}{\left(1+x^2\right)^2}$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=\frac{2x}{1+x^2}$$
$$Q=\frac{1}{\left(1+x^2\right)^2}$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{\int\frac{2x}{1+x^2}dx}$$

$\Rightarrow I.F = e^{2\log|\sec x|}$

$\Rightarrow I.F = e^{\log(1+x^2)}$

$\Rightarrow I.F = 1+x^2$

General solution is of the form

$y(I.F) = \int (Q \times I.F)\,dx + C$

$\Rightarrow y(1+x^2) = \int \left(\frac{1}{(1+x^2)^2}\right)(1+x^2)\,dx + C$

$\Rightarrow y(1+x^2) = \int \frac{1}{1+x^2}dx + C$

$\Rightarrow y(1+x^2) = \tan^{-1} x + C$

Given $y=0$ when $x=1$

$0(1+1) = \tan^{-1}(1) + C$

$\Rightarrow C + \frac{\pi}{4} = 0$

$\Rightarrow C = -\frac{\pi}{4}$

Therefore the particular solution will be

$\Rightarrow y(1+x^2) = \tan^{-1} x - \frac{\pi}{4}$

The particular solution for given differential equation satisfying the given conditions is

$y(1+x^2) = \tan^{-1} x - \frac{\pi}{4}$

15. Find particular solution for $\frac{dy}{dx} - 3y\cot x = \sin 2x$ satisfying $y=2$ when $x = \frac{\pi}{2}$

Solution:

The given differential equation is $\frac{dy}{dx}-3y\cot x=\sin 2x$

$$\frac{dy}{dx}+\left(-3\cot x\right)y=\sin 2x$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=-3\cot x$$
$$Q=\sin 2x$$

Calculate the integrating factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{-3\int \cot x dx}$$

$$\Rightarrow I.F=e^{-3\log|\sin x|}$$

$$\Rightarrow I.F=e^{\log\left(\frac{1}{\sin^3 x}\right)}$$

$$\Rightarrow I.F=\frac{1}{\sin^3 x}$$

General solution is of the form

$$y\left(I.F\right)=\int\left(Q\times I.F\right)dx+C$$

$$\Rightarrow y\left(\frac{1}{\sin^3 x}\right)=\int\left(\sin 2x\right)\left(\frac{1}{\sin^3 x}\right)dx+C$$

$$\Rightarrow y\left(\frac{1}{\sin^3 x}\right)=2\int\left(\sin x\cos x\right)\left(\frac{1}{\sin^3 x}\right)dx+C$$

$$\Rightarrow \frac{y}{\sin^3 x}=2\int\left(\frac{\cos x}{\sin^2 x}\right)dx+C$$

$$\Rightarrow \frac{y}{\sin^3 x}=2\int \cot x\cos ecx\,dx+C$$

$$\Rightarrow \frac{y}{\sin^3 x} = -2\operatorname{cosec} x + C$$

$$\Rightarrow y = -2\sin^2 x + C\sin^3 x$$

Given $y = 2$ when $x = \frac{\pi}{2}$

$$2 = -2\sin^2\left(\frac{\pi}{2}\right) + C\sin^3\left(\frac{\pi}{2}\right)$$

$$\Rightarrow C - 2 = 2$$

$$\Rightarrow C = 4$$

Therefore the particular solution will be

$$\Rightarrow y = -2\sin^2 x + 4\sin^3 x$$

The particular solution for given differential equation satisfying the given conditions is

$$y = -2\sin^2 x + 4\sin^3 x$$

16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point

Solution:

According to question the slope of tangent $\frac{dy}{dx}$ is equal to sum of the coordinate

$$\frac{dy}{dx} = x + y$$

The given differential equation is

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx}+(-1)y=x$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$p=-1$
$Q=x$

Calculate he integration factor

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{-1\int dx}$$

$$\Rightarrow I.F=e^{-x}$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)dx+C$$

$$\Rightarrow y(e^{-x})=\int x(e^{-x})dx+C$$

$$\Rightarrow ye^{-x}=-xe^{-x}+\int(e^{-x})dx+C$$

$$\Rightarrow ye^{-x}=-xe^{-x}-e^{-x}+C$$

$$\Rightarrow y=-x-1+Ce^{x}$$

$$\Rightarrow y+x+1=Ce^{x}$$

Given $y=0$ when $x=0$ as it passes through origin

$$0+0+1=Ce^{0}$$

$$\Rightarrow C=1$$

Therefore the equation of the required curve is $y+x+1=e^{x}$

17. Find the equation of a curve passing through the point (0,2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5

Solution:

Let the slope of tangent be $\frac{dy}{dx}$

According to question

$$x+y=\frac{dy}{dx}+5$$

The given differential equation is

$$x+y=\frac{dy}{dx}+5$$

$$\frac{dy}{dx}+(-1)y=x-5$$

It is differential equation of the form $\frac{dy}{dx}+py=Q$, with

$$p=1$$
$$Q=x-5$$

Calculate the integrating factor

$$I.\mathrm{F}=\mathrm{e}^{\int pdx}$$

$$\Rightarrow I.\mathrm{F}=\mathrm{e}^{-1\int dx}$$

$$\Rightarrow I.\mathrm{F}=\mathrm{e}^{-x}$$

General solution is of the form

$$y(I.F)=\int(Q\times I.F)dx+C$$

$$y\left(e^{-x}\right)=\int(x-5)\left(e^{-x}\right)dx+C$$

$$\Rightarrow y\left(e^{-x}\right)=\int x\left(e^{-x}\right)dx-\int e^{-x}\,dx+C$$

$$\Rightarrow y\left(e^{-x}\right)=x\int e^{-x}dx-\int\left((x)'\int e^{-x}dx\right)dx-5\int e^{-x}dx+C$$

$$\Rightarrow ye^{-x}=-xe^{-x}+\int\left(e^{-x}\right)dx+5e^{-x}+C$$

$$\Rightarrow ye^{-x}=-xe^{-x}+4e^{-x}+C$$

$$\Rightarrow y=-x+4+Ce^{-x}$$

$$\Rightarrow y+x-4=Ce^{-x}$$

Given as it passes through (0,2)

$$2+0-4=Ce^{0}$$

$$\Rightarrow C=-2$$

Therefore the equation of the required curve is $y+x+4=-2e^{x}$

18. Find the integrating factor of the differential equation $x\dfrac{dy}{dx}-y=2x^2$

Solution:

Given differential equation is $x\dfrac{dy}{dx}-y=2x^2$

$$\Rightarrow \frac{dy}{dx}-\left(\frac{1}{x}\right)y=2x^2$$

Thus it is a linear differential equation of the form $\dfrac{dy}{dx}+py=Q$

$$p=-\frac{1}{x}$$

$$I.F=e^{\int pdx}$$

$$\Rightarrow I.F=e^{-\int\frac{1}{x}dx}$$

$\Rightarrow I.F = e^{-\log|x|}$

$\Rightarrow I.F = e^{\log x^{-1}}$

$\Rightarrow I.F = \frac{1}{x}$

Therefore integrating factor is $\frac{1}{x}$.

Thus the correct option is (C)

19. Find the integrating factor of the differential equation

$\left(1-y^2\right)\frac{dx}{dy}+yx=ay\left(-1<y<1\right)$

Solution:

Given differential equation is $\left(1-y^2\right)\frac{dx}{dy}+yx=ay$

$\Rightarrow \frac{dx}{dy}+\left(\frac{y}{1-y^2}\right)x=\frac{ay}{\left(1-y^2\right)}$

Thus it is a linear differential equation of the form $\frac{dx}{dy}+px=Q$

$p=\frac{y}{1-y^2}$

$I.F = e^{\int pdy}$

$\Rightarrow I.F = e^{\int \frac{y}{1-y^2}dy}$

$\Rightarrow I.F = e^{-\frac{1}{2}\int \frac{-2y}{1-y^2}dy}$

$\Rightarrow I.F = e^{-\frac{1}{2}\log\left(1-y^2\right)}$

$$\Rightarrow I.F = e^{\log\left(1-y^2\right)^{-\frac{1}{2}}}$$

$$\Rightarrow I.F = \frac{1}{\sqrt{1-y^2}}$$

Therefore integrating factor is $\dfrac{1}{\sqrt{1-y^2}}$

Thus the correct option is (D)